

On Some Problems of James Miller

B. BHOWMIK, S. PONNUSAMY

Department of Mathematics, Indian Institute of Technology Madras,

Chennai-600 036, India

emails: ditya@iitm.ac.in,

samy@iitm.ac.in

AND

K.-J. WIRTHS

Institut für Analysis,

TU Braunschweig, 38106 Braunschweig, Germany

email: kjwirths@tu-bs.de

ABSTRACT

We consider the class of meromorphic univalent functions having a simple pole at $p \in (0, 1)$ and that map the unit disc onto the exterior of a domain which is starlike with respect to a point $w_0 \neq 0, \infty$. We denote this class of functions by $\Sigma^*(p, w_0)$. In this paper, we find the exact region of variability for the second Taylor coefficient for functions in $\Sigma^*(p, w_0)$. In view of this result we rectify some results of James Miller.

RESUMEN

Consideramos la clase de funciones univalentes meromorforficos teniendo un polo simple en $p \in (0, 1)$ y la aplicación del disco unitario sobre el exterior de un dominio el cual es estrellado con respecto al punto $w_0 \neq 0, \infty$. Denotamos esta clase de funciones por $\Sigma^*(p, w_0)$. En este artículo encontramos la región exacta de variabilidad del segundo coeficiente de Taylor para funciones in $\Sigma^*(p, w_0)$. En vista de estos resultados nosotros rectificamos algunos resultados de James Miller.

Key words and phrases: *Starlike, meromorphic, and Schwarz' functions, Taylor coefficient.*

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1 Introduction

Let $\mathbb{D} := \{z : |z| < 1\}$ be the unit disc in the complex plane \mathbb{C} . Let Σ^* denote the class of functions

$$g(z) = \frac{1}{z} + d_0 + d_1z + d_2z^2 + \dots$$

which are univalent and analytic in \mathbb{D} except for the simple pole at $z = 0$ and map \mathbb{D} onto a domain whose complement is starlike with respect to the origin. Functions in this class is referred to as the meromorphic starlike functions in \mathbb{D} . This class has been studied by Clunie [4] and later an extended version by Pommerenke [10], and many others. Another related class of our interest is the class $S(p)$ of univalent meromorphic functions f in \mathbb{D} with a simple pole at $z = p$, $p \in (0, 1)$, and with the normalization $f(z) = z + \sum_{n=2}^{\infty} a_n(f)z^n$ for $|z| < p$. If $f \in S(p)$ maps \mathbb{D} onto a domain whose complement with respect to $\overline{\mathbb{C}}$ is convex, then we call f a concave function with pole at p and the class of these functions is denoted by $Co(p)$. In a recent paper, Avkhadiiev and Wirths [2] established the region of variability for $a_n(f)$, $n \geq 2$, $f \in Co(p)$ and as a consequence two conjectures of Livingston [7] in 1994 and Avkhadiiev, Pommerenke and Wirths [1] were settled.

In this paper, we consider the class $\Sigma^*(p, w_0)$ of meromorphically starlike functions f such that $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is a starlike set with respect to a finite point $w_0 \neq 0$ and have the standard normalization $f(0) = 0 = f'(0) - 1$. We now recall the following analytic characterization for functions in $\Sigma^*(p, w_0)$.

Theorem A. *$f \in \Sigma^*(p, w_0)$ if and only if there is a probability measure $\mu(\zeta)$ on $\partial\mathbb{D} = \{\zeta : |\zeta| = 1\}$ so that*

$$f(z) = w_0 + \frac{pw_0}{(z-p)(1-zp)} \exp\left(\int_{\partial\mathbb{D}} 2\log(1-\zeta z)d\mu(\zeta)\right)$$

where

$$w_0 = -\frac{1}{p + 1/p - 2 \int_{|\zeta|=1} \zeta d\mu(\zeta)}.$$

The necessary part of Theorem A has been proved by Miller [9] while the sufficiency part has been established by Yuh Lin [6, Theorem 1]. In [8, 9], Miller discussed a numbers of properties of the class $\Sigma^*(p, w_0)$. See also [3, 6, 11] for some other basic results such as bounds for $|f(z) - w_0|$.

We may state an equivalent formulation of Theorem A (see also [11]). A function f is said to be in $\Sigma^*(p, w_0)$ if and only if there exists an analytic function $P(z)$ in \mathbb{D} with $P(0) = 1$ and

$$\operatorname{Re} P(z) > 0, \quad z \in \mathbb{D}, \tag{1.1}$$

where

$$P(z) = \frac{-zf'(z)}{f(z) - w_0} - \frac{p}{z - p} + \frac{pz}{1 - pz}. \tag{1.2}$$

We may write $P(z)$ in the following power series form

$$P(z) = 1 + b_1z + b_2z^2 + \dots.$$

Also, each $f \in \Sigma^*(p, w_0)$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f)z^n, \quad |z| < p. \tag{1.3}$$

To recall the next result, we need to introduce a notation. Let $\mathcal{P}(b_1)$ denote the class of analytic functions $P(z)$ satisfying $P(0) = 1$, $P'(0) = b_1$ and $\operatorname{Re} P(z) > 0$ in \mathbb{D} .

In 1972, Miller [8] obtained estimations for the second Taylor coefficient $a_2(f)$. Indeed, he showed that

Theorem B. *If $f(z) \in \Sigma^*(p, w_0)$, then the second coefficient is given by*

$$a_2(f) = \frac{1}{2}w_0 \left(b_2 - p^2 - \frac{1}{p^2} - \frac{1}{w_0^2} \right)$$

where b_2 is the second coefficient of a function in $\mathcal{P}(b_1)$, i.e. the region of variability for $a_2(f)$ is contained in the disc

$$\left| a_2(f) + \frac{1}{2}w_0 \left(p^2 + \frac{1}{p^2} + \frac{1}{w_0^2} \right) \right| \leq |w_0|. \tag{1.4}$$

Further there is a p_0 , $0.39 < p_0 < 0.61$, such that if $p < p_0$, then $\operatorname{Re} a_2(f) > 0$.

In 1980, Miller [9, Equation (9)] also proved a sharp estimate regarding the second Taylor coefficient. In fact, he showed that

$$\left| a_2(f) - \frac{1 + p^2}{p} - w_0 \right| \leq |w_0|, \quad f \in \Sigma^*(p, w_0). \tag{1.5}$$

The aim of this paper is to find the region of variability for the second coefficient $a_2(f)$ of functions in $\Sigma^*(p, w_0)$ for any fixed pair (p, w_0) . Also we find the exact region of variability for $a_2(f)$ for fixed p , and as a consequence of this we show that $\operatorname{Re} a_2(f) > 0$ for all values of $p \in (0, 1)$ which Miller did not seem to expect as we see in the last part of Theorem B.

2 Region of Variability of Second Taylor Coefficients for Functions in $\Sigma^*(p, w_0)$

Theorem 2.1. *Let $f \in \Sigma^*(p, w_0)$ having the expansion (1.3). Then for a fixed pair (p, w_0) , the exact region of variability of the second Taylor coefficient $a_2(f)$ is the disc determined by the*

inequality

$$\left| a_2(f) - \left(p + \frac{1}{p} + w_0 \right) + \frac{1}{4} w_0 \left(p + \frac{1}{p} + \frac{1}{w_0} \right)^2 \right| \leq |w_0| \left(1 - \frac{1}{4} \left| p + \frac{1}{p} + \frac{1}{w_0} \right|^2 \right). \quad (2.2)$$

Proof. The proof uses the representation formula (1.1), i.e. $f \in \Sigma^*(p, w_0)$ if and only if $\operatorname{Re} P(z) > 0$ in \mathbb{D} with $P(0) = 1$, where P is given by (1.2). Since it is convenient to work with the class of Schwarz functions, we can write each such P as

$$P(z) = \frac{1 + \omega(z)}{1 - \omega(z)}, \quad z \in \mathbb{D}, \quad (2.3)$$

where $\omega: \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic with $\omega(0) = 0$ so that $\omega(z)$ has the form

$$\omega(z) = c_1 z + c_2 z^2 + \dots. \quad (2.4)$$

Using (1.2) and the power series representations of $P(z)$ and $f(z)$, it is easy to compute

$$\begin{cases} b_1 = p + \frac{1}{p} + \frac{1}{w_0}, \text{ and} \\ b_2 = p^2 + \frac{1}{p^2} + \frac{1}{w_0^2} + \frac{2a_2(f)}{w_0}. \end{cases} \quad (2.5)$$

Now eliminating w_0 from (2.5), we get

$$b_2 = p^2 + \frac{1}{p^2} + \left[b_1 - \left(p + \frac{1}{p} \right) \right]^2 + 2a_2(f) \left[b_1 - \left(p + \frac{1}{p} \right) \right]. \quad (2.6)$$

Using the power series representations of $P(z)$ and $\omega(z)$, it follows by comparing the coefficients of z and z^2 on both sides that

$$b_1 = 2c_1 \quad \text{and} \quad b_2 = 2(c_1^2 + c_2).$$

Inserting the above two relations in (2.6), we get

$$2(c_1^2 + c_2) = p^2 + \frac{1}{p^2} + \left[2c_1 - \left(p + \frac{1}{p} \right) \right]^2 + 2a_2(f) \left[2c_1 - \left(p + \frac{1}{p} \right) \right].$$

Now solving the above equation for $a_2(f)$, we get

$$a_2(f) = \frac{1}{p} + p \left(\frac{c_1^2 - c_2 + p^2 - 2c_1 p}{1 + p^2 - 2c_1 p} \right). \quad (2.7)$$

Now, since w_0 and p are fixed, we have c_1 fixed. Hence using the well known estimate $|c_2| \leq 1 - |c_1|^2$ for unimodular bounded function $\omega(z)$, the last equation results the following estimate

$$\left| a_2(f) - \frac{1}{p} - p \left(\frac{c_1^2 + p^2 - 2c_1 p}{1 + p^2 - 2c_1 p} \right) \right| \leq \frac{p(1 - |c_1|^2)}{|1 + p^2 - 2c_1 p|}.$$

Now, as $b_1 = 2c_1$, substituting $c_1 = \frac{1}{2}(p + 1/p + 1/w_0)$ in the above equation we get the required estimate as given in (2.2). A point on the boundary of the disc described by (2.2) is attained for the unique extremal functions given by (1.2) and (2.3), where

$$\omega(z) = \frac{z(c_1 + cz)}{1 + \overline{c_1}cz}, \quad |c| = 1.$$

The points in the interior of the disc described in (2.2) are attained for the same functions, but with $|c| < 1$. □

Remark. Comparison of Theorem B and Theorem 2.8 below, shows that the exact region of variability of $a_2(f)$ found by Miller is for the case $c_1 = 0$ only. A little computation reveals that both variability regions are the same for $c_1 = 0$, i.e.,

$$\left| a_2(f) - \frac{1 + p^2 + p^4}{p(1 + p^2)} \right| \leq \frac{p}{1 + p^2}.$$

This also shows that (1.5) gives the precise region of variability only for the case $c_1 = 0$. In all other cases, the boundaries of the discs described by (1.4) and (1.5) have only one point in common with the disc described by (2.2) because, in the both cases, on the boundaries of the discs described by (1.4) and (1.5), we need $|b_2| = 2$. Now, as $b_2 = 2(c_2 + c_1^2)$, this means that $|c_2 + c_1^2| = 1$. According to the coefficients bounds for unimodular bounded function, this is only possible for a unique c_2 if $c_1 \neq 0$.

In the following theorem, we describe the exact region of variability of the second Taylor coefficient of $f \in \Sigma^*(p, w_0)$, where only p is fixed.

Theorem 2.8. *Let $f \in \Sigma^*(p, w_0)$ having the expansion (1.3) and let p be fixed. Then the exact set of variability of the second Taylor coefficient $a_2(f)$ is given by*

$$|a_2(f) - 1/p| \leq p. \tag{2.9}$$

Proof. We may rewrite (2.7) as

$$a_2(f) = \frac{1}{p} + pM, \tag{2.10}$$

where

$$M = \frac{c_1^2 - c_2 + p^2 - 2c_1p}{1 + p^2 - 2c_1p}.$$

We wish to prove that $|M| \leq 1$. Since $\omega'(0) = c_1$, we have $|c_1| \leq 1$.

Now we fix $c_1 \in \overline{\mathbb{D}}$. Then $c_1^2 - c_2$ varies in the closed disc

$$\Delta(c_1) := \{z : |z - c_1^2| \leq 1 - |c_1|^2\}.$$

The map

$$T(\zeta) = \frac{\zeta + p^2 - 2c_1p}{1 + p^2 - 2c_1p}$$

maps the disc $\Delta(c_1)$ onto the disc with center

$$\frac{c_1^2 + p^2 - 2c_1p}{1 + p^2 - 2c_1p}$$

and radius

$$\frac{1 - |c_1|^2}{|1 + p^2 - 2c_1p|}.$$

Therefore, in order to prove $|M| \leq 1$, it suffices to show that

$$\left| \frac{c_1^2 + p^2 - 2c_1p}{1 + p^2 - 2c_1p} \right| + \frac{1 - |c_1|^2}{|1 + p^2 - 2c_1p|} \leq 1.$$

This is equivalent to

$$|c_1 - p|^2 + 1 - |c_1|^2 = \operatorname{Re}(1 + p^2 - 2c_1p) \leq |1 + p^2 - 2c_1p|.$$

We see that equality is attained in the above inequality if and only if c_1 is real. Now for real c_1 , we have

$$T(\Delta(c_1)) = \overline{\mathbb{D}} \text{ if and only if } c_1 = p \text{ or } w_0 = \frac{-p}{1 - p^2}.$$

Hence the extremal functions for the inequality (2.9) are given by (1.2) with $P(z)$ as in (2.3) with

$$\omega(z) = \frac{z(p + cz)}{1 + pcz}, \quad |c| = 1,$$

and the points in the interior of the disc described by (2.9) are attained for the same functions, but with $|c| < 1$. We observe that for real c_1 we can obtain $M = 1$ only for $c_2 = c_1^2 - 1$. This results in other starlike centers, but the extremal function is always the same, since $a_2(f) = p + 1/p$ is attained in the class $S(p)$ only for $f(z) = z/((1 - zp)(1 - z/p))$, see for instance [5]. \square

Remark. This result ensures us that $\operatorname{Re} a_2(f) > 0$ for all $p \in (0, 1)$. In the article [8, Theorem 1], Miller hoped for a possibility that for $p > .61$, the real part of $a_2(f)$ may be negative. But in view of our theorem we conclude that his hope was in vain.

Remark. In [9], Miller has obtained an estimate for the real part of the third coefficient $a_3(f)$ for all p . However, in geometric function theory, the classical question of finding the exact region of variability for $a_n(f)$, $n \geq 3$, $f \in \Sigma^*(p, w_0)$, remains an open problem.

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