Well-Posedness Results for Anisotropic Nonlinear Elliptic Equations with Variable Exponent and L^1-Data

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ABSTRACT

We study the anisotropic boundary value problem $-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial}{\partial x_i} u\right) = f$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a smooth open bounded domain in $\mathbb{R}^N$ ($N \geq 3$) and $f \in L^1(\Omega)$. We prove the existence and uniqueness of an entropy solution for this problem.

RESUMEN

Estudiamos el problema de valores en la frontera anisotropico $-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial}{\partial x_i} u\right) = f$ en Ω, $u = 0$ sobre $\partial\Omega$, donde Ω es un dominio abierto suave do $\mathbb{R}^N$ ($N \geq 3$) y $f \in L^1(\Omega)$. Proveamos la existencia y unicidad de una solución de entropía para este problema.

Key words and phrases: *Anisotropic; variable exponent; entropy solution; electrorheological fluids.*

Math. Subj. Class.: *35J20, 35J25, 35D30, 35B38, 35J60.*

1 Introduction

Let Ω be an open bounded domain of $\mathbb{R}^N$ ($N \geq 3$) with smooth boundary. Our aim is to prove the existence and uniqueness of an entropy solution to the anisotropic nonlinear elliptic problem

$$
\begin{cases}
-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial}{\partial x_i} u\right) = f \text{ in } \Omega \\
\\
u = 0 \quad \text{on } \partial\Omega,
\end{cases}
\tag{1.1}
$$

where the right-hand side $f \in L^1(\Omega)$.

We assume that for $i = 1, ..., N$ the function $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and satisfies the following conditions: $a_i(x, \xi)$ is the continuous derivative with respect to ξ of the mapping $A_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $A_i = A_i(x, \xi)$, i.e. $a_i(x, \xi) = \frac{\partial}{\partial \xi} A_i(x, \xi)$ such that

the following equality and inequalities holds

$$
A_i(x, 0) = 0,
\tag{1.2}
$$

for almost every $x \in \Omega$.

There exists a positive constant C_1 such that

$$
|a_i(x, \xi)| \leq C_1(j_i(x) + |\xi|^{p_i(x)-1})
\tag{1.3}
$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a nonnegative function in $L^{p_i'(\cdot)}(\Omega)$, with $1/p_i(x) + 1/p_i'(x) = 1$.

There exists a positive constant C_2 such that

$$
(a_i(x, \xi) - a_i(x, \eta)) \cdot (\xi - \eta) \geq
\begin{cases}
C_2 |\xi - \eta|^{p_i(x)} & \text{if } |\xi - \eta| \geq 1 \\
C_2 |\xi - \eta|^{p_i^-} & \text{if } |\xi - \eta| < 1,
\end{cases}
\tag{1.4}
$$

for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}$, with $\xi \neq \eta$,

and

$$
|\xi|^{p_i(x)} \leq a_i(x, \xi) . \xi \leq p_i(x) A_i(x, \xi)
\tag{1.5}
$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$.

We also assume that the variable exponent $p_i(.) : \overline{\Omega} \to [2, N)$ are continuous functions for all $i = 1, ..., N$ such that:

$$
\frac{\bar{p}(N-1)}{N(\bar{p}-1)} < p_i^- < \frac{\bar{p}(N-1)}{N - \bar{p}}, \quad \sum_{i=1}^{N} \frac{1}{p_i^-} > 1 \text{ and } \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\bar{p} - N}{\bar{p}(N-1)},
\tag{1.6}
$$

where $\frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i^-}$, $p_i^- := ess\inf_{x \in \Omega} p_i(x)$ and $p_i^+ := ess\sup_{x \in \Omega} p_i(x)$.

We introduce the numbers

$$
q = \frac{N(\bar{p}-1)}{N-1}, \; q^* = \frac{Nq}{N-q} = \frac{N(\bar{p}-1)}{N-\bar{p}}.
\tag{1.7}
$$

A prototype example that is covered by our assumptions is the following anisotropic $(p_1(.), ..., p_N(.))$-harmonic equation:

Set $A_i(x, \xi) = (1/p_i(x)) |\xi|^{p_i(x)}$, $a_i(x, \xi) = |\xi|^{p_i(x)-2} \xi$ where $p_i(x) \geq 2$. Then we get the following equation:

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial}{\partial x_i} u \right|^{p_i(x)-2} \frac{\partial}{\partial x_i} u \right) = f$$

which, in the particular case when $p_i = p$ for any $i \in \{1, ..., N\}$ is the anisotropic $p(.)$-Laplace equation.

For the proof of existence of entropy solutions of (1.1), we follow [2] and derive a priori estimates for the approximated solutions u_n and the partial derivatives $\frac{\partial u_n}{\partial x_i}$ in the Marcinkiewicz spaces $\mathcal{M}^{q^*}$ and $\mathcal{M}^{p_i^- q/\bar{p}}$ respectively (see section 2 or [2] for definition of Marcinkiewicz spaces).

The study of nonlinear elliptic equations involving the $p-$Laplace operator is based on the theory of standard Sobolev spaces $W^{m,p}(\Omega)$ in order to find weak solutions. For the nonhomogeneous $p(.)$-Laplace operators, the natural setting for this approach is the use of the variable exponent Lebesgue and Sobolev spaces $L^{p(.)}(\Omega)$ and $W^{m,p(.)}(\Omega)$.

Variable exponent Lebesgue spaces appeared in the literature for the first time in a 1931 article by Orlicz[21]. After [21], Orlicz abandoned the study of variable exponent spaces to concentrate on the theory of function spaces that now bears his name (Orlicz spaces). After Orlicz's work (cf. [21]), H. Hudzik [14], and J. Musielak [20] investigated the variable exponent Sobolev spaces.

Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers, notably Sharapudinov[26] and Tsenov [27]. The next major step in the investigation of variable exponent Lebesgue and Sobolev spaces was the comprehensive paper by O. Kovacik and J. Rakosnik in the early 90's [16]. This paper established many of basic properties of Lebesgue and Sobolev spaces with variables exponent. Variable Sobolev spaces have been used in the last decades to model various phenomena. In [5], Chen, Levine and Rao proposed a framework for image restoration based on a Laplacian variable exponent. Another application which uses non-homogeneous Laplace operators is related to the modelling of electrorheological fluids. The first major discovery in electrorheological fluids was due to Willis Winslow in 1949 (cf. [28]). These fluids have the interesting property that their viscosity depends on the electric field in the fluid. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. For some technical applications, consult Pfeiffer et *al* [22]. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories. For more information on properties, modelling and the application of variable exponent spaces to these fluids, we refer to Diening [6], Rajagopal and Ruzicka [22], and Ruzicka [24].

In this paper, the operator involved in (1.1) is more general than the $p(.)-$Laplace operator. Thus, the variable exponent Sobolev space $W^{1,p(.)}(\Omega)$ is not adequate to study nonlinear problems of this type. This leads us to seek entropy solutions for problems (1.1) in a more general variable exponent Sobolev space which was introduced for the first time by Mihaïlescu et *al* [18].

As the right-hand side in (1.1) is in $L^1(\Omega)$, the suitable notion of solution is a notion of entropy solution (cf. [3]).

The remaining part of this paper is organized as follows: Section 2 is devoted to mathematical preliminaries including, among other things, a brief discussion of variable exponent Lebesgue, Sobolev, anisotropic spaces and Marcinkiewicz spaces. Existence of weak energy solution for (1.1) where $f \in L^\infty(\Omega)$ was proved in [14]; We will also briefly recall the results of [14] in section 2. The main existence and uniqueness result is stated and proved in section 3.

2 Mathematical Preliminaries

In this section, we define Lebesgue, Sobolev and anisotropic spaces with variable exponent and give some of their properties. Roughly speaking, anistropic Lebesgue and Sobolev spaces are functional spaces of Lebesgue's and Sobolev's type in which different space directions have different roles.

Given a measurable function $p(.) : \Omega \to [1, \infty)$. We define the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ as the set of all measurable function $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} \, dx$$

is finite. If the exponent is bounded, i.e., if $p_+ < \infty$, then the expression

$$|u|_{p(.)} := \inf \left\{ \lambda > 0 : \rho_{p(.)}(u/\lambda) \le 1 \right\}$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(.)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $p_- > 1$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left| \int_\Omega uv dx \right| \le \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) |u|_{p(.)} |v|_{p'(.)}, \tag{2.1}$$

for all $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$.

Now, let

$$W^{1,p(.)}(\Omega) := \left\{ u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega) \right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(.)} := |u|_{p(.)} + |\nabla u|_{p(.)}.$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(.)}(\Omega)$.

Next, we define $W_0^{1,p(.)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$ under the norm $\|u\|_{1,p(.)}$.

Set $C_+(\overline{\Omega}) = \left\{ p(.) \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} p(x) > 1 \right\}$.

Furthermore, if $p(.) \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(.)}(\Omega)$,

that is $H_0^{1,p(.)}(\Omega) = W_0^{1,p(.)}(\Omega)$ (cf. [13]). Since Ω is an open bounded set and $p(.) \in C_+(\overline{\Omega})$ is logarithmic Hölder, the $p(.)-$Poincaré inequality

$$|u|_p \leq C \, |\nabla u|_{p(.)}$$

holds for all $u \in W_0^{1,p(.)}(\Omega)$, where C depends on p, $|\Omega|$, diam(Ω) and N (see [13]), and so

$$\|u\| := |\nabla u|_{p(.)} \,,$$

is an equivalent norm in $W_0^{1,p(.)}(\Omega)$. Of course also the norm

$$\|u\|_{p(.)} := \sum_{i=1}^{N} \left| \frac{\partial}{\partial x_i} u \right|_{p(.)}$$

is an equivalent norm in $W_0^{1,p(.)}(\Omega)$. Hence the space $W_0^{1,p(.)}(\Omega)$ is a separable and reflexive Banach space.

Let us present a natural generalization of the variable exponent Sobolev space $W_0^{1,p(.)}(\Omega)$ (cf. [18]) that will enable us to study the problem (1.1) with sufficient accuracy.

First of all, we denote by $\overrightarrow{p}(.) : \overline{\Omega} \rightarrow \mathbb{R}^N$ the vectorial function $\overrightarrow{p} = (p_1(.), ..., p_N(.))$. The *anisotropic variable exponent Sobolev space* $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}(.)} := \sum_{i=1}^{N} \left| \frac{\partial}{\partial x_i} u \right|_{p_i(.)} .$$

The space $\left(W_0^{1,\overrightarrow{p}(.)}(\Omega), \|u\|_{\overrightarrow{p}(.)} \right)$ is a reflexive Banach space (cf. [18]).

Let us introduce the following notations:

$$\overrightarrow{P}_+ = (p_1^+, ..., p_N^+), \overrightarrow{P}_- = (p_1^-, ..., p_N^-),$$

$$P_+^+ = max\left\{ p_1^+, ..., p_N^+ \right\}, P_-^+ = max\left\{ p_1^-, ..., p_N^- \right\}, P_-^- = min\left\{ p_1^-, ..., p_N^- \right\},$$

and

$$P_-^* = \frac{N}{\displaystyle\sum_{i=1}^{N} \frac{1}{p_i^-} - 1}, \ P_{-,\infty} = max\left\{ P_-^+, P_-^* \right\}.$$

We have the following result (cf. [18]):

Theorem 2.1. *Assume $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary. Assume relation (1.6) is fulfilled. For any $q \in C(\overline{\Omega})$ verifying*

$$1 < q(x) < P_{-,\infty} \text{ for all } x \in \overline{\Omega},$$

then the embedding

$$W_0^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$$

is continuous and compact.

We Also recall the result of the study of problem (1.1) for the right-hand side $f \in L^\infty(\Omega)$ (cf. [15]). We first recall the definition of weak energy solution of (1.1) for the right-hand side more regular i.e $f \in L^\infty(\Omega)$.

Definition 2.2. *Let $f \in L^\infty(\Omega)$; a weak energy solution of (1.1) is a function $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ such that*

$$\int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}\varphi dx = \int_\Omega f(x)\varphi dx, \text{ for all } \varphi \in W_0^{1,\overrightarrow{p}(.)}(\Omega). \tag{2.2}$$

We have proved among other results in [15] the following Theorem

Theorem 2.3. *Assume (1.2)-(1.6) and $f \in L^\infty(\Omega)$. Then there exists a unique weak energy solution of (1.1).*

Finally, in this paper, we will use the Marcinkiewicz spaces $\mathcal{M}^q(\Omega)(1 < q < \infty)$ with constant exponent. Note that the Marcinkiewicz spaces $\mathcal{M}^{q(.)}(\Omega)$ in the variable exponent setting was introduced for the first time by Sanchon and Urbano (see [25]).

Marcinkiewicz spaces $\mathcal{M}^q(\Omega)(1 < q < \infty)$ contain the measurable functions $g : \Omega \to \mathbb{R}$ for which the distribution function

$$\lambda_g(\gamma) = |\{x \in \Omega : |g(x)| > \gamma\}|, \ \gamma \geq 0,$$

satisfies an estimate of the form

$$\lambda_g(\gamma) \leq C\gamma^{-q}, \text{ for some finite constant } C > 0.$$

The space $\mathcal{M}^q(\Omega)$ is a Banach space under the norm

$$\|g\|^*_{\mathcal{M}^q(\Omega)} = \sup_{t>0} t^{1/q} \left(\frac{1}{t} \int_0^t g^*(s)ds \right),$$

where g^* denotes the nonincreasing rearrangement of f:

$$g^*(t) = \inf \{\gamma > 0 : \lambda_g(\gamma) \leq t\}.$$

We will use the following pseudo norm

$$\|g\|_{\mathcal{M}^q(\Omega)} = \inf \{C : \lambda_g(\gamma) \leq C\gamma^{-q}, \ \forall \gamma > 0\},$$

which is equivalent to the norm $\|g\|^*_{\mathcal{M}^q(\Omega)}$ (see [2]).

We have the following Lemma (for the proof, see [2, proof of Lemma 2.2]).

Lemma 2.4. *Let g be a nonnegative function in $W_0^{1,\overrightarrow{P}_-}(\Omega)$. Assume $\bar{p} < N$, and that there exists a constant c such that*

$$\sum_{i=1}^N \int_{\{|g|\leq\gamma\}} \left| \frac{\partial g}{\partial x_i} \right|^{p_i^-} dx \leq c(\gamma + 1), \forall \gamma > 0. \tag{2.3}$$

Then there exists a constant C, depending on c, such that

$$\|g\|_{\mathcal{M}^{\frac{N(\bar{p}-1)}{N-\bar{p}}}(\Omega)} \leq C.$$

3 Existence and Uniqueness of Entropy Solution

In this section, we study the problem (1.1) for a right-hand side $f \in L^1(\Omega)$. In the L^1 setting, the suitable notion of solution for the study of (1.1) is the notion of entropy solution (cf. [3]).

We first define the troncation function T_t by $T_t(s) := max\{-t, min\{t, s\}\}$.

Definition 3.1. *A measurable function u is an entropy solution to problem (1.1) if, for every $t > 0$, $T_t(u) \in W_0^{1, \overrightarrow{p}(.)}(\Omega)$ and*

$$\int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i} u) . \frac{\partial}{\partial x_i} T_t(u - \varphi) dx \leq \int_\Omega f(x) T_t(u - \varphi) dx, \qquad (3.1)$$

for all $\varphi \in W_0^{1, \overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.2. A function u such that $T_t(u) \in W_0^{1, \overrightarrow{p}(.)}(\Omega)$ for all $t > 0$ does not necessarily belong in $W_0^{1,1}(\Omega)$. However, it is possible to define its weak gradient, still denoted by ∇u.

Our main result in this section is the following:

Theorem 3.3. *Assume (1.2)-(1.6) and $f \in L^1(\Omega)$. Then there exists a unique entropy solution u to problem (1.1).*

Proof.

The proof of this Theorem will be done in three steps.

∗**Step 1. A priori estimates**. We start with the existence of the weak gradient for every measurable function u such that $T_t(u) \in W_0^{1, \overrightarrow{p}(.)}(\Omega)$ for all $t > 0$.

Proposition 3.4. *If u is a measurable function such that $T_t(u) \in W_0^{1, \overrightarrow{p}(.)}(\Omega)$ for all $t > 0$, then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$v \chi_{\{|u|<t\}} = \nabla T_t(u) \text{ for a.e. } x \in \Omega, \text{ and for all } t > 0,$$

where χ_A denotes the characteristic function of a measurable set A. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the standard distributional gradient of u.

Proof. As $T_t(u) \in W_0^{1, \overrightarrow{p}(.)}(\Omega) \hookrightarrow W_0^{1, \overrightarrow{P}^-}(\Omega) \subset W_0^{1,1}(\Omega)$ for all $t > 0$ since $1 < p_i^-$ for all $i = 1, ..., N$, then by Theorem 1.5 in [1], the result follows.

Proposition 3.5. *Assume (1.2)-(1.6) and $f \in L^1(\Omega)$. Let u be an entropy solution of (1.1). If there exists a positive constant M such that*

$$\sum_{i=1}^N \int_{\{|u|>t\}} t^{q_i(x)} dx \leq M, \text{ for all } t > 0, \qquad (3.2)$$

then

$$\sum_{i=1}^N \int_{\left\{\left|\frac{\partial}{\partial x_i} u\right|^{\alpha_i(.)} > t\right\}} t^{q_i(x)} dx \leq \|f\|_1 + M \text{ for all } t > 0,$$

where $\alpha_i(.) = p_i(.)/(q_i(.) + 1).$, *for all* $i = 1, ..., N$.

Proof. Take $\varphi = 0$ in (3.1), we have

$$\sum_{i=1}^{N} \int_{\Omega} a_i\left(x, \frac{\partial}{\partial x_i} T_t(u)\right).\frac{\partial}{\partial x_i} T_t(u) dx \leq \int_{\Omega} f(x) T_t(u) dx.$$

We deduces from inequality above that

$$\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial}{\partial x_i} T_t(u) \right|^{p_i(x)} dx \leq t \left\| f \right\|_1, \text{ for all } t > 0.$$

Therefore, defining $\psi := T_t(u)/t$, we have, for all $t > 0$,

$$\sum_{i=1}^{N} \int_{\Omega} t^{p_i(x)-1} \left| \frac{\partial}{\partial x_i} \psi \right|^{p_i(x)} dx = \frac{1}{t} \int_{\Omega} \left| \frac{\partial}{\partial x_i} T_t(u) \right|^{p_i(x)} dx \leq \left\| f \right\|_1.$$

From the above inequality, the definition of $\alpha_i(.)$ and (3.2), we have

$$\sum_{i=1}^{N} \int_{\left\{ \left| \frac{\partial}{\partial x_i} u \right|^{\alpha_i(.)} > t \right\}} t^{q_i(x)} dx \leq \int_{\left\{ \left| \frac{\partial}{\partial x_i} u \right|^{\alpha_i(.)} > t \right\} \cap \{|u| \leq t\}} t^{q_i(x)} dx + \int_{\{|u| > t\}} t^{q_i(x)} dx$$

$$\leq \int_{\{|u| \leq t\}} t^{q_i(x)} \left(\frac{\left| \frac{\partial}{\partial x_i} u \right|^{\alpha_i(x)}}{t} \right)^{\frac{p_i(x)}{\alpha_i(x)}} dx + M \leq \left\| f \right\|_1 + M, \text{ for all } t > 0.$$

Proposition 3.6. *Assume (1.2)-(1.6) and* $f \in L^1(\Omega)$. *Let* u *be an entropy solution of (1.1), then*

$$\frac{1}{h} \sum_{i=1}^{N} \int_{\{|u| \leq h\}} \left| \frac{\partial}{\partial x_i} T_h(u) \right|^{p_i(x)} dx \leq M$$

for every $h > 0$, *with* M *a positive constant. More precisely, there exists* $D > 0$ *such that*

$$meas\, \{|u| > h\} \leq D^{P_-^-} \frac{1+h}{h^{P_-^-}}.$$

Proof. Taking $\varphi = 0$ in the entropy inequality (3.1) and using (1.5), we obtain

$$\sum_{i=1}^{N} \int_{\{|u| \leq h\}} \left| \frac{\partial}{\partial x_i} T_h(u) \right|^{p_i(x)} dx \leq h \left\| f \right\|_1 \leq Mh$$

for all $h > 0$.

Next,

$$\sum_{i=1}^{N} \int_{\{|u| \leq h\}} \left| \frac{\partial}{\partial x_i} T_h(u) \right|^{p_i(x)} dx \leq Mh \Rightarrow \sum_{i=1}^{N} \int_{\{|u| \leq h\}} \left| \frac{\partial}{\partial x_i} T_h(u) \right|^{P_-^-} dx \leq C(1+h).$$

We can write the above inequality as

$$\sum_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} T_h(u) \right\|_{P_-}^{P_-} \leq C(1+h) \text{ or } \|T_h(u)\|_{W_0^{1,P_-}(\Omega)} \leq [C(1+h)]^{\frac{1}{P_-}}.$$

By the Poincaré inequality in constant exponent, we obtain

$$\|T_h(u)\|_{L^{P_-}(\Omega)} \leq D(1+h)^{\frac{1}{P_-}}.$$

The above inequality imply that

$$\int_{\Omega} |T_h(u)|^{P_-} \, dx \leq D^{P_-}(1+h),$$

from which we obtain

$$meas\,\{|u| > h\} \leq D^{P_-} \frac{1+h}{h^{P_-}}.$$

∗ **Step 2. Uniqueness of entropy solution**. The proof of uniqueness of entropy solutions follow the technics by Bénilan et *al* [3] (see also [25]). Let $h > 0$ and u, v two entropy solutions of (1.1). We write the entropy inequality (3.1) corresponding to the solution u, with $T_h v$ as test function, and to the solution v, with $T_h u$ as test function. Upon addition, we get

$$\begin{cases} \displaystyle\int_{\{|u-T_h v|\leq t\}} \sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i} u).\frac{\partial}{\partial x_i}(u - T_h v)dx+ \\[3mm] \displaystyle\int_{\{|v-T_h u|\leq t\}} \sum_{i=1}^{N} a(x, \frac{\partial}{\partial x_i} v).\frac{\partial}{\partial x_i}(v - T_h u)dx \\[3mm] \displaystyle \leq \int_{\Omega} f(x)(T_t(u - T_h v) + T_t(v - T_h u))dx. \end{cases} \tag{3.3}$$

Define

$$E_1 := \{|u - v| \leq t, |v| \leq h\}, \ E_2 := E_1 \cap \{|u| \leq h\}, \text{ and } E_3 := E_1 \cap \{|u| > h\}.$$

We start with the first integral in (3.3). By (1.5), we have

$$
\begin{cases}
\displaystyle\int_{\{|u-T_hv|\le t\}} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}(u-T_hv)dx = \\[4mm]
\displaystyle\int_{\{|u-T_hv|\le t\}\cap(\{|v|\le h\}\cup\{|v|>h\})} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}(u-T_hv)dx = \\[4mm]
\displaystyle\int_{\{|u-T_hv|\le t,|v|\le h\}} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}(u-T_hv)dx+ \\[4mm]
\displaystyle\int_{\{|u-T_hv|\le t,|v|>h\}} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}(u-T_hv)dx \\[4mm]
\displaystyle= \int_{\{|u-v|\le t,|v|\le h\}} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}(u-v)dx + \int_{\{|u-T_hv|\le t,|v|>h\}} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}u\,dx \\[4mm]
\displaystyle\ge \int_{\{|u-v|\le t,|v|\le h\}} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}(u-v)dx = \int_{E_1} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}(u-v)dx \\[4mm]
\displaystyle= \int_{E_1\cap(\{|u|\le h\}\cup\{|u|>h\})} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}(u-v)dx = \int_{E_2} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}(u-v)dx \\[4mm]
\displaystyle+ \int_{E_3} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}(u-v)dx = \int_{E_2} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}(u-v)dx+ \\[4mm]
\displaystyle\int_{E_3} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}u\,dx - \int_{E_3} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}v\,dx \\[4mm]
\displaystyle\ge \int_{E_2} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}(u-v)dx - \int_{E_3} \sum_{i=1}^{N} a(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}v\,dx.
\end{cases}
$$

$$(3.4)$$

Using (1.3) and (2.1), we estimate the last integral in (3.4) as follows

$$
\begin{cases}
\left| \int_{E_3} \sum_{i=1}^{N} a(x, \frac{\partial}{\partial x_i} u) . \frac{\partial}{\partial x_i} v dx \right| \leq C_1 \int_{E_3} \sum_{i=1}^{N} \left(j_i(x) + \left| \frac{\partial}{\partial x_i} u \right|^{p_i(x)-1} \right) \left| \frac{\partial}{\partial x_i} v \right| dx \\[4mm]
\leq C \sum_{i=1}^{N} \left(|j|_{p_i'(.)} + \left\| \left| \frac{\partial}{\partial x_i} u \right|^{p_i(x)-1} \right\|_{p_i'(.), \{h < |u| \leq h+t\}} \right) \left| \frac{\partial}{\partial x_i} v \right|_{p_i(.), \{h-t < |v| \leq h\}}
\end{cases}
\tag{3.5}
$$

where $\left\| \left| \frac{\partial}{\partial x_i} u \right|^{p_i(x)-1} \right\|_{p_i'(.), \{h < |u| \leq h+t\}} = \left\| \left| \frac{\partial}{\partial x_i} u \right|^{p_i(x)-1} \right\|_{L^{p_i'(.)}(\{h < |u| \leq h+t\})}$.

For all $i = 1, ..., N$, the quantity $\left(|j_i|_{p_i'(.)} + \left\| \left| \frac{\partial}{\partial x_i} u \right|^{p_i(x)-1} \right\|_{p_i'(.), \{h < |u| \leq h+t\}} \right)$ is finite, since

$u \in W_0^{1, \overrightarrow{p}(.)}(\Omega)$ and $j_i \in L^{p_i'(.)}(\Omega)$; then by Proposition 3.6, the last expression converges to zero as h tends to infinity. Therefore, from (3.4) and (3.5), we obtain

$$
\int_{\{|u - T_h v| \leq t\}} \sum_{i=1}^{N} a(x, \frac{\partial}{\partial x_i} u) . \frac{\partial}{\partial x_i} (u - T_h v) dx \geq I_h + \int_{E_2} \sum_{i=1}^{N} a(x, \frac{\partial}{\partial x_i} u) . \frac{\partial}{\partial x_i} (u - v) dx,
\tag{3.6}
$$

where I_h converges to zero as h tends to infinity. We may adopt the same procedure to treat the second term in (3.3) to obtain

$$
\int_{\{|v - T_h u| \leq t\}} \sum_{i=1}^{N} a(x, \frac{\partial}{\partial x_i} v) . \frac{\partial}{\partial x_i} (v - T_h u) dx \geq J_h - \int_{E_2} \sum_{i=1}^{N} a(x, \frac{\partial}{\partial x_i} v) . \frac{\partial}{\partial x_i} (u - v) dx,
\tag{3.7}
$$

where J_h converges to zero as h tends to infinity.
Next, consider the right-hand side of inequality (3.3). Noting that

$$
T_t(u - T_h v) + T_t(v - T_h u) = 0 \text{ in } \{|u| \leq h, |v| \leq h\};
$$

we obtain

$$
\left| \int_{\Omega} f(x)(T_t(u - T_h v) + T_t(v - T_h u)) dx \right| =
$$

$$
\left| \int_{\{|u| > h\}} f(x)(T_t(u - T_h v) + T_t(v - T_h u)) dx + \int_{\{|u| \leq h\}} f(x)(T_t(u - T_h v) + T_t(v - T_h u)) dx \right| =
$$

$$
\left| \int_{\{|u| > h\}} f(x)(T_t(u - T_h v) + T_t(v - T_h u)) dx + \int_{\{|u| \leq h, |v| > h\}} f(x)(T_t(u - T_h v) + T_t(v - T_h u)) dx \right|
$$

$$
\leq 2t \left(\int_{\{|u| > h\}} |f| dx + \int_{\{|v| > h\}} |f| dx \right).
$$

According to Proposition 3.6, both $meas \{|u| > h\}$ and $meas \{|v| > h\}$ tend to zero as h goes to infinity, then by the inequality above, the right-hand side of inequality (3.3) tends to zero as h

goes to infinity. From this assertion, (3.3), (3.6) and (3.7), we obtain, letting $h \to +\infty$,

$$\int_{\{|u-v|\leq t\}} \sum_{i=1}^{N} (a(x, \frac{\partial}{\partial x_i} u) - a(x, \frac{\partial}{\partial x_i} v)).\frac{\partial}{\partial x_i}(u-v)dx \leq 0, \text{ for all } t > 0.$$

By assertion (1.4), we conclude that $\frac{\partial}{\partial x_i} u = \frac{\partial}{\partial x_i} v$, for all $i = 1, .., N$ a.e. in Ω.
We deduce that

$$\|u-v\|_{\overrightarrow{p}(.)} = \sum_{i=1}^{N} \left| \frac{\partial}{\partial x_i} u - \frac{\partial}{\partial x_i} v \right|_{p_i(.)} = 0,$$

and hence $u = v$, a.e. in Ω.

∗ **Step 3. Existence of entropy solutions**. Let $(f_n)_n$ be a sequence of bounded functions, strongly converging to $f \in L^1(\Omega)$ and such that

$$\|f_n\|_1 \leq \|f\|_1, \text{ for all } n. \tag{3.8}$$

We consider the problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial}{\partial x_i} u_n) = f_n \text{ in } \Omega \\ \\ u_n = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.9}$$

It follows from Theorem 2.3 (cf. [15]) that problem (3.9) has a unique weak energy solution $u_n \in W_0^{1, \overrightarrow{p}(.)}(\Omega)$ since $f_n \in L^\infty(\Omega)$. Our interest is to prove that these approximated solutions u_n tend, as n goes to infinity, to a measurable function u which is an entropy solution of the limit problem (1.1). To this end, we derive a priori estimates for u_n and $\frac{\partial}{\partial x_i} u_n$ in the Marcinkiewicz spaces $\mathcal{M}^{q^*}$ and $\mathcal{M}^{p_i^- q/\bar{p}}$ where q^* and $\bar{p}$ are defined in (1.6) and (1.7).

Let $\gamma > 0$, denote T_γ the corresponding truncation function. Note that

$$DT_\gamma(r) = \begin{cases} 1 \text{ if } |r| < \gamma \\ \\ 0 \text{ if } |r| > \gamma. \end{cases}$$

In particular, we have

$$a_i(x, \xi)DT_\gamma(r)\xi \geq a_i(x, \xi)\xi\chi_{\{|r|<\gamma\}}, \text{for all } i = 1, ..., N. \tag{3.10}$$

Lemma 3.7. There exists a constant c, not depending on n, such that

$$\sum_{i=1}^{N} \int_{\{|u_n|\leq\gamma\}} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i^-} dx \leq c(\gamma+1), \text{ for all } \gamma > 0. \tag{3.11}$$

Proof. Inserting $\varphi = T_\gamma(u_n)$ into (2.2), we have

$$\int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i} u_n) DT_\gamma(u_n) \frac{\partial}{\partial x_i} u_n dx = \int_\Omega f_n T_\gamma(u_n) dx.$$

Using (3.10) and the coercivity condition (1.5), we obtain (3.11).

Lemma 3.8. There exists a constant C, not depending on n, such that

$$\|u_n\|_{\mathcal{M}^{q*}(\Omega)} \leq C$$

and

$$\left\| \frac{\partial u_n}{\partial x_i} \right\|_{\mathcal{M}^{p_i^- q/\bar{p}}(\Omega)} \leq C, \text{ for all } i = 1, ..., N.$$

Proof. The result of Lemma 3.8 is a direct consequence of Lemma 3.7 (cf. [2, proof of Lemma 3.3]).

In view of Lemma 3.8 and following [2] (see also [4, Lemma A.2]), we deduce that u_n is uniformly bounded in $L^{k_0}(\Omega)$ for some $k_0 < q^*$ with $k_0 > p_i^- q/\bar{p}$ and $\frac{\partial u_n}{\partial x_i}$ is uniformly bounded in $L^{k_i}(\Omega)$ for some $k_i > 1$ with $p_i^- - 1 < k_i < p_i^- q/\bar{p}$, for all $i = 1, ..., N$. From this, we get that u_n is uniformly bounded in the isotropic Sobolev space $W_0^{1,k_{min}}(\Omega)$, where $k_{min} = min(k_1, ..., k_N)$. Consequently, we can assume without loss of generality (see also [2, Lemma 3.4]) that as $n \to 0$,

$$
\begin{cases}
u_n \to u \text{ a.e in } \Omega \text{ and in } L^{k_{min}}(\Omega) \\
\\
u_n \rightharpoonup u \text{ in } W_0^{1,k_{min}}(\Omega) \\
\\
\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \text{ in } L^1(\Omega), \text{ for all } i = 1, ..., N. \\
\\
a_i(x, \frac{\partial u_n}{\partial x_i}) \to a_i(x, \frac{\partial u}{\partial x_i}) \text{ a.e in } \Omega \text{ and in } L^1(\Omega), \text{ for all } i = 1, ..., N.
\end{cases}
\tag{3.12}
$$

Now, fix $t > 0$, $\varphi \in W_0^{1, \overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$, and choose $T_t(u_n - \varphi)$ as a test function in (2.2), with u replaced by u_n to obtain

$$\int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i} u_n) . \frac{\partial}{\partial x_i} T_t(u_n - \varphi) dx = \int_\Omega f_n(x) T_t(u_n - \varphi) dx.$$

Note that this choice can be made using a standard density argument. We now pass to the limit in the previous identity. For the right-hand side, the convergence is obvious since f_n converges strongly in L^1 to f and $T_t(u_n - \varphi)$ converges weakly-* in L^∞, and a.e., to $T_t(u - \varphi)$.

Next, we write the left hand side as

$$\int_{\{|u_n - \varphi| \leq t\}} \sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i} u_n) . \frac{\partial}{\partial x_i} u_n dx - \int_{\{|u_n - \varphi| \leq t\}} \sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i} u_n) . \frac{\partial}{\partial x_i} \varphi dx \tag{3.13}$$

By (3.12), the second integral of (3.13) converges to

$$\int_{\{|u-\varphi|\leq t\}} \sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}\varphi dx.$$

For the first integral in (3.13), as $\sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i}u_n).\frac{\partial}{\partial x_i}u_n$ is nonnegative by (1.5), we obtain by using (3.12) and Fatou's Lemma

$$\int_{\{|u-\varphi|\leq t\}} \sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}u dx \leq \lim_{n\to\infty} \inf \int_{\{|u_n-\varphi|\leq t\}} \sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i}u_n).\frac{\partial}{\partial x_i}u_n dx.$$

Gathering results, we obtain

$$\int_{\Omega} \sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}T_t(u - \varphi)dx \leq \int_{\Omega} f(x)T_t(u - \varphi)dx,$$

i.e., u is an entropy solution of (1.1).

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