

Evolution of probability measures by cellular automata on algebraic topological Markov chains

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ABSTRACT

In this paper we review some recent results on the evolution of probability measures under cellular automata acting on a fullshift. In particular we discuss the crucial role of the attractiveness of maximal measures. We enlarge the context of the results of a previous study of topological Markov chains that are Abelian groups; the shift map is an automorphism of this group. This is carried out by studying the dynamics of Markov measures by a particular additive cellular automata.

Many of these topics were within the focus of Francisco Varela's mathematical interests.

Key terms: Topological Markov chains, cellular automata, maximal measure, entropy, additive cellular automata.

INTRODUCTION

Let A be a finite set of symbols and $M = (M_{a,b} : a, b \in A)$ be a $\{0,1\}$ -valued matrix indexed by the alphabet A . We consider the set of allowed two-sided sequences of symbols in A or configurations, $X_M = \{x = (x_i)_{i \in \mathbb{Z}} \cap i \in \mathbb{Z}, M_{x_i x_{i+1}} = 1\}$

The matrix M specifies a set of constraints of the configurations, $M_{a,b=0}$ means that the path from a to b is forbidden, and on the contrary, when $M_{a,b=1}$ it is allowed. In the event that all entries of M are equal to one, the set X_M , is called fullshift and denoted by $A^{\mathbb{Z}}$.

A cellular automaton (CA) defined on X_M is a map $F : X_M \rightarrow X_M$ such that for $x \in X_M$ and $i \in \mathbb{Z}$, $(Fx)_i = f(x_{i-r} \dots x_{i+r})$ where $r \in \mathbb{N}$ is the radius and $f : A^{2r+1} \rightarrow A$ is a given local rule (see Figure 1). When the local rule does not depend on left coordinates x_{i-r}, \dots, x_{i-i} , we say it is a one-sided CA. In this case the local rule is defined by $f : A^{r+1} \rightarrow A$ and $(Fx)_i = f(x_{i-r} \dots x_{i+r})$.

Among CA we distinguish the shift map $\sigma_M : X_M \rightarrow X_M$ such that $(\sigma_M x)_i = x_{i+1}$. This map codes time in the sense that x_i represents the

state of system at time i . Observe that σ_M is one-sided, and it is induced by the local rule with $r = 1, f : A^2 \rightarrow A, f(x_0 x_1) = x_1$. It is evident that $F \circ \sigma_M = \sigma_M \circ F$ for any CA $F; X_M \rightarrow X_M$. When X_M is a fullshift, we use the notation σ_A instead of σ_M .

Cellular automata have served as models of systems which present self organization, and the study of their dynamics has been the object of several papers over the last three decades. One approach to the study of such behavior is to consider a 'natural' probability measure μ of full support in X_M (which represents a realistic amount of initial conditions) and the sequence of iterated measures $(F^n \mu : n \in \mathbb{N})$, where $(F^n \mu)(B) = \mu(F^n(B))$ for every measurable set B . The behavior of such a sequence has been studied in particular contexts in recent years: (Lind, 1984; Mass and Martínez, 1998; Ferrari et al, 2000; Pivato and Yassawir, 2001, 2002). One of the primary objectives of this study is to link the limit behavior of this sequence with probability measures of maximal entropy, in other words, the way in which the CA goes to

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a maximal excited state in equilibrium.

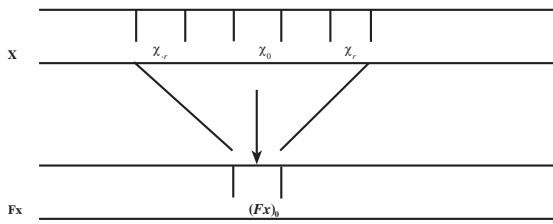


FIGURE 1: CA given by F

It is easy to verify that the above iteration admits several limit behaviors. To illustrate, consider in the fullshift $\{0,1\}^{\mathbb{Z}}$ the mod 2 CA given by $(Fx)_i = x_i + x_{i+1}$, where addition is modulo 2. Then: $(F^{2^n}x)_i = x_i + x_{i+2^n}$, and

$$(F^{2^n-1}x)_i = \sum_{j=0}^{2^n-1} x_{i+j},$$

so for any Bernoulli measure μ the limits of $(F^n \mu : n \in \mathbb{N})$ are different along these subsequences.

In order to study the equilibrium, a natural point of view is to consider the convergence of the temporal mean

$$\frac{1}{N} \sum_{n=0}^{N-1} F^n \mu$$

for a class of translation invariant probability measures μ on X_M . In fact, any limit for this sequence is also invariant for the CA. The main existing results concern fullshifts. This study began with Lind's pioneer (1984) work for the mod 2 CA and initial Bernoulli measures, where it is proven that the temporal mean converges to the uniform Bernoulli measure, which is the maximal entropy measure for F and $\sigma_{\{0,1\}}$. The same result was shown for larger classes of group cellular automata and more general initial probability measures in (Mass and Martínez, 1998; Ferrari et al, 2000).

Ferrari et al (2000) used regeneration theory of stochastic processes with finite state space. Lind's point of view is that of harmonic analysis: the cellular automaton is viewed as an endomorphism of the product group and the uniform Bernoulli measure is its Haar measure. This approach was followed in Pivato and Yassawir (2001 and 2002), where the authors proved the same kind of results for the class of probability measures they called harmonically mixing, and the family of

additive cellular automata verifying a diffusive condition. The previous results motivated us to study those sensitive cellular automata which can be decomposed as the product of an additive cellular automaton with one having an equi-continuous direction (Host et al, 2002).

Let us provide some notation and background.

ALPHABET, WORDS, SEQUENCES

A finite set of symbols A is called alphabet, and by A we mean the set of finite sequences or words $w = w_0 \dots w_{n-1}$ with letters in A and $|w|$ denotes the length of $w \in A^+$. Given $x = (x_j)_{j \in \mathbb{Z}} \in A^{\mathbb{Z}}$ and $i \leq j$ in \mathbb{Z} , or $x = x_0 \dots x_n \in A^+$ and $i \leq j$ in $\{0, \dots, n\}$, we denote by $x[i, j] = x_i \dots x_j$ the finite word in x between coordinates i and j .

Let $X_M = \{x = (x_j)_{j \in \mathbb{Z}} \in A^{\mathbb{Z}} : \cap_{j \in \mathbb{Z}} M_{x_j, x_{j+1}} = I\}$ for M a $\{0,1\}$ -valued matrix indexed by A . This is the topological Markov chain induced by M .

For any $n \in \mathbb{N}$ we define the set of words of length n in X_M by $L_n(X_M) = \{w \in A^* : \exists x \in X_M, x[0, n-1] = w\}$

The set of all words or language of X_M is $L(X_M) = \cup_{n \in \mathbb{N}} L_n(X_M)$. Given $w \in L(X_M)$ and $i \in \mathbb{N}$, the cylinder set starting in coordinate i with word w is $[w]_i = \{x \in X_M : x[i, i + |w| - 1] = w\}$.

Cylinder sets are a fundamental base for the topology of X_M , which is metrizable and compact.

Let $F : X_M \rightarrow X_M$ be a CA. We can apply its local rule f (of radius r) to words. Given $w = w_0 \dots w_n \in L_n(X_M)$ with $|w| \geq 2r + 1$, then $f(w) = f(w_0 \dots w_{2r}) f(w_{2r+1} \dots w_{2r+1+2r}) f(w_{n-2r} \dots w_n)$.

This allows us to define by induction the N -th power of f , $f^N : L_{2Nr+1}(X_M) \rightarrow L_1(X_M)$, by $f^N(w_{-Nr} \dots w_{Nr}) = f^{N-1}(f(w_{-Nr} \dots w_{-Nr+r}) \dots f(w_{Nr-r} \dots w_{Nr}))$.

CLASSES OF CA

A CA $F : X_M \rightarrow X_M$ is said to be right permutative if for every $w_n \in L_{r_2}(X_M)$ the map $f(w, \cdot)$ defined by $f(w, \cdot)(a) = f(wa)$, when $w_n a \in L_{r_2+1}(X_M)$, is one-to-one. Analogously we define left permutative and bipermutative CA.

Assume $(X_M, +)$ is a group. We say that F is additive if for every $x, y \in XL_M$ $F(x+y) = F(x) + F(y)$. By Kitchens' results (1987), any

topological Markov chain that is a group and such that the shift σ_M is an automorphism, is equivalent to one induced by a group $(L_1(X_M), +)$ in such a way that $x+y = (x_i + y_i)_{i \in \mathbb{Z}}$. Therefore the local rule of an additive CA F satisfies $f(w+w') = f(w) + f(w')$ for $w, w' \in L_{r+1}(X_M)$. A particular class of CA additive is that whose local rules f can be written $f(w_0 w_1) = w_0 + w_1$ for every $w_0 w_1 \in L_2(X_M)$, which is equivalent to $F^{(x)} = x + \sigma_{M(x)}$. These CA are right permutative.

Another class of right permutative CA are additive-translation CA. They are defined on a product space by $F \times G : X_M \times X_{M'} \rightarrow X_M \times X_{M'}$, where $(X_M, +)$ is an Abelian group, $F : X_M \rightarrow X_M$ is additive, and $G : X_{M'} \rightarrow X_{M'}$ is defined by $(Gx)_i = g(x_{i+1})$ with $g : L_1(X_{M'}) \rightarrow (X_{M'})$ a one-to-one local rule. This class of CA appear naturally as the representation of right permutative CA whose local rules verify some algebraic conditions in [Host et al, 2002].

Probabilistic background

Relevant shift invariant probability measures are Bernoulli and Markov measures. Let us describe them in $A^{\mathbb{Z}}$. It is enough to give its value in the set of cylinder sets.

Let $\pi = (\pi_a : a \in A)$ be a probability vector; the Bernoulli measure μ is given by $\mu_\pi([a_0, \dots, a_n]_i) = \prod_{j=0}^n \pi_{a_j}$. Under this measure all coordinates are independent, so there is no memory.

Markov measures have memory one, and they are defined by a stochastic matrix P and its stationary probability vector π and we denote it by $\mu_{\pi, P}$. It is given by

$$\mu_{\pi, P}([a_0, \dots, a_n]_i) = \pi_{a_0} P_{a_0, a_1} \dots P_{a_{n-1}, a_n}$$

This measure is supported by $X_M \subseteq A^{\mathbb{Z}}$ where $M_{a,b} = 1$ if and only if $P_{a,b} > 0$. Observe that if $P_{a,b} = \pi_b$ for any $a, b \in A$ then $\mu_{\pi, P} = \mu_\pi$.

Let us introduce a relevant class of infinite memory probability measures. Denote $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $-\mathbb{N}^* = \{-i : i \in \mathbb{N}^*\}$. Let μ be a shift invariant probability measure on $A^{\mathbb{Z}}$. For $\underline{w} \in A^{-\mathbb{I}^*}$ let $\mu_{\underline{w}}$ be the conditional measure on $A^{\mathbb{N}}$ given \underline{w} that is for any $m \geq 0$ and $a_0, \dots, a_m \in A$, $\mu_{\underline{w}}\{x_0 = a_0, \dots, x_m = a_m\} = \mu\{x_0 = a_0, \dots, x_m = a_m | x_i = w_i, i < 0\}$.

The measure μ is said to have *complete connections* if it satisfies

$$\forall a_0, \dots, a_m \in A, \forall \underline{w} \in A^{-\mathbb{I}^*}, \mu_{\underline{w}}\{x_0 = a_0, \dots, x_m = a_m\} > 0.$$

In this case define for $m \geq 0$

$$\gamma_m = \sup \left\{ \left| \frac{\mu_{\underline{w}}\{x_0 = a\}}{\mu_{\underline{v}}\{x_0 = a\}} - 1 \right| : a \in A, \underline{v}, \underline{w} \in A^{-\mathbb{I}^*}, v_i = w_i, i \in [-m, -1] \right\}.$$

The measure μ has summable decay if $\sum_{m=0}^{\infty} \gamma_m < \infty$. This is a uniform continuity condition on $\mu_{\underline{w}}$ as a function of \underline{w} .

Entropy

One of the fundamental invariants in ergodic theory is entropy, both topological and measure-theoretical. Here we introduce them in the context of cellular automata. Let $F : X_M \rightarrow X_M$ be a CA and μ a F -invariant probability measure defined on measurable sets of X_M , that is, $\mu(C) = \mu(F^{-1}(C))$ for every cylinder set C . Let $\alpha_n = \{[w]_{-n} : w \in L_{2n+1}(X_M)\}$ be the partition of centered cylinders of length $2n + 1$ in X_M . The dynamics of this partition α_n under the transformation F is given by the sequence of partitions $(\alpha_n(F, N))_{N \in \mathbb{I}}$, where $\alpha_n(F, N) = \{B_0 \cap F^{-1}B_1 \cap \dots \cap F^{-N+1}B_{N-1} : B_j \in \alpha_n, j \in \{0, \dots, N-1\}\}$.

The entropy of F with respect to μ is

$$h_\mu(F) = \lim_{n \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{B \in \alpha_n(F, N)} (-\mu(B) \log \mu(B)).$$

If $F = \sigma_M$ it holds that

$$h_\mu(\sigma_M) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{B \in \alpha_n} (-\mu(B) \log \mu(B)).$$

There is a topological counterpart. The topological entropy of F is

$$h_{top}(F) = \lim_{n \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} \# \{B \in \alpha_n(F, N) : B \neq \emptyset\}.$$

The variational principle links both notions of entropy in the following way:

$$h_{top}(F) = \sup \{h_\mu(F) : \mu \text{ is a } F\text{-invariant probability measure}\}.$$

A probability measure μ satisfying $h_{top}(F) = h_\mu(F)$ is said to be of maximal entropy. These measures carry the maximal amount of information of the system.

Maximal entropy measures for topological Markov chains X_M such that $M^m > 0$ for some $m \in \mathbb{I}$ can be explicitly described in terms of algebraic characteristics of the matrix. Let u, v be left and right eigenvectors of M associated to the eigenvalue λ of maximum modulus, that is, $uM = \lambda u$, $Mv = \lambda v$. By Perron-

Frobenius theorem $\lambda > 0$ and also $u > 0, v > 0$. Assume $\sum_{a \in A} u_a v_a = 1$. Define the stochastic matrix P_M by

$$(P_M)_{a,b} = \frac{v_b}{\lambda v_a} M_{a,b}, \quad a, b \in A.$$

The probability vector $\pi_M = ((\pi_M)_a : a \in A)$ given by $(\pi_M)_a = u_a v_a$ is the stationary vector for this matrix, that is, $\pi_M P_M = \pi_M$. Then the associated stationary Markov measure μ_{π_M, P_M} supported in X_M verifies

$$\mu_{\pi_M, P_M}([a_0 \dots a_{n-1}]_i) = \lambda^{-n} u_{a_0} v_{a_{n-1}} M_{a_0, a_1} \dots M_{a_{n-2}, a_{n-1}}.$$

If X_M is the fullshift $A^{\mathbb{Z}}$, the maximal measure is the uniform Bernoulli measure $\lambda_A^{\mathbb{E}}$ such that

$$\lambda_A^{\mathbb{E}}([a_0 \dots a_{n-1}]_i) = (\#A)^{-n}.$$

Temporal mean of iteration of measures by CA

Let $F: X_M \rightarrow X_M$ be a CA and μ be a shift invariant probability measure on X_M . As usual, $F^n \mu$ denotes the measure given by $(F^n \mu)(B) = \mu(F^{-n} B)$ for B a measurable set. We will discuss the convergence in distribution of the temporal mean measure

$$M_\mu^N(F) = \frac{1}{N} \sum_{n=0}^{N-1} F^n \mu.$$

If this limit exists as $N \rightarrow \infty$, we denote it by $M\mu(F)$.

CONVERGENCE OF TEMPORAL MEANS TO MAXIMAL ENTROPY MEASURE

Let $F: X_M \rightarrow X_M$ be a CA. In this section we will supply conditions ensuring the convergence of $M_\mu^N(F)$ to a maximal entropy measure for σ_M . This is shown for different kinds of translation invariant probability measures μ . In particular we will consider the case of stationary Markov probability measures on X_M .

Convergence of temporal means for additive CA

Let $(K, +)$ be a finite Abelian group and $F: K^{\mathbb{Z}} \rightarrow K^{\mathbb{Z}}$ be an additive CA. In a recent article, Pivato and Yassawir (2002) used a harmonic

analysis framework to study the convergence of $M_\mu^N(F)$ to the uniform Bernoulli measure of $K^{\mathbb{Z}}$. For this purpose they introduce the notion of harmonically mixing measures. A shift invariant probability measure μ on $K^{\mathbb{Z}}$ is called harmonically mixing if for all $\varepsilon > 0$, there is some $m > 0$ so that for every character $\chi \in \hat{K}^{\mathbb{Z}}$ with $\text{rank}(\chi) > m$ it holds $|\int_{K^{\mathbb{Z}}} \chi d\mu| < \varepsilon$, where $\text{rank}(\chi)$ is the number of coordinates of dependence of χ .

The statement (2.1) $\lim_{N \rightarrow \infty} M_\mu^N(F) = \lambda_K^{\mathbb{E}}$ was shown to hold for any additive CA and any harmonically mixing measure in Pivato and Yassawir (2001) (a previous proof for $K = \mathbb{Z}_p, p$ a prime number, was shown in Pivato and Yassawir (2002)). The proof is based on the fact that for every non-trivial character $\chi \in \hat{K}^{\mathbb{Z}}$, there is some subset $D_\chi \subset \hat{\mathbb{I}}$ of density one, so that $\text{rank}(\chi \circ F^j)$ goes to ∞ when j tends to ∞ inside D_χ . Moreover, Pivato and Yassawir (2002) also proved that Bernoulli measures and Markov measures whose transition matrices have strictly positive entries are harmonically mixing.

Our approach to (2.1) was probabilistic. In Ferrari et al (2000) we show that (2.1) holds for any measure with complete connections and summable decay memory for groups where $\#K$ is a power of a prime number. In addition, in Host et al (2002) we extend the result to any group by showing that measures with complete connections and summable decay are harmonically mixing.

In addition to these results, in an ergodic context, when $\#K = p$ a prime number, if all weak limits of $(M_\mu^N(F) : N \in \hat{\mathbb{I}})$ are ergodic for the shift map and have positive entropy for F , we prove in Host et al (2002) that (2.1) holds.

Convergence of temporal means for additive-translation CA

In Host et al (2002) we consider more general classes of CA, those which admit a product representation as an additive-translation CA. They contain one-sided right permutative CA of radius 1 acting on a fullshift such that the local rules $f: A^2 \rightarrow A$ are associative, that is, there is a one-to-one map $\psi: A \rightarrow A$ such that $\psi[f(a,b,c)] = f(a,\psi(c))$; or satisfy a *N-scaling condition*, that is,

$$f^N(x_0, \dots, x_N) = f(x_0, x_N)$$

where f^N is the N -th power of f . Therefore temporal means can be computed for classes of CA that *a priori* do not have any algebraic structure. A main point in this decomposition is that in one coordinate we have a group structure and in the other a translation CA for which temporal means are periodic.

Let $F \times G : (K \times A)^{\mathbb{Z}} \rightarrow (K \times A)^{\mathbb{Z}}$ be an additive-translation CA and let μ be a shift invariant probability measure on $(K \times A)^{\mathbb{Z}}$. Consider a weak limit $\tilde{\mu}$ of $(M_{\mu}^N(F \times G) : N \in \mathbb{N})$. This measure has two natural projections in each coordinate, $\pi_K \tilde{\mu}$ and $\pi_A \tilde{\mu}$, but *a priori* it is not true that $\tilde{\mu} = \pi_K \tilde{\mu} \times \pi_A \tilde{\mu}$. We prove in Host et al, (2002) that if $\pi_K \tilde{\mu} = \lambda_K^Z$ and for some $L \in \mathbb{Z}$ the CA $G \circ \sigma_A^L$ has zero entropy for $\pi_A \tilde{\mu}$, then $\tilde{\mu} = \lambda_K^Z \times \pi_A \tilde{\mu}$.

A corollary of the previous result and the properties described in previous subsection is: for any shift invariant probability measure μ with complete connections and summable decay it holds

$$M_{\mu}(F \times G) = \lambda_K^{\mathbb{Z}} \times \nu,$$

where $\nu = M_{\pi_A \mu}(G)$. In fact, $G \circ \sigma_A^{-1}$ has zero entropy because it is a CA of radius 0.

Convergence of temporal means for a particular additive CA

In this section we announce a new result on convergence of temporal means to the maximal entropy measure, in the context of additive CA acting on an Abelian group $(X_M, +)$. We will not give a proof of the result but we will illustrate it on a particular example constructed on $(X_M, +)$ given in Kitchens (1987).

In the example, the alphabet is $A = \mathbb{Z}_4 \times \mathbb{Z}_2 = \{0,1,2,3\} \times \{0,1\}$, which is an Abelian group with coordinate operations addition mod 4 and addition mod 2 respectively (that we call $+$ in the sequel for simplicity). The matrix M verifies $M_{(a_1, a_2), (b_1, b_2)} = 1$ if and only if $b_2 = a_2 + a_2 \pmod 2$. It can be verified that $(X_M, +)$ is a group and $\sigma_M : X_M \rightarrow X_M$ is an automorphism of $(X_M, +)$. In fact $M_{(a_1+a_1', a_2+a_2'), (b_1+b_1', b_2+b_2')} = 1$ whenever $b_2 = a_2 + a_2' \pmod 2$ and $b_1 = a_1 + a_1' \pmod 2$. Finally we consider the additive CA $F: X_M \rightarrow X_M$, given by $F: (x) = x + \sigma_M(x)$ (it is called group CA in Ferrari et al, (2000, 2000)). Below we

denote $\sigma_{\mathbb{Z}} = \sigma_2$ and $\sigma_{\mathbb{Z}_4} = \sigma_4$.

Let μ_M be the measure of maximal entropy for σ_M defined in Section 2. It is the Markovian measure defined by the transition matrix

$$P_M = \begin{bmatrix} 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \end{bmatrix}$$

and its stationary vector is

$$\pi_M = [1/8 \ 1/8 \ 1/8 \ 1/8 \ 1/8 \ 1/8 \ 1/8 \ 1/8],$$

where we order the alphabet as

$\{(0,0), (0,1), (1,0), (1,1), (2,0), (2,1), (3,0), (3,1)\}$

Consider another shift invariant Markov probability measure $\mu_{\pi, P}$ defined by a stochastic matrix P and its stationary probability vector π full supported by X_M , that is, $M_{a,b} > 0$ if and only if $P_{a,b} > 0$.

Theorem 1

We have $M_{\mu_{\pi, P}}(F) = \mu_M$.

Proof. The elements of X_M are denoted by $z = (x, y)$. We set $G : \mathbb{Z}_4^{\mathbb{Z}} \rightarrow \mathbb{Z}_4^{\mathbb{Z}}$ to be the CA induced by F on $\mathbb{Z}_4^{\mathbb{Z}}$. It can be proven inductively that

$$F^n z = \sum_{i=0}^n \binom{n}{i} \sigma_M^i z = z + \sum_{i=1}^n \binom{n}{i} (\sigma_4^i x, \sigma_4^{i-1} x \pmod 2)$$

Since F is right permutative it can be shown that for any allowed word $a_0, \dots, a_{m-1} \in L_m(X_M)$ and $n \geq 1$ there are two words $b_0 \dots b_m, c_0 \dots c_m \in \mathbb{Z}_4^*$ such that the following equality of sets holds

$$\{z \in X_M : F^n z[0, m-1] = a_0 \dots a_{m-1}\} = \{z \in X_M : G^{n-1} x[0, m] = b_0 \dots b_m\} \cup \{z \in X_M : G^{n-1} x[0, m] = c_0 \dots c_m\},$$

the union being disjoint.

We compute

$$\mu_{\pi, P}(\{z \in X_M : G^n x[0, m] = b_0 \dots b_m\})$$

and

$$\mu_{\pi, P}(\{z \in X_M : G^n x[0, m] = c_0 \dots c_m\})$$

for $n \geq 0$. Since $\mu_{\pi, P}$ is Markovian, the same arguments used in the fundamental construction of Ferrari et al (2000) allow us to prove

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu_{\pi, P}(\{z \in X_M : G^n x[0, m] = b_0 \dots b_m\}) = \left(\frac{1}{4}\right)^{m+1}$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu_{\pi, P}(\{z \in X_M : G^n x[0, m] = c_0 \dots c_m\}) = \left(\frac{1}{4}\right)^{m+1}$$

Therefore

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu_{\pi, P}(\{z \in X_M : F^n z[0, m-1] = a_0 \dots a_{m-1}\}) \\ = \frac{1}{8} \left(\frac{1}{4}\right)^{m-1} = \mu_M([a_0 \dots a_{m-1}]_0). \end{aligned}$$

This shows the result.

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