

On Semisubmedian Functions and Weak Plurisubharmonicity

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ABSTRACT

In this note subharmonic and plurisubharmonic functions on a complex space are studied intrinsically. For applications subharmonicity is characterized more effectually in terms of properties that need be verified only locally off a thin analytic subset; these include the submean-value inequalities, the spherical (respectively, solid) monotonicity, near as well as weak subharmonicity. Several results of Gunning [9, K and L] are extendable via regularity to complex spaces. In particular, plurisubharmonicity amounts (on a normal space) essentially to regularized weak plurisubharmonicity, and similarly for subharmonicity (on a general space). A generalized Hartogs' lemma and constancy criteria for certain matrix-valued mappings are given.

RESUMEN

En esta nota son estudiadas intrínsecamente las funciones subarmonicas y plurisubarmonicas sobre un espacio complejo. Para aplicaciones, subarmonicidad es caracterizada mas eficientemente en términos de propiedades que necesitan ser verificadas solamente localmente en un subconjunto analítico delgado; estas aplicaciones incluyen la desigualdad del valor-submedio, la monotonicidad esférica (respectivamente, sólida), bien como subarmonicidad debil. Varios resultados de Gunning [9, K and L] son extendibles vía regularidad a espacios complejos. En particular, plurisubarmonicidad (sobre un espacio normal)

¹Supports by the "Globale Methoden in der komplexen Geometrie" Grant of the German research society DFG and the Faculty Improvement Grant of Minnesota State University, Mankato, are gratefully acknowledged.

importa esencialmente para plurisubarmonicidad débil regularizada y similarmente para subarmonicada (sobre un espacio general). Son dados un lema de Hartogs generalizado y un criterio de constancia para ciertas aplicaciones matriz-valuada.

Key words and phrases: *Subharmonicity, seminear subharmonicity, Jensen function, weak subharmonicity, weak plurisubharmonicity*

2000 Math. Subj. Class.: *Primary: 31C05; Secondary: 31C10, 32C15*

1 Introduction

It is well-known that subharmonic functions are widely used in potential theory and partial differential equations, and their complex analogue, the plurisubharmonic functions, have played a significant role in the development of higher dimensional complex analysis. In this work such functions are studied on a general complex space from an intrinsic viewpoint, the main purpose here being to show that they admit intrinsic distributional representations despite the presence of singularities. In what follows let Y denote a (reduced) complex space of pure dimension $m > 0$. This means that Y is a Hausdorff space which admits a countable basis of open sets and an open covering $\{U_j\}$ together with homeomorphisms $\alpha_j : U_j \rightarrow V_j$ where V_j is a pure m -dimensional analytic subset in some open subset of \mathbb{C}^{n_j} such that each mapping $\alpha_j \circ \alpha_k^{-1} : \alpha_k(U_j \cap V_k) \rightarrow \alpha_j(U_j \cap V_k)$ is biholomorphic. A *Riemann covering* of an m -dimensional complex space Y at a point $a \in Y$ is a holomorphic map of a neighborhood of a into \mathbb{C}^m with discrete fibers. A mapping $\phi : Y \rightarrow [-\infty, \infty)$ is called *subharmonic* ($\phi \in SH(Y)$) if: (i) ϕ is upper semicontinuous ($\phi \in C^{\text{usc}}(Y)$); (ii) for any compact set $K \subset Y$ and any function $h : K \rightarrow \mathbb{R}$ which is continuous in K and (locally) semiharmonic ([24]) in $\text{int}(K)$, and $h(z) \geq \phi(z)$ for every $z \in \partial K$, it follows that $h(a) \geq \phi(a)$ for every $a \in K$ ([17, p. 1][10, p. 16]). Subharmonicity is both a global and a local property (Lemma 3.2), the link between the two being rested with a maximum principle for almost everywhere *solidly submedian* functions (Proposition 3.1). In fact, via local Riemann coverings characterizations of subharmonicity can be given, as in the Euclidean case, in terms of the local property of being *solidly*, respectively, *spherically*, *submedian*. With the continuity assumption subharmonicity is also equivalent, by virtue of a maximum principle for *seminearly subharmonic functions* (Proposition 3.3), to *near subharmonicity*, *spherical* (respectively, *solid*) *monotonicity*. For applications it gains in usefulness if such a property needs only be verified locally off a thin analytic subset (Theorems 3.1 and 3.3). Especially, in the definition of subharmonicity, it suffices to require the condition (ii) to hold for closed pseudoballs K lying in Y off some thin analytic subset, instead of arbitrary compact subsets of Y (by Lemma 3.1 and Theorem 3.1; cf. [1, p. 36]).

A locally integrable (or an upper semicontinuous) function $\psi : Y \rightarrow [-\infty, \infty)$ is said to be *regularized at* $a \in Y$ (respectively, *regularized in* Y) if

$$\nu_{p_\Delta}(a) \psi(a) = \lim_{\varepsilon \rightarrow 0} \langle \psi \rfloor \Delta \rangle_{a, \varepsilon} \quad (1.1)$$

for some standard domain Δ at a (respectively, the said condition holds for each $a \in Y$). An upper semicontinuous function $\psi : Y \rightarrow [-\infty, \infty)$ is said to be *strictly regularized in* Y , if ψ satisfies not only the condition (1.1) for any standard domain Δ at every point of Y , but also the condition

$$\nu_{p_\Delta}(a) \psi(a) = \lim_{\varepsilon \rightarrow 0} [\psi] \Delta_{a,\varepsilon}, \tag{1.2}$$

for each pseudoball Δ at every point a of Y . Here $\nu_p(a)$ denotes the *multiplicity* of a light holomorphic map p at a ([21, p. 22]; for a topological treatment see Radò and Reichelderfer [18]), and $\langle \psi | \Delta \rangle_{a,\varepsilon}$ (respectively, $[\psi] \Delta_{a,\varepsilon}$) the solid (respectively, spherical) mean-value function of ψ ((2.3)-(2.4)). Every plurisubharmonic function is strictly regularized (Propositions 3.2 and 4.2).

A mapping $\phi : Y \rightarrow [-\infty, \infty)$ is *plurisubharmonic* ($\phi \in PSH(Y)$) in the sense of Lelong and Oka) iff $\phi \in C^{\text{usc}}(Y)$ and the pull-back of ϕ by any holomorphic map from the unit disc $\Delta \subset \mathbb{C}$ into Y is subharmonic in Δ . This definition turns out, owing to a notable result of Fornæss and Narasimhan ([5, p. 64]), to be equivalent to one in an apparently stronger sense ([15, p. 356]): a mapping $\phi : Y \rightarrow [-\infty, \infty)$ is plurisubharmonic if ϕ has a plurisubharmonic extension into the ambient space of a local embedding of Y at every point. Consequently plurisubharmonicity of a mapping is a local property. In contrast to "subharmonicity", such a mapping is necessarily *radially submedian* ((4.2)) and, if not identically equal to $-\infty$ on any open subset, a *Jensen function* (Proposition 4.2). On a normal complex space the (possibly weaker) property of being *semiradially submedian* suffices to characterize plurisubharmonicity (Theorem 4.1).

A locally integrable function with nonnegative local distributional Laplacian is called *weakly subharmonic*. With C^2 -differentiability such a function can be regarded as possessing (off a thin analytic subset) nonnegative local spherical mean-radial derivative. By a similar token the requirement of semidefinite positivity of the distributional *Levi form* of locally integrable functions gives rise to a class $\mathcal{PSH}_{\text{w}}(Y)$ of *weakly plurisubharmonic* functions. Denote by $SH_{\{\neq -\infty\}}(Y)$ the set of all subharmonic functions in Y not identically equal to $-\infty$ on any open subset of Y . It turns out that a weakly subharmonic function ϕ belongs to $SH_{\{\neq -\infty\}}(Y)$ if and only if ϕ is regularized in Y (Theorem 3.4). Owing to the strict regularity of plurisubharmonic functions, several results of Gunning [9, K and L] are extendable to complex spaces. Some of these are indicated in §4 and §5. Especially, given $\phi \in \mathcal{PSH}_{\text{w}}(Y)$, there exists a unique locally integrable, regularized function $\psi : Y \rightarrow [-\infty, \infty)$ such that $\psi = \phi$ almost everywhere and $\psi \in SM_{\text{rad}}(Y) \cap PSH_{\{\neq -\infty\}}(Y_{\text{reg}})$; *if further Y is normal, then a function $\phi : Y \rightarrow [-\infty, \infty)$ is weakly plurisubharmonic and regularized in Y if and only if $\phi \in PSH_{\{\neq -\infty\}}(Y)$* (Theorem 4.2 and Corollary 4.1). As applications, a generalized Hartogs' lemma, conditions for the constancy of certain matrix-valued mappings (including extensions of two results of Bochner and Montgomery [2, p. 155]), and the maximum modulus principle for weakly real-analytic, subharmonic mappings, are given. Also, some lemmas and examples of recurring use on subharmonicity and plurisubharmonicity are gathered in an Appendix.

The author is indebted to the referee for suggestions which led to the improvement of this paper.

2 Preliminaries

Denote by $\|z\|$ the Euclidean norm of $z = (z_1, \dots, z_m) \in \mathbb{C}^m$, where each component $z_j = x_j + i y_j$. Let the space \mathbb{C}^m be oriented so that the Eulidean Kähler form $v^m := ((i/2\pi) \partial \bar{\partial} \|z\|^2)^m$ is positive. In what follows let X, Y denote (reduced) complex spaces of pure dimension $m > 0$, and let $p : Y \rightarrow \mathbb{C}^m$ be a holomorphic map. Set $a' := p(a)$, $p^{[a]} := p - a'$, and $r_a := \|p^{[a]}\|$ for each $a \in Y$. If $U \subseteq Y$ is an open set, $a \in U$ and $r > 0$, set $U_{[a]}(r) := \{z \in U \mid r_a(z) < r\}$, and $U_{[a]}[r] := \{z \in U \mid r_a(z) \leq r\}$.

Denote by $\mathbb{B}_{[a']}(r)$ the open ball in \mathbb{C}^m with center a' and radius r , omitting the subscript if $a' = 0$. Let dv (respectively, $d\sigma_r$) be the Euclidean volume element of \mathbb{C}^m (respectively, the sphere $\mathbb{S}(r) = \partial\mathbb{B}(r)$) and $|\mathbb{B}(r)|$ (respectively, $|\mathbb{S}|$) the volume of $\mathbb{B}(r)$ (respectively, $|\mathbb{S}(1)|$).

A complex space X together with a holomorphic map $p : X \rightarrow \Omega$, where Ω is an open, connected subset of \mathbb{C}^m , is called a *semi-Riemann domain* (of dimension $m > 0$) if there exists a thin analytic set Σ in Ω such that $\Sigma_p := p^{-1}(\Sigma)$ is thin in X , and the restriction $p : X^0 := X \setminus \Sigma_p \rightarrow \Omega_0 := \Omega \setminus \Sigma$ has discrete fibers; the map $p = (p_1, \dots, p_m) : X \rightarrow \Omega$ is called a *Riemann semicovering*. If $\Sigma = \emptyset$, then (X, p) is called a *Riemann domain* and the map p a *Riemann covering*. If $p : X \rightarrow \Omega$ is, in addition, a local homeomorphism, then (X, p) is said to be *unramified*. Every proper modification of an affine algebraic variety is a semi-Riemann domain (as such it is parabolic in the sense of [22, pp. 73–74]). Riemann domains, in particular, analytic coverings of \mathbb{C}^m , play a fundamental role in complex analysis; in fact, every pure m -dimensional complex space is locally an analytic covering of a domain in \mathbb{C}^m (hence a Riemann domain).

Let (X, p) be a semi-Riemann domain; denote by X^* the largest open subset of X on which p is locally biholomorphic. For each open subset $D \subseteq X$, set $D^0 := D \cap X^0$ and $D^* := D \cap X^*$. Suppose that $p : \tilde{U} \rightarrow \Omega$, is a holomorphic map defined on an open neighborhood \tilde{U} of a point $a \in Y$ such that: i) $p^{-1}(a') \cap \tilde{U} = \{a\}$; ii) for a sufficiently small ball $U' := \mathbb{B}_{[a']}(r)$ in \mathbb{C}^m , $U_a := p^{-1}(U') \cap \tilde{U} = p^{-1}(U') \cap \tilde{U}$ is connected and the mapping $p|_{U_a} : U_a \rightarrow U'$ is an analytic covering (biholomorphic if $a \in D^*$); iii) every branch V_a^k , $k = 1, \dots, s_a$, of U_a contains a ; and iv)

$$s_a = \deg(p|_{U_a}) = \nu_p^y(a) \tag{2.1}$$

([21, Proposition 1.3]). Such an U_a is called, for convenience, a *pseudoball* (of radius r) at a ([24, p. 557]). A *pseudopolydisc* $\Delta_{[a]}(R)$ over a polydisc (of polyradius R) in \mathbb{C}^m is similarly defined. An open neighborhood Δ of a point a in a complex space Y is called a *standard domain at a* if and only if there exists a Riemann covering $p : \Delta \rightarrow \Omega$ exhibiting Δ as either a pseudoball (of radius R) or a pseudopolydisc (of polyradius R) at a . If $p : Y \rightarrow \Omega$ is a Riemann semicovering and $a \in Y^0$, define $d(a)$ (respectively, $d_{\text{pd}}(a)$) := the supremum of $R > 0$ (respectively, $R = (R_1, \dots, R_m) > (0, \dots, 0)$) such that a pseudoball (respectively, polydisc) $\Delta_{[a]}(R)$ exists.

The notions of C^k -differential forms, the exterior differentiation d , the operators $\partial, \bar{\partial}$ and $d^c := (1/4\pi i)(\partial - \bar{\partial})$, are well-defined on a complex space Y despite the presence of singularities ([23, Chapter 4]). If $G \subseteq Y$ is open subset, denote by dG the (maximal) *boundary manifold* of G_{reg} in Y_{reg} , the manifold of simple points of Y , oriented to the exterior of G_{reg} ([23, p. 218]). If $p : G \rightarrow \mathbb{C}^m$ is a holomorphic map and $a \in G$, the form

$$v_p := dd^c r_a^2 = \frac{i}{2\pi} \partial \bar{\partial} r_a^2 \tag{2.2}$$

is nonnegative ([23, §4]) and independent of a . The *Poincaré form*

$$\sigma_a := \frac{2 d^c r_a}{r_a^{2m-1}} \wedge v_p^{m-1}$$

is d -closed. Also set $d\tilde{v} := p^*dv$, $d\sigma_{[a],r} := (p^{[a]})^*d\sigma_r$, where dv (respectively, $d\sigma_r$) denotes the Euclidean volume element of \mathbb{C}^m (respectively, $\mathbb{S}(r)$). If $p : Y \rightarrow \mathbb{C}^m$ is a Riemann semicovering and $\phi \in C^{\text{usc}}(U)$, where $U = U_{[a]}(r_0) \subset Y$, the associated *spherical mean-value function* is defined by

$$[\phi]U]_{a,r} := \int_{dU_{[a]}(r)} \phi \sigma_a, \quad 0 < r < r_0, \quad (2.3)$$

([24, (3.2)]). Similarly, if $\phi \in C^{\text{usc}}(\Delta)$, where $\Delta = \Delta_{[a]}(R_0)$ is a standard domain in Y , the associated *solid mean-value function* is defined by

$$\langle \phi \rangle \Delta]_{a,R} := \frac{1}{\text{vol}(\Delta(a', R))} \int_{\Delta_a(R)} \phi(z) d\tilde{\nu}(z), \quad 0 < R < R_0, \quad (2.4)$$

where $\text{vol}(\Delta(a', R))$ denotes the Euclidean volume of the induced domain $\Delta(a', R) := p(\Delta_a(R))$. Let $j_{dG} : dG \rightarrow Y$ denote the inclusion mapping and $d\sigma = d\sigma_{dG}$ the (Lebesgue) surface measure on $Y^* \cap dG$ induced by the local patches $p_U := p : U \rightarrow \mathbb{B}_{[a]}(r)$ on an unramified neighborhood U of a point $a \in dG$.

3 Subharmonicity

An upper semicontinuous map $\phi : Y \rightarrow [-\infty, \infty)$ is called (locally) *semispherically submedian* in Y ($\phi \in SM_{\text{sm.sph}}(Y)$), if there exists a thin analytic subset A of Y such that: (i) at every $z \in Y \setminus A$, there is a pseudoball Δ of radius r_0 for which the inequality

$$\nu_{p_\Delta}(z) \phi(z) \leq [\phi] \Delta]_{z,r}, \quad 0 < r < r_0, \quad (3.1)$$

holds; (ii) if ϕ admits an *exceptional peak point*, that is, a point $z_* \in A$ with $\phi(z_*) = \sup_Y \phi$, then $\phi(z_*) \leq \liminf_{n \rightarrow \infty} \phi(a_n)$ for all sequences $\{a_n\}$ in $\Delta \setminus A$ converging to z_* . Similarly, the set $SM_{\text{sm.sol}}(Y)$ of (locally) *semisolidly submedian* maps is defined by requiring (in place of (3.1)) the inequality

$$\nu_{p_\Delta}(z) \phi(z) \leq \langle \phi \rangle \Delta]_{z,R}, \quad 0 < R < R_0, \quad (3.2)$$

to hold for some pseudoball $\Delta = \Delta_z(R_0)$. If in the preceding the set A can be taken to be the empty set, then ϕ is called *spherically submedian* (respectively, *solidly submedian*) in Y .

An upper semicontinuous map $\phi : Y \rightarrow [-\infty, \infty)$ is said to be *weakly Jensen* if there exists a thin analytic subset A of Y such that at each $z \in Y_0(\phi) := Y \setminus A$, the inequality (3.2) holds with respect to some standard domain Δ , where, in the case of a pseudoball (respectively, pseudopolydisc), R_0 denotes the radius (respectively, the polyradius) of Δ . (In view of Proposition 4.1 and Theorem 3.1) the following extension of [24, Proposition 3.2] generalizes the maximum principle for subharmonic (and plurisubharmonic) functions:

Proposition 3.1. (Maximum principle for weakly Jensen functions) *Let D be a domain in Y and $\phi : D \rightarrow [-\infty, \infty)$ a weakly Jensen function with finite supremum M . Assume that (i) the peak set $\mathcal{P} := \{z \in D \mid \phi(z) = M\}$ is nonempty; (ii) if no peak point of ϕ lies in $D_0 := D_0(\phi)$, then every sequence $\{a_n\}$ in D_0 converging to a point $z_0 \in \mathcal{P}$ satisfies the inequality $\phi(z_0) \leq \liminf_{n \rightarrow \infty} \phi(a_n)$. Then $\phi = \text{constant}$.*

Proof. Let $F := \{z \in D \mid \phi(z) < M\}$. Assume at first that ϕ admits a peak point in D_0 . Choose a standard domain $\Delta \subseteq D_0$ at such a point a for which the inequality (3.2) holds. Then the proof of [24, Proposition 3.2] carries over here and yields the desired conclusion. Now consider the case where ϕ has no peak point in D_0 . Let z_0 be any peak point of ϕ , and $\{a_n\}$ a sequence in D_0 converging to z_0 . Choose a sequence of standard domains $\Delta_{[a_n]}(R_n)$ (relative to a Riemann covering p_n) with norm $\|R_n\| \rightarrow 0$ as $n \rightarrow \infty$ such that the solid submean-value property (3.2) holds. Suppose that $F \cap B_{[z_0]}(R_k) \neq \emptyset$ for every standard domain $B_{[z_0]}(R_k)$. Then $\phi(z) < M$ for all z in a neighborhood of each point in $F \cap B_{[z_0]}(R_k)$. Since $\Delta_{[a_n]}(R_n) \cap B_{[z_0]}(R_k) \neq \emptyset$ for sufficiently large n and $a_n \in F$, it follows from [23, Proposition 5.2.2] and [24, (2.4)] that

$$\begin{aligned} \text{vol}(\Delta(a'_n, R_n)) \nu_{p_n}(a_n) \phi(a_n) &\leq \int_{\Delta_{a_n}(R_n)} \phi(z) d\tilde{\nu}(z) \\ &< \int_{\Delta_{a_n}(R_n)} M d\tilde{\nu}(z) = M \text{vol}(\Delta(a'_n, R_n)) \deg(p_n \lfloor \Delta). \end{aligned}$$

This inequality and the relation [24, (2.5)] imply that

$$\phi(z_0) \leq \liminf_{n \rightarrow \infty} \phi(a_n) < M,$$

hence a contradiction. Therefore $F \cap B_{[z_0]}(R_k) = \emptyset$ for some $B_{[z_0]}(R_k)$. Thus the set \mathcal{P} is open and nonvoid. Then by the connectedness of D one must have $F = \emptyset$. Consequently $\phi(z) \equiv M$ in D . \square

In the following, if V^k is a branch of a standard domain Δ at point $a \in Y$, and $\phi \in L^1(\Delta)$, denote by $\tilde{\phi}_k$ a locally integrable function on Δ' such that $p^* \tilde{\phi}_k = \phi$ on V^k . Since each branch V^k contains a , the function $\tilde{\phi}_k$ can be chosen to be continuous in Δ' , provided so is ϕ in Δ .

Lemma 3.1. *Let (X, p) be a Riemann domain. Assume that $\phi : X \rightarrow [-\infty, \infty)$ is upper semicontinuous, and for each pseudoball $U \subset X$ and every continuous function $h : \bar{U} \rightarrow \mathbb{R}$ with $h \geq \phi \lfloor \partial U$ such that h is semiharmonic in U , one has $h \geq \phi \lfloor U$. Then $\nu_p(a) \phi(a) \leq [\phi]_{a,r}$ for all $a \in X$ and all $r \in (0, d(a))$, (in particular, ϕ is spherically submedian).*

Proof. Let U be a pseudoball at $a \in X$, and $r \in (0, d(a))$. There exists, for each branch V^k , $k = 1, \dots, l$, of U , a decreasing sequence of continuous functions $\tilde{\phi}_k^n$ on $U'_{a'}[r]$ converging pointwise to $\tilde{\phi}_k$. The Poisson integral $h_n := P_{a',r}(\tilde{\phi}_k^n)$ is continuous and harmonic in $U'_{a'}(r)$, and $h_n = \tilde{\phi}_k^n \geq \tilde{\phi}_k$ on $\partial U'_{a'}(r)$. By the subharmonicity of $\tilde{\phi}_k$ and the formula [25, (4.16)], one has

$$\tilde{\phi}_k(z') \leq h_n(z') = \int_{\partial U'_{[a']}(r)} P_{a',r}(z', \zeta') \tilde{\phi}_k^n(\zeta') d\sigma_{[a'],r},$$

for each $z' \in U'_{a'}(r)$. Hence by the monotone convergence theorem,

$$\tilde{\phi}_k(z') \leq \lim_{n \rightarrow \infty} h_n(z') = \int_{\partial U'_{[a']}(r)} P_{a',r}(z', \zeta') \tilde{\phi}_k(\zeta') d\sigma_{[a'],R}.$$

In particular, by the definition (2.3),

$$\tilde{\phi}_k(a') \leq [\tilde{\phi}_k]_{a',r} = \int_{\partial U'_{[a']}(r)} \frac{1}{r} \frac{r^2}{|\mathbb{S}| \|a' - \zeta'\|^{2m}} \tilde{\phi}_k(\zeta') d\sigma_{[a'],r}.$$

Consequently ϕ satisfies the spherical submean-value inequality (3.1) for all $r \in (0, d(a))$. □

Observe that for each $\psi \in L^1(\mathbb{B}_{[a']}(r_0))$, the Fubini type formula holds:

$$\int_{\mathbb{B}_{[a']}(r)} \psi dv_{[a']} = \int_0^r \left(\int_{\mathbb{S}_{[a']}(t)} \psi d\sigma_{[a'],t} \right) dt, \quad 0 < r < r_0. \tag{3.3}$$

Lemma 3.2. *Let Y , be a pure m -dimensional complex space and $\phi : Y \rightarrow [-\infty, \infty)$. Then $\phi \in SH(Y)$ if and only if ϕ is locally subharmonic in Y .*

Proof. Using the formula (3.3), the following preliminary assertion can be proved (in the same way as in [24, Lemma 3.1]): For every pseudoball $\Delta = \Delta_{[a]}(R_0) \subset Y$ and for each $\phi \in L^1(\overline{\Delta})$, the inequality " $[\phi]\Delta_{a,R} \geq (\text{constant}) C$ for all $R \in (0, R_0)$ " implies that " $\langle \phi \rangle_{a,R} \geq C$ for all $R \in (0, R_0)$ ". Suppose that ϕ is locally subharmonic in Y . Let $K \subset Y$ be a compact subset and $h : K \rightarrow \mathbb{R}$ a continuous function which is semiharmonic in $\text{int}(K)$ and $h(z) \geq \phi(z)$ for all $z \in \partial K$. It suffices to show that the difference $\psi := \phi - h$ satisfies the maximum principle on each connected component G of $\text{int}(K)$. Suppose now that $\sup_G \psi$ is attained at a point of G . By assumption, at each point $a \in G$ there is an open neighborhood Q such that $\phi \in SH(Q)$. Then by Lemma 3.1 and the preliminary assertion, ϕ is solidly submedian in a neighborhood U of $a \in Q$. Thus by virtue of the Gauss mean-value formula for semiharmonic functions one has $\nu_p(a) \psi(a) \leq \langle \psi \rangle_{U,a,R}$ for all $R \in (0, R_0)$. Therefore, by the maximum principle for solidly submedian functions (Proposition 3.1), $\psi = (\text{constant}) C$ in G , so that for every sequence $\{a_n\} \subset G$ converging to a point $a \in \partial G$, one must have

$$0 \geq (\phi - h)(a) \geq \limsup_{n \rightarrow \infty} (\phi - h)(a_n) = C.$$

Consequently $h(z) \geq \phi(z)$ for all $z \in K$, as desired. □

Theorem 3.1. (1) $SM_{\text{sm.sph}}(Y) = SM_{\text{sm.sol}}(Y) = SH(Y)$. (2) *If $\phi \in SH(Y)$, then ϕ is solidly submedian (relative to any pseudoball) at every point of Y .*

Proof. The preliminary assertion in Lemma 3.2 shows that $SM_{\text{sm.sph}}(Y)$ is a subset of $SH_{\text{sm.sol}}(Y)$. As in the proof of Lemma 3.2, the inclusion " $SM_{\text{sm.sol}}(Y) \subseteq SH(Y)$ " is a consequence of Proposition 3.1. Finally the assertion (2) follows from Lemma 3.1 and the preliminary assertion in Lemma 3.2. □

Remark 3.1. *The second assertion of the above theorem gives a generalization of Lemma 15.2 of [7] to complex spaces. Also, it will be seen later (Theorem 3.4) that if ϕ is only weakly subharmonic, then the solid submean-value inequality holds almost everywhere in Y .*

Corollary 3.1. *If $\phi : Y \rightarrow \mathbb{C}$ is semiharmonic, then there exists $\psi \in C^0(Y) \cap SH(Y)$ such that $\psi = \phi$ almost everywhere in Y .*

Proof. Since ϕ is semiharmonic in any given pseudoball $U \subset Y$, there exists, by [24, Theorems 4.2], a continuous function ψ_U in U such that $\psi_U = \phi$ almost everywhere and ψ_U has the solid mean-value property. Hence Theorem 3.1 implies that ψ_U is subharmonic in U . The conclusion follows then from Lemma 3.2. □

An upper semicontinuous function $\phi : Y \rightarrow [-\infty, \infty)$ is said to be *spherically* (respectively, *solidly*) *monotone at* $z \in Y$ if there exists a pseudoball U at z of radius r_0 such that

$$[\phi]U]_{z,r} \leq [\phi]U]_{z,s}, \quad \text{if } 0 < r \leq s < r_0, \tag{3.4}$$

(respectively,

$$\langle \phi \rangle U]_{z,r} \leq \langle \phi \rangle U]_{z,s}, \quad \text{if } 0 < r \leq s < r_0). \tag{3.5}$$

Let $p_U : U \rightarrow \Omega$ be an unramified Riemann covering at a point $a \in Y_{\text{reg}}$. The *radial derivative* of $\phi \in C^1(U)$ at $z \in dU_a(r) \setminus p^{-1}(a')$ (for sufficiently small $r > 0$) is defined by

$$j_{a,r}^*(d^c \phi \wedge v_{p_U}^{m-1})(z) = \frac{r^{2m-1}}{2} (\mathfrak{R}_{p_U,a} \phi)(z) \sigma_a, \tag{3.6}$$

where $j_{a,r} : dU_a(r) \rightarrow X$, denotes the inclusion (for more detail see [24, p. 567]). The *spherical mean radial derivative*

$$[\mathfrak{R}_{p_U,z}(\phi)]_{z,r} := \int_{dU_{[z]}(r)} \mathfrak{R}_{p_U,z}(\phi) \sigma_z, \quad z \in U,$$

exists for sufficiently small $r > 0$. It follows from the identity (3.6) (and [24, p. 571]) that

$$\frac{d}{d\rho} [\phi]U]_{z,\rho} \Big|_{\rho=r} = [\mathfrak{R}_{p_U,z}(\phi)]_{z,r}, \quad \forall z \in U, \tag{3.7}$$

for sufficiently small $r > 0$. For a real-valued $\phi \in C^2(Y_{\text{reg}})$ the spherical monotonicity at a point $a \in Y_{\text{reg}}$ amounts to the condition that ϕ possesses (locally) nonnegative spherical mean radial derivative at a , that is, there is an unramified Riemann covering p_U at a such that $[\mathfrak{R}_{p_U,a}(\phi)]_{a,r} \geq 0$ for sufficiently small $r > 0$. Let Δ_{p_U} denote the pull-back (under p_U) of the Euclidean Laplace operator on \mathbb{C}^m . By differentiating under the integral sign and using the divergence theorem, it is easy to show that

$$\frac{d}{d\rho} [\phi]B]_{a,\rho} \Big|_{\rho=r} = \frac{1}{|\mathbb{S}^{2m-1}|} \int_{B_{[a]}(r)} \Delta_{p_U} \phi \, dv_{[a],r}.$$

Consequently one has, for each $z \in U$,

$$[\mathfrak{R}_{p_U,z}(\phi)]_{z,r} = \frac{r}{2m} \langle \Delta_{p_U} \phi \rangle_{z,r}, \tag{3.8}$$

for sufficiently small $r > 0$.

Lemma 3.3. *If $p_U : U \rightarrow \Omega$ is an unramified Riemann covering and $\phi \in C^2(U)$ is real-valued and solidly submedian, then $\Delta_{p_U} \phi \geq 0$.*

Proof. Suppose that $\Delta_{p_U} \phi < 0$ in $U_{[a]}(r_0)$ for some $a \in U$ and $r_0 > 0$. By the relations (3.7)-(3.8) one has

$$\frac{d}{d\rho} [\phi]U_{a,\rho} \Big|_{\rho=r} < 0, \quad \forall r \in (0, r_0).$$

Hence it follows from the relation (1.2) that

$$\nu_{p_U}(a) \phi(a) > \frac{1}{|\mathbb{S}| \rho^{2m-1}} \int_{dX_{[a]}(\rho)} \phi d\sigma_{[a],\rho}, \quad \forall \rho \in (0, r_0). \tag{3.9}$$

Thus, multiplying (3.9) by $|\mathbb{S}| \rho^{2m-1}$ and integrating over $(0, r)$, $0 < r < r_0$, yields

$$|\mathbb{B}(r)| \nu_{p_U}(a) \phi(a) > \int_{X_{[a]}(r)} \phi d\tilde{\nu},$$

which implies that $\nu_{p_U}(a) \phi(a) > \langle \phi]U \rangle_{a,r}$, a contradiction. Therefore the Lemma is proved. \square

An upper semicontinuous map $\phi : Y \rightarrow [-\infty, \infty)$ is called *nearly subharmonic at $z \in Y$* if there exists a pseudoball U at z of radius r_0 such that

$$\langle \phi]U \rangle_{z,r} \leq [\phi]U_{z,r} \quad \forall r \in (0, r_0); \tag{3.10}$$

ϕ is said to be *seminearly subharmonic in Y* ($\phi \in SM_{\text{sm.near}}(Y)$) if there exists at each $a \in Y$ an open neighborhood Q and a thin analytic subset A_Q of Q such that ϕ is nearly subharmonic at every point of $Q \setminus A_Q$. The class $\mathfrak{M}_{\text{sm.sph}}(Y)$ (respectively, $\mathfrak{M}_{\text{sm.sol}}(Y)$) of functions *semispherically monotone* (respectively, *semisolidly monotone*) in Y is similarly defined. Note that by Lemma 3.1 and Theorem 3.1-(2), if $\phi \in SH(Y)$, then ϕ is spherically monotone and nearly subharmonic relative to every pseudoball at any point of Y .

Lemma 3.4. *If ϕ is spherically monotone at $a \in Y$ (with U and r_0 as in (3.4) for $z = a$), and ϕ is not identically equal to $-\infty$ in U , then the function " $r \mapsto [\phi]U_{a,r}$ " is continuous on $(0, r_0)$.*

Proof. Since

$$\int_{dU_{[a]}(r)} \phi d\sigma_{[a],r} = \sum_{k=1}^{s_a} \int_{dV_{[a]}^k(r)} \phi d\sigma_{[a],r} = \sum_{k=1}^{s_a} \int_{\mathbb{S}_{[a']}^k(r)} \hat{\phi}_k d\sigma_{[a'],r}, \quad \forall r \in (0, r_0),$$

it suffices to prove that the function " $r \mapsto [\phi]V^k_{a,r}$ " is continuous on $(0, r_0)$. Let ψ denote the induced function of $\phi]V^k$ on U' . There exists a monotonically decreasing sequence $\{\psi^{(\mu)}\}$ of C^∞ subharmonic functions (with increasing domains) converging pointwise on U' to ψ . Let $\varepsilon > 0$ and $t \in (0, r_0)$ be given. Suppose that there exists a decreasing sequence $\{t_n\}$ converging to t such that $[\psi]U'_{a',t} + \varepsilon \leq [\psi]U'_{a',t_n}$ for each n . Then, for a sufficiently large N ,

$$[\psi]U'_{a',t} + \varepsilon \leq [\psi^{(\mu)}]U'_{a',t_n} \xrightarrow{n \rightarrow \infty} [\psi^{(\mu)}]U'_{a',t},$$

where the last limit relation follows from the Stokes Theorem. Applying the monotone convergence theorem to the sequence $\{\psi^{(N)} - \psi^{(\mu)}\}$ for a sufficiently large N , one obtains

$$[\psi \rfloor U']_{a',t} + \varepsilon \leq \lim_{\mu \rightarrow \infty} [\psi^{(\mu)} \rfloor U']_{a',t} = [\psi \rfloor U']_{a',t},$$

a contradiction. The conclusion now follows from the spherical monotonicity of ϕ . □

If U is a pseudoball at $a_0 \in Y$ of radius r_0 , and $\phi \in L^1(\overline{U})$, applying the formula (3.3) to the restriction of ϕ to each branch of U , one obtains, for every $r \in (0, r_0)$, the relation

$$|\mathbb{B}(r)| \langle \phi \rfloor U \rangle_{a_0,r} = \int_0^r [\phi \rfloor U]_{a_0,t} |\mathbb{S}| t^{2m-1} dt. \tag{3.11}$$

Theorem 3.2. $SH(Y) \subseteq \mathfrak{M}_{\text{sph}}(Y) \subseteq \mathfrak{M}_{\text{sol}}(Y) = SM_{\text{near}}(Y)$.

Proof. It will be first shown that, for a given $\phi \in C^{\text{usc}}(Y)$, "subharmonicity" implies "spherical monotonicity". Let U be a pseudoball at $a \in Y$ of radius r_0 such that ϕ is subharmonic in U . Let W_ε and the smooth approximations $\tilde{\phi}_{k,\varepsilon}$ of $\tilde{\phi}_k$ (extended to be 0 outside W_ε), $k = 1, \dots, l$, be defined as in the proof of Theorem 4.2 of [24]. It will be shown that each $\tilde{\phi}_{k,\varepsilon}$ is solidly submedian in W_ε . Let $\mathbf{m}_{\mathbb{B}(r)} := \frac{1}{|\mathbb{B}(r)|} \chi_{\mathbb{B}(r)}$ be the measure defined by a uniform unit mass distribution over the ball $\mathbb{B}(r)$. Take $z' \in U'$. There exists $r_{z'} > 0$ such that

$$\begin{aligned} \langle \tilde{\phi}_k \rfloor U' \rangle_{z',r} &= \frac{1}{|\mathbb{B}(r)|} \int_{\mathbb{B}_{[z']}(r)} \tilde{\phi}_k(y) dv(y) \\ &= (\mathbf{m}_{\mathbb{B}(r)} * \tilde{\phi}_k)(z'), \quad 0 < r < r_{z'}. \end{aligned}$$

Thus for every $z \in V^j \setminus \{a\}$,

$$\tilde{\phi}_{k,\varepsilon}(z') = (h_\varepsilon * \tilde{\phi}_k)(z') \leq h_\varepsilon * (\mathbf{m}_{\mathbb{B}(r)} * \tilde{\phi}_k)(z'), \quad \forall r \in (0, r_{z'}).$$

Hence (following an argument of [14, p. 205]) the inequality

$$\tilde{\phi}_{k,\varepsilon}(z') \leq (\mathbf{m}_{\mathbb{B}(r)} * \tilde{\phi}_{k,\varepsilon})(z') = \langle \tilde{\phi}_{k,\varepsilon} \rfloor U' \rangle_{z',r}$$

holds at each $z' \in W_\varepsilon \setminus \{a'\}$ for sufficiently small $r > 0$. Thus by Lemma 3.3, $\Delta_{p_U} \tilde{\phi}_{k,\varepsilon}$ is nonnegative in $W_\varepsilon \setminus \{a'\}$, and therefore by the formulas (3.7)-(3.8), the function " $r \mapsto [\tilde{\phi}_{k,\varepsilon} \rfloor U']_{z',r}$ " is increasing on $(0, r_0 - \varepsilon)$. Moreover, for a fixed $r \in (0, r_0)$,

$$[\tilde{\phi}_k \rfloor U']_{z',r} = \lim_{\varepsilon \rightarrow 0} [\tilde{\phi}_{k,\varepsilon} \rfloor U']_{z',r}, \quad \forall z' \in U'.$$

It follows from this that the function " $r \mapsto [\phi \rfloor U]_{z,r}$ " is increasing on $(0, r_0)$ for every $z \in U$. Thus ϕ is spherically monotone at every point of Y . By approximating the last integral in the formula (3.11) by its Riemann sums, it follows easily that the "spherical monotonicity" of ϕ at a point $z \in Y$ implies its "solid monotonicity" at the same point.

To prove the assertion that ϕ is solidly monotone at a point $z \in Y$ if and only if it is nearly subharmonic at the same point, it may be assumed without loss of generality that ϕ is not identically

equal to $-\infty$ in U . Then by Lemma 3.4, differentiation of the formula (3.11) yields for each $z \in U$ and $r \in (0, r_0)$,

$$\frac{r}{2m} \frac{d}{d\rho} \langle \phi \rangle U_{z,\rho} \Big|_{\rho=r} = [\phi]U_{z,r} - \frac{1}{|\mathbb{B}(r)|} \int_0^r [\phi]U_{z,t} t^{2m-1} dt.$$

Therefore the desired equivalence follows.

To prove that "spherical monotonicity" at a point implies "near subharmonicity" at the same point, suppose that (with Q and U as above) the inequality (3.4) holds for each $z \in Q$. Then the relation (3.11) implies that

$$|\mathbb{B}(r)| \langle \phi \rangle U_{z,r} \leq [\phi]U_{z,r} \frac{|\mathbb{S}| r^{2m}}{2m}, \quad \forall r \in (0, r_0),$$

so that the inequality (3.13) holds, thus proving the desired claim. \square

An upper semicontinuous function $\psi : Y \rightarrow [-\infty, \infty)$ is called: (1) *strongly regularized in Y* , if ψ satisfies both the conditions (1.1) and (1.2) for all pseudoballs Δ at every point of Y ; (2) a *Jensen function* ($\psi \in J(Y)$), if at each $z \in Y$ the inequality (3.2) holds for ψ with respect to every standard domain $\Delta = \Delta_z(R_0)$, where, in the case of a pseudoball (respectively, pseudopolydisc), $R_0 = d_{\text{pd}}(a)$ (respectively, $R_0 = d(a)$). Observe that the expression [24, (3.9)] (where the proof, with slight modifications, remains valid in the case of a pseudopolydisc) shows that every continuous function $\psi : Y \rightarrow \mathbb{R}$ is strictly regularized in Y . This property shall be extended to Jensen (hence plurisubharmonic) functions. If $\psi : Y \rightarrow [-\infty, \infty)$, define

$$\psi^{(*)}(a) := \limsup_{z \rightarrow a} \psi(z), \quad \forall a \in Y.$$

Since every subharmonic function is weakly Jensen, it follows from the maximum principle (Proposition 3.1) that $\psi = \psi^{(*)}$ for all $\psi \in SH(Y)$ (compare [9, Theorem J2(c)]).

Proposition 3.2. (1) *If $\psi \in SH(Y)$, then ψ is strongly regularized in Y . In particular, if $\psi \in SH(Y)$ is not identically zero, then the set $\{z \in Y \mid \psi(z) \neq 0\}$ is of positive measure at some point of Y .* (2) *If $\psi \in J(Y)$, then ψ is strictly regularized in Y .*

Proof. Consider first the case $\psi \in SH(Y)$. For any pseudoball Δ at a point $a \in Y$ and a fixed small positive r ,

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \langle \psi \rangle \Delta_{a,\varepsilon} &\leq \limsup_{\varepsilon \rightarrow 0} \left\langle \sup_{z \in \Delta_a(r) \setminus \{a\}} \psi(z) \right\rangle_{a,\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \langle 1 \rangle \Delta_{a,\varepsilon} \sup_{z \in \Delta_a(r) \setminus \{a\}} \psi(z) \\ &= \nu_{p_\Delta}(a) \sup_{z \in \Delta_a(r) \setminus \{a\}} \psi(z). \end{aligned}$$

Here (on the above right-hand side) the relation (3.9) of [24] is invoked. Thus

$$\limsup_{\varepsilon \rightarrow 0} \langle \psi \rangle \Delta_{a,\varepsilon} \leq \nu_{p_\Delta}(a) \psi(a). \tag{3.12}$$

It follows then from the solid submean-value property of ψ that the limit relation (1.1) holds. The relation (1.2) can be similarly proved.

If $\psi \in J(Y)$, then the inequality (3.12) remains valid for any standard domain Δ at each point $a \in Y$, hence the limit relation (1.1) follows from the solid submean-value property (3.2) \square

A maximum principle for continuous nearly subharmonic functions in \mathbb{C}^m was proved in [6, Lemma 3.1, p. 5]. This result is generalized and strengthened below for later applications:

Proposition 3.3. (Maximum principle for seminearly subharmonic functions) *Assume ϕ is a real-valued, continuous, seminearly subharmonic function in a domain $D \subseteq Y$. If ϕ is bounded above and admits a peak point in D , then $\phi = \text{constant}$ in D .*

Proof. Let $\mathcal{P} := \{z \in D \mid \phi(z) = M\}$ be the peak set of ϕ . There exists at every $a \in D$ an open neighborhood Q and a thin analytic subset A_Q of Q such that ϕ is nearly subharmonic at every point of $Q \setminus A_Q$. Let $\hat{D} := \cup(Q \setminus A_Q)$. Consider first the case $C_0 \cap \mathcal{P} \neq \emptyset$ for some component C_0 of \hat{D} . Let C_0 be any such component and U a pseudoball of radius r_0 at a point $a_0 \in C_0 \cap \mathcal{P}$ such that the inequality

$$\langle \phi \rangle U_{z,r} \leq [\phi] U_{z,r} \quad \forall r \in (0, r_0), \quad (3.13)$$

holds for each $z \in U$. Observe that the mapping $r \mapsto [\phi] U_{a_0,r}$ is continuous in r . Suppose there exists $t_1 \in (0, r_0)$ such that $[\phi] U_{a_0,t_1} < M - \frac{1}{N}$ (for $N > \frac{1}{|M|}$). Following an idea of [6, p. 5 (last paragraph)], it will be shown that this assumption leads to a contradiction to the inequality (3.13). Let ρ^* be the infimum of all $t \in (0, t_1)$ such that the spherical means $[\phi] U_{a_0,t}$ are bounded above by $[\phi] U_{a_0,t_1}$. Then $\rho^* > 0$, for, otherwise, there is a sequence $\{t_\mu\}$ converging to 0 with $[\phi] U_{a_0,t_\mu} \leq [\phi] U_{a_0,t_1}$ for each μ , such that, by virtue of [24, Proposition 3.1],

$$\deg(p_U) \phi(a_0) \leq [\phi] U_{a_0,t_1} < M - \frac{1}{N}. \quad (3.14)$$

Hence a contradiction results. The Fubini type formula (3.3) implies that

$$\begin{aligned} |\mathbb{B}(\rho^*)| \langle \phi \rangle U_{a_0,\rho^*} &= \sum_{j=1}^l \int_{V_{[a_0]}^j(\rho^*)} \phi \, dv_{[a_0]} \\ &= \int_0^{\rho^*} \left(\sum_{j=1}^l \int_{\mathbb{S}_{[a_0]}(t)} \tilde{\phi}_j \, d\sigma_{[a_0],t} \right) dt \\ &= \int_0^{\rho^*} [\phi] U_{a_0,t} |\mathbb{S}| t^{2m-1} \, dt > [\phi] U_{a_0,t_1} \frac{|\mathbb{S}|(\rho^*)^{2m}}{2m}. \end{aligned}$$

This leads to a contradiction $\langle \phi \rangle U_{a_0,\rho^*} > [\phi] U_{a_0,\rho^*}$. Therefore one must have $[\phi] U_{a_0,r} \geq M$ for every $r \in (0, r_0)$. Then the formula (3.11) yields that $\langle \phi \rangle U_{a_0,r} \geq M$ for such r . Since $\phi \leq M$, this implies that $\phi = M$ in a neighborhood of a_0 . Thus $C_0 \subseteq \mathcal{P}$. Consequently, by the continuity of ϕ , $\phi = M$ also in D .

Now suppose that $C_0 \cap \mathcal{P} = \emptyset$ for all components C_0 of \hat{D} . Let z_* be any point of \mathcal{P} . Then $z_* \in A_Q$ for every Q (chosen as above). There exists a sequence $\{a_n\}$ in $Q \setminus (A_Q \cup \mathcal{P})$ (for a fixed Q) converging to z_* . Choose a sequence of pseudoballs $U_{[a_n]}(r_n)$ such that ϕ is nearly submedian at each point of $U_n = U_{[a_n]}(r_n)$. Then (as shown in the equation (3.14)),

$$\deg(p_U)\phi(a_n) \leq [\phi]U_n|_{a_n, t_n} < M - \frac{1}{N},$$

(where t_n depends on a_n). Thus by the continuity of ϕ , one has $\phi(z_*) < M$, a contradiction! Hence the preceding argument showing that $\phi = M$ in a neighborhood of a_0 implies the same for a_n . Consequently the sequence $\{a_n\}$ lies in \mathcal{P} , again a contradiction! Therefore $C_0 \cap \mathcal{P} \neq \emptyset$ for some component C_0 of \hat{D} , and thereby the proposition is proved. \square

Theorem 3.3. *For a continuous function $\phi : Y \rightarrow \mathbb{R}$ the following conditions are equivalent: (i) $\phi \in \mathfrak{M}_{\text{sm.sph}}(Y)$; (ii) $\phi \in \mathfrak{M}_{\text{sm.sol}}(Y)$; (iii) $\phi \in SM_{\text{sm.near}}(Y)$; (iv) $\phi \in SH(Y)$.*

Proof. In view of Theorem 3.2, it suffices to prove that every element $\phi \in SM_{\text{sm.near}}(Y)$ is subharmonic in Y . Let $K \subset Y$ be a compact set and $h : K \rightarrow \mathbb{R}$ a continuous function such that h is semiharmonic in $\text{int}(K)$ and $h(z) \geq \phi(z)$ for every $z \in \partial K$. Suppose that ϕ is seminearly subharmonic in Y . Then at each point a in a connected component G of $\text{int}(K)$, there is an open neighborhood Q and a thin analytic subset A_Q of Q such that ϕ is nearly subharmonic at each point of $Q \setminus A_Q$. Thus given $z \in Q \setminus A_Q$, there is a pseudoball $U = U_z(r_0)$ in which the inequality (3.13) holds, and, consequently,

$$\langle \phi - h \rangle U|_{z,r} \leq [\phi - h]U|_{z,r}, \quad \forall r \in (0, r_0).$$

Thus by Proposition 3.3, the function $\phi - h$ satisfies the maximum principle on G . It follows that $\phi - h \leq 0$ in $\text{int}(K)$. This proves the subharmonicity of ϕ . \square

A locally Lipschitz function is semiharmonic if it has almost everywhere vanishing local radial derivative (this will be shown in a later work). With C^2 -differentiability, "subharmonicity" betokens similarly the nonnegativity (off a thin analytic subset) of the local spherical mean radial derivative:

Proposition 3.4. (1) *Let $\phi \in C^0(Y) \cap C^2(Y_{\text{reg}})$ be real-valued. Assume that for some thin analytic subset A of Y there exists, at each $a \in Y \setminus A$, an unramified Riemann covering $p_U : U \rightarrow \Omega$ satisfying one of the following: (a) ϕ has (locally) nonnegative spherical mean radial derivative at each point of U ; (b) $\Delta_{p_U}\phi \geq 0$ in U . Then ϕ is subharmonic in Y . (2) If $\phi \in SH(Y) \cap C^2(Y_{\text{reg}})$, then both the conditions (a) and (b) are valid for any unramified Riemann covering (on an open set in Y_{reg}).*

Proof. Let $p_U : U \rightarrow \Omega$ be an unramified Riemann covering at $a \in Y_{\text{reg}}$. Clearly for any $\phi \in C^2(U)$ the relation (3.8) implies that the conditions (a) and (b) are equivalent; moreover, by virtue of the formula (3.7) each of these implies that ϕ is spherically monotone in U . Thus the first assertion follows from Theorem 3.3. Since each subharmonic function is solidly submedian, the second assertion is a consequence of the preceding and Lemma 3.3. \square

Definition 3.1. *A locally integrable function $\phi : Y \rightarrow [-\infty, \infty)$ is called weakly subharmonic in Y ($\phi \in SH_{\text{w}}(Y)$) if there exists at each $a \in Y$ a standard domain $\Delta = \Delta_a(R)$ such that for all nonnegative $u \in C_0^2(\Delta)$,*

$$\int_{\Delta_a(r)} \phi dd^c u \wedge v_{p_U}^{m-1} \geq 0, \quad \forall r \in (0, R). \tag{3.15}$$

Observe that, if $p : D \rightarrow \Omega$, is a Riemann semicovering on an open set $D \subseteq Y$ and if $\phi \in \mathcal{SH}_w(D)$, then the inequality (3.15) holds with D in place of $\Delta_a(r)$ for all nonnegative $u \in C_0^2(D)$.

Theorem 3.4. (1) *There exists a bijection between $\mathcal{SH}_w(Y)$ and $SH_{\{\neq -\infty\}}(Y)$ taking each $\phi \in \mathcal{SH}_w(Y)$ to a unique element $\hat{\phi} \in SH_{\{\neq -\infty\}}(Y)$ such that $\hat{\phi} = \phi$ almost everywhere in Y .* (2) *A function $\phi : Y \rightarrow [-\infty, \infty)$ is weakly subharmonic and regularized in Y if and only if $\phi \in SH_{\{\neq -\infty\}}(Y)$.*

Proof. Suppose that $\phi \in \mathcal{SH}_w(Y)$. Let $a \in Y$ and $\tilde{\phi}_{k,\varepsilon}$ be defined as in the proof of Theorem 3.2 (for $U = \Delta$ as in the above definition). The weak subharmonicity of ϕ implies that the induced function $\tilde{\phi}_k \in \mathcal{SH}_w(\Delta')$, hence the function $\tilde{\phi}_{k,\varepsilon}$ is weakly subharmonic in W_ε for every $\varepsilon > 0$ (by the proof of the assertion "(1) implies (2)" in [24, Theorem 4.2, p. 564]). Suppose that the form $dd^c \tilde{\phi}_{k,\varepsilon} \wedge v_{[a']}^{m-1}$ is negative at some $z' \in W_\varepsilon$. Then $dd^c \tilde{\phi}_{k,\varepsilon} \wedge v_{[a']}^{m-1} < 0$ in a neighborhood $Q \subseteq W_\varepsilon$ of z' , hence for all $u \in C_0^2(Q)$ with $u \geq 0$ and $u = 1$ in a neighborhood of z' , one has

$$\int_{\Delta'} \tilde{\phi}_{k,\varepsilon} dd^c u \wedge v_{[a']}^{m-1} = \int_{\Delta'} u dd^c \tilde{\phi}_{k,\varepsilon} \wedge v_{[a']}^{m-1} < 0,$$

a contradiction. Thus $\Delta_{id} \tilde{\phi}_{k,\varepsilon} \geq 0$ in W_ε , so that (by the proof of Theorems 3.2) the function $\tilde{\phi}_{k,\varepsilon}$ is spherically monotone, and for every $z' \in W_\varepsilon \setminus \{a'\}$,

$$\tilde{\phi}_{k,\varepsilon}(z') \leq \langle \tilde{\phi}_{k,\varepsilon} \rangle_{z',r} \tag{3.16}$$

for sufficiently small $r > 0$. It follows from the argument of [13, p. 20] that the function $\varepsilon \mapsto \tilde{\phi}_{k,\varepsilon}(z')$ is nondecreasing for fixed $z' \in W_\varepsilon$. Define

$$\psi_\Delta(z) := \min_k \lim_{\varepsilon \rightarrow 0} \{ \tilde{\phi}_{k,\varepsilon}(z') \mid V^k \ni z \}, \quad \forall z \in \Delta. \tag{3.17}$$

Then ψ_Δ is upper semicontinuous in Δ . Moreover $\psi_\Delta(z) = \phi(z)$ for almost every $z \in \Delta$. Also, the formula (3.17) implies that for each $z \in \Delta$,

$$\begin{aligned} \psi_\Delta(z) &\leq \min_k \{ \tilde{\phi}_{k,\varepsilon}(z') \mid V^k \ni z \} \\ &\leq \min_k \frac{1}{|\mathbb{B}(r)|} \int_{\mathbb{B}_{[z']}(r)} \tilde{\phi}_{k,\varepsilon}(y) dv(y), \end{aligned}$$

for sufficiently small positive ε and r , where the last inequality follows from (3.16). Letting $\varepsilon \rightarrow 0$ this inequality implies that

$$\nu_p(z) \psi_\Delta(z) \leq \langle \psi_\Delta \rangle_{z,r},$$

for sufficiently small $r > 0$. Therefore by Theorem 3.1, $\psi_\Delta \in SH_{\{\neq -\infty\}}(\Delta)$. It follows from Proposition 3.2 that setting $\psi := \psi_\Delta$ on each Δ (as above) defines a subharmonic function $\psi : Y \rightarrow [-\infty, \infty)$ which coincides with ϕ almost everywhere in Y .

Conversely, given $\hat{\phi} \in SH_{\{\neq -\infty\}}(Y)$, it will first be shown that $\hat{\phi}$ is locally integrable in Y . For this purpose a preliminary observation is needed: if $U_z(r) \subset Y$, $0 < r < d(z)$, is a pseudoball at z and $\hat{\phi}(z) > -\infty$, then $\hat{\phi}$ is integrable over $U_z(r)$. This is an immediate consequence of the local solid submean-value inequality for $\hat{\phi}$. Let I be the set of all points of Y where $\hat{\phi}$ is locally integrable. Clearly I is an open set and since locally $\hat{\phi} \neq -\infty$, it follows from the preliminary observation that I is nonempty. On the other hand, if $b \in Y \setminus I$, the preliminary observation implies that $\hat{\phi} \equiv -\infty$ almost everywhere in some pseudoball $U_b(R)$; for, otherwise, $\hat{\phi}(z) > -\infty$ for some point $z \in U_b(r)$ for each (sufficiently small) $r > 0$; hence there is a sequence $\{z_n\} \subset Y$ converging to b such that $\hat{\phi}(z_n) > -\infty$ for every n . If $0 < R < d(b)/2$ and n is sufficiently large, one has $\|z_n - b\| < R$ and $\hat{\phi}$ is locally integrable over $\overline{U}_{z_n}(R)$; but since $b \in U_{z_n}(R)$, it follows that $b \in I$, a contradiction. Thus $U_b(R) \subset Y \setminus I$, so that $Y \setminus I$ is open. Applied to each component Y_μ of Y , this implies that $Y_\mu \setminus I = \emptyset$, hence $\hat{\phi}$ is locally integrable in Y .

It remains to show that $\hat{\phi}$ satisfies the condition (3.15). Let Δ be a standard domain at $a \in Y$ and denote by $\tilde{\phi}^k$ the function on Δ' induced by $\hat{\phi}|V^k$. Then $\tilde{\phi}^k$ is subharmonic, hence solidly submedian, in Δ' . It follows as in the proof of Theorem 3.2 that the smooth approximations $\tilde{\phi}_\varepsilon^k$ of $\tilde{\phi}^k$ are solidly submedian in W_ε . Consequently by Lemma 3.3, $\Delta_{\text{id}} \tilde{\phi}_\varepsilon^k \geq 0$ in W'_ε (for a given open set $\Delta'' \Subset \Delta'$). Then by the L^1 -convergence of $\tilde{\phi}_\varepsilon^k$ to $\tilde{\phi}^k$, for each nonnegative $g \in C_0^2(\Delta')$,

$$\int_{\Delta'} \tilde{\phi}^k dd^c g \wedge v_{[a']}^{m-1} = \lim_{\varepsilon \rightarrow 0} \int_{\Delta'} g dd^c \tilde{\phi}_\varepsilon^k \wedge v_{[a']}^{m-1} \geq 0. \tag{3.18}$$

Let $u \in C_0^2(\Delta)$ with $u \geq 0$. Since the function \tilde{u}^k induced by $u|V^k$ has compact support in Δ' , there exists a sequence $f_n \in C_0^2(\Delta')$ such that $f_n \rightarrow \Delta_{\text{id}} \tilde{u}^k$ in $L^1(\Delta')$. By solving the Dirichlet problem

$$dd^c (w v_{[a']}^{m-1}) = \frac{1}{4m} f_n v_{[a']}^m \text{ in } \Delta', \quad w = 0 \text{ on } \partial D_n,$$

where D_n is an open neighborhood of $\text{Spt}(f_n)$ with smooth boundary, one obtains a sequence $\{w_n\}$ in $C_0^2(\Delta')$ such that $\Delta_{\text{id}} w_n$ tends to $\Delta_{\text{id}} \tilde{u}^k$ in $L^1(\Delta')$. Without loss of generality it may be assumed that each w_n is nonnegative. Then by the inequality (3.18),

$$\int_{V^k} \hat{\phi} dd^c u \wedge v_p^{m-1} = \int_{\Delta' \setminus \{a'\}} \tilde{\phi}^k dd^c \tilde{u}^k \wedge v_{[a']}^{m-1} \geq 0.$$

The proof of the assertion (1) is thereby completed. The second assertion follows from the preceding proof and the regularity of a subharmonic function. □

4 Plurisubharmonicity

Let $\tilde{\pi} : \mathbb{C}^m \setminus \{0\} \rightarrow \mathbb{P}^{m-1}(\mathbb{C})$ denote the natural projection. The induced map $\pi : \mathbb{S}^{2m-1} \rightarrow \mathbb{P}^{m-1}(\mathbb{C})$ defines a fiber bundle with fiber $\pi^{-1}([\mathbf{a}]) = \iota_{\mathbf{a}} := \{\zeta \mathbf{a} \mid \zeta \in \mathbb{C}, \|\zeta\| = 1\}$ over $[\mathbf{a}] \in \mathbb{P}^{m-1}(\mathbb{C})$, where

$\|\mathbf{a}\| = 1$. Let $j_{(r)} : \mathbb{S}^{2m-1} \rightarrow \mathbb{C}^m$ be the dilation: $j_{(r)}(\mathfrak{z}) = r\mathfrak{z}$. The Fubini-Study form $\tilde{\omega}$ on $\mathbb{P}^{m-1}(\mathbb{C})$ pulls back under $\tilde{\pi}$ to the projective form $\omega := dd^c \log \|z\|^2$ on $\mathbb{C}^m \setminus \{0\}$. Thus

$$\pi^*(\tilde{\omega}^{m-1}) = j_{(r)}^*(\tilde{\pi}^*(\tilde{\omega}^{m-1})) = j_{(r)}^*(\omega^{m-1}). \tag{4.1}$$

An upper semicontinuous function $\phi : Y \rightarrow [-\infty, \infty)$ is said to be (locally) *radially submedian* ($\phi \in SM_{\text{rad}}(Y)$) if there exists at every point $a \in Y$ a pseudoball U of radius r_0 such that for all $\mathbf{a} \in \mathbb{S}^{2m-1}$,

$$\nu_{p_U}(a)\phi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \sum_k \tilde{\phi}_k(a' + re^{i\theta}\mathbf{a}) d\theta, \quad 0 < r < r^0, \tag{4.2}$$

(where the sum ranges over all branches of U) for some $r^0 = r^0(a, \mathbf{a}) \leq r_0$. An upper semicontinuous function $\phi : Y \rightarrow [-\infty, \infty)$ is said to be (locally) *semiradially submedian* ($\phi \in SM_{\text{sm.rad}}(Y)$) if ϕ belongs, locally at every $a \in Y$, to $SM_{\text{rad}}(Q \setminus A_Q)$ for some thin analytic subset A_Q in an open neighborhood Q of a . A peak point of $\phi|_Q$ belonging to A_Q is called a *local exceptional peak point*.

Proposition 4.1. $PSH(Y) \subseteq SM_{\text{rad}}(Y) \subseteq SH(Y)$.

Proof. Let U be a pseudoball at $a \in Y$. The first inclusion relation follows from the fact that given $\phi \in PSH(Y)$, each induced function $\tilde{\phi}_k$ is subharmonic at a'_0 when restricted to the complex line $\mathfrak{L}_{\mathbf{a}} = \{z \in \mathbb{C}^m \mid z = a'_0 + \zeta\mathbf{a}, \zeta \in \mathbb{C}\}$, for each $\mathbf{a} \in \mathbb{S}^{2m-1}$. To prove the second inclusion relation, suppose that $\phi \in SM_{\text{rad}}(Y)$. The expression (4.1) allows one to evaluate the spherical means of each $\tilde{\phi}_k : U' \rightarrow \mathbb{R}$ by integrating along fibers of the bundle map π (as in [20, p. 202]). The detail is given below for completeness.

Without loss of generality assume that $a'_0 = 0$. Write $\mathfrak{z} = e^{i\theta}\mathbf{a}$, $\mathbf{a} \in \mathbb{S}^{2m-1}$, $0 \leq \theta \leq 2\pi$. Let $i_{\mathbf{a}} : \mathfrak{L}_{\mathbf{a}} \hookrightarrow \mathbb{C}^m$, $j_{\mathbf{a}} : \mathfrak{l}_{\mathbf{a}} \hookrightarrow \mathfrak{L}_{\mathbf{a}}$, $\mathfrak{J}_{\mathbf{a}} : \mathfrak{l}_{\mathbf{a}} \hookrightarrow \mathbb{S}^{2m-1}$ be the inclusion mappings. Then

$$\mathfrak{J}_{\mathbf{a}}^* j_{\mathbf{a}}^*(d^c \ln \|z\|^2) = j_{\mathbf{a}}^* i_{\mathbf{a}}^*(d^c \ln \|z\|^2) = j_{\mathbf{a}}^*(d^c \ln \|\zeta\|^2) = \frac{1}{2\pi} d\theta.$$

Therefore for small $r > 0$ one has

$$\begin{aligned} \nu_p(a_0)\phi(a_0) &= \int_{[\mathbf{a}] \in \mathbb{P}^{m-1}(\mathbb{C})} \nu_p(a_0)\phi(a_0)\tilde{\omega}^{m-1} \\ &\leq \int_{[\mathbf{a}] \in \mathbb{P}^{m-1}(\mathbb{C})} \left[\int_0^{2\pi} \sum_{j=1}^l \tilde{\phi}_j(a'_0 + re^{i\theta}\mathbf{a}) \frac{d\theta}{2\pi} \right] \tilde{\omega}^{m-1} \\ &= \sum_{j=1}^l \int_{\mathbb{S}^{2m-1}} \tilde{\phi}_j(a'_0 + r\mathfrak{z}) j_{r\mathfrak{z}}^*(\sigma_0) = \sum_{j=1}^l \int_{dV_{[\mathbf{a}_0]}^j(r)} \phi \sigma_{a_0} \\ &= [\phi]U_{a_0,r} \end{aligned}$$

Thus ϕ is spherically submedian in Y . Therefore, by Theorem 3.1, $\phi \in SH(Y)$. □

Theorem 4.1. (1) If $\phi \in SM_{\text{sm.rad}}(Y)$ and is continuous at every local exceptional peak point (if any), then ϕ is subharmonic. (2) If Y is a normal space, then $SM_{\text{sm.rad}}(Y) = PSH(Y)$.

Proof. By Proposition 4.1 and Theorem 3.1, a function $\phi \in SM_{\text{sm.rad}}(Y)$ that is continuous at every local exceptional peak point is semisolidly submedian, hence subharmonic, in Y . To prove the second assertion, note that in the definition of the set $SM_{\text{sm.rad}}(Y)$, the local analytic set A_Q at a point z can be chosen (without loss of generality) so that $Q \setminus A_Q \subseteq Q_{\text{reg}}$. The desired conclusion follows then from the plurisubharmonicity test in \mathbb{C}^m ([26, p. 73]) and the extension theorem of Grauert and Remmert for plurisubharmonic functions on a normal complex space ([8, Satz 3, p. 181]). \square

A function $\phi : Y \rightarrow [0, \infty)$ is called *logarithmically subharmonic* (respectively, *logarithmically plurisubharmonic*), if $\log \phi \in SH(Y)$ (respectively, $\log \phi \in PSH(Y)$); for C^2 -functions in \mathbb{C}^m , another definition of logarithmic plurisubharmonicity is given in [3, p. 200]). If $f = (f_1, \dots, f_N) : Y \rightarrow \mathbb{C}^N$ and $c = (c_1, \dots, c_N)$, where each c_j is an arbitrary positive constant, set

$$\lfloor f \rfloor^c := \|f_1\|^{c_1} + \dots + \|f_N\|^{c_N}.$$

If f is a holomorphic map, then the function $\lfloor f \rfloor^c$ is logarithmically plurisubharmonic (see Example 6.3 of the Appendix); furthermore, $\log \lfloor f \rfloor^c$ is a Jensen function, provided f has thin zero set. This is in fact a special case of the next proposition (which generalizes the classical *Jensen's inequality*):

Proposition 4.2. $PSH(Y)_{\{\neq -\infty\}} \subseteq J(Y)$.

Proof. Let $\phi \in PSH_{\{\neq -\infty\}}(Y)$. By Theorems 4.1 and 3.4, ϕ is subharmonic and locally integrable in Y . The submean-value inequality relative to pseudoballs follows from Theorem 3.1. Suppose now that $U = \Delta_a(R) \Subset Y$ is a pseudopolydisc of polyradius R . Each induced function $\tilde{\phi}_k$ is plurisubharmonic in $p(U) \setminus \{a'\}$, hence also in $p(U)$ (by [8, Satz 3, p. 181]), so that it is subharmonic in each variable z_j , $1 \leq j \leq m$. Repeated integration yields the inequality

$$\phi(a) \leq \text{vol}(\Delta(a', R))^{-1} \int_{\Delta(a', R)} \tilde{\phi}_k(z') \, dv(z').$$

From this and the identity (2.1) the the submean-value inequality (3.2) follows. \square

Definition 4.1. A locally integrable function $\psi : Y \rightarrow [-\infty, \infty)$ is said to be *weakly plurisubharmonic* in Y ($\psi \in \mathcal{PSH}_{\text{w}}(Y)$) if at each point $a \in Y$ there exists a standard domain Δ such that the Levi form of ψ , $\mathcal{L}(\psi, \lambda)$, is positive semidefinite in Δ , that is, for some thin analytic subset A of Δ (with $\Delta \setminus A \subseteq \Delta^*$) and for all nonnegative $u \in C_0^2(\Delta)$,

$$\langle \mathcal{L}(\psi, \lambda), u \rangle_{\Delta} := \int_{\Delta \setminus A} \psi \sum_{\mu, \nu} \frac{\partial^2 u}{\partial p_{\mu} \partial \bar{p}_{\nu}} \lambda_{\mu} \bar{\lambda}_{\nu} \, d\bar{v} \geq 0, \quad \forall \lambda \in \mathbb{C}^m. \tag{4.3}$$

Similarly, a locally integrable function $\psi : Y \rightarrow [-\infty, \infty)$ is said to be *weakly pluriharmonic* in Y ($\psi \in \mathcal{PH}_{\text{w}}(Y)$) if at each $a \in Y$ there exists a standard domain Δ in which the above Levi form $\mathcal{L}(\psi, \lambda) \equiv 0$.

Theorem 4.2. (Compare [13, Theorem 3, p. 21] and [9, Theorem K15]) (1) $PSH_{\{\neq -\infty\}}(Y) \subseteq \mathcal{PSH}_{\text{w}}(Y)$. (2) Assume that $\phi \in \mathcal{PSH}_{\text{w}}(Y)$. Then there exists a unique locally integrable, regularized function $\psi : Y \rightarrow [-\infty, \infty)$ such that $\psi = \phi$ almost everywhere and $\psi \in SM_{\text{rad}}(Y) \cap PSH_{\{\neq -\infty\}}(Y_{\text{reg}})$.

Proof. Let $\phi \in PSH_{\{\neq -\infty\}}(Y)$. By Theorems 4.1 and 3.4, ϕ is locally integrable in Y . Consider first the special case of a C^2 -functions $\phi : Y \rightarrow [-\infty, \infty)$. Suppose that Y is nonsingular. For any standard domain $\Delta \subset Y$ and $\alpha \in C^2(\Delta)$, denote by $\tilde{\alpha}$ the function on Δ' induced by α and set

$$L(\alpha, \lambda)(z) := \sum_{\mu, \nu} \frac{\partial^2 \tilde{\alpha}}{\partial z_\mu \partial \bar{z}_\nu}(z') \lambda_\mu \bar{\lambda}_\nu, \quad z' \in \Delta', \lambda \in \mathbb{C}^m.$$

Then $\phi \in PSH(U)$ if and only if $L(\phi, \lambda)(z) \geq 0$ for every $z \in \Delta$ and every $\lambda \in \mathbb{C}^m$; but that is clearly equivalent to the requirement that there exists a thin analytic subset A of Δ such that for all nonnegative $u \in C_0^2(\Delta)$,

$$\int_{\Delta \setminus A} u L(\phi, \lambda) v_p^m \geq 0.$$

By [9, Lemma K14], the latter condition is in turn equivalent to the requirement (4.3) (with $D = \Delta$). If Y is singular, then by using local desingularizations of Y , it is easily seen that a C^2 -function $\phi : Y \rightarrow \mathbb{R}$ belongs to $PSH(Y)$ precisely when it satisfies the condition (4.3) of weak plurisubharmonicity. Thus the C^2 case of the assertion (1) is established.

For the general case, observe that each induced function $\tilde{\phi}_k : \Delta' \rightarrow \mathbb{R}$ is plurisubharmonic in $U' := \Delta'$, hence so are the smooth approximations $\tilde{\phi}_{k,\varepsilon}$ in W_ε . Let $u \in C_0^2(\Delta)$ and denote by \tilde{u}_k the function induced by $u \rfloor V^k$ in Δ' . Since the support of \tilde{u}_k lies in W_ε whenever ε is sufficiently small, and for such values ε it follows from the first part of the proof that

$$\int_{\Delta'} \tilde{\phi}_{k,\varepsilon} L(\tilde{u}_k, \lambda) dv(z) \geq 0, \quad \lambda \in \mathbb{C}^m.$$

As $\varepsilon \rightarrow 0$, the function $\tilde{\phi}_{k,\varepsilon}$ converges to $\tilde{\phi}_k$ in L^1 -norm on the support of \tilde{u}_k . Consequently ϕ satisfies the condition (4.3). Thus the assertion (1) is established in the general case.

Now suppose that $\phi \in \mathcal{PSH}_w(Y)$. Let Δ be a pseudoball at $a \in Y$ relative to which the condition (4.3) holds. Define $\psi_\Delta : \Delta \rightarrow [-\infty, \infty)$ by

$$\nu_{p_\Delta}(z) \psi_\Delta(z) := \lim_{r \rightarrow 0} \sum \{ \langle \phi \rfloor V^k \rangle_{z,r} \mid V^k \ni z \}, \quad \forall z \in \Delta.$$

Then ψ_Δ is regularized in Δ and $\psi_\Delta = \phi$ almost everywhere; moreover, the Levi form $\mathcal{L}(\psi_\Delta, \lambda) \geq 0$ in Δ . Consider the smooth approximations $\tilde{\psi}_{k,\varepsilon}$ in W_ε of the induced function $\tilde{\psi}_k$ of $\psi_k := \psi_\Delta \rfloor V^k$. The second half of the proof of [9, Theorem K15] shows that $\tilde{\psi}_k$ is plurisubharmonic in $\Delta' \setminus \{a'\}$, hence also in Δ' (by Grauert and Remmert's extension theorem [8, Satz 3, p. 181]). It follows from Proposition 3.2 that the function ψ given by $\psi := \psi_\Delta$ on each Δ is well-defined. Also, ψ is locally integrable, radially submedian and regularized in Y . Since ψ_Δ is plurisubharmonic in each open set $\Delta \setminus \{a\}$, so is the function ψ in Y_{reg} . □

Remark 4.1. *The above theorem and its proof (of the first assertion) imply that, if $D \subseteq Y$ is a normal, open subset and $\phi \in PSH_{\{\neq -\infty\}}(D)$, then for any Riemann semicovering $p : D \rightarrow \Omega$, the inequality (4.3) remains valid: $\langle \mathcal{L}(\phi, \lambda), u \rangle_D \geq 0$ for all nonnegative $u \in C_0^2(D)$ and every $\lambda \in \mathbb{C}^m$.*

Corollary 4.1. *Let Y be a normal space. A function $\phi : Y \rightarrow [-\infty, \infty)$ is weakly plurisubharmonic and regularized in Y if and only if $\phi \in PSH_{\{\neq -\infty\}}(Y)$.*

Proof. Suppose that $\phi \in \mathcal{PSH}_w(Y)$ is regularized in Y . Then by Theorem 4.2, ϕ is plurisubharmonic in Y_{reg} . The Grauert and Remmert's extension theorem ([8, Satz 3, p. 181]) asserts that $\phi|_{Y_{\text{reg}}}$ admits a plurisubharmonic extension ψ to Y , which, by subharmonicity, must coincide with ϕ (Proposition 3.2). Conversely, every element of $PSH_{\{\neq -\infty\}}(Y)$ is weakly plurisubharmonic and regularized in Y by Theorem 4.2-(1) and Proposition 3.2. \square

Remark 4.2. (Compare [9, Corollary K18]) *Let Y be a normal space. A function $\phi : Y \rightarrow [-\infty, \infty)$ is pluriharmonic if and only if ϕ is weakly pluriharmonic and regularized in Y .*

5 Applications

As a consequence of Theorem 3.1, subharmonicity and plurisubharmonicity are closed under a basic sequential limiting process (Lemma 6.1). By allowing to decrease the values of a subharmonic (respectively, plurisubharmonic) function on a null set, the failure of the preservation of subharmonicity (respectively, plurisubharmonicity) under more general limit operations can be remedied. A mapping $\phi : Y \rightarrow [-\infty, \infty)$ is called *presubharmonic* ($\phi \in \mathcal{SH}[Y]$) (respectively, *preplurisubharmonic*, ($\phi \in \mathcal{PSH}[Y]$)) if ϕ admits a subharmonic (respectively, plurisubharmonic) majorant $\hat{\phi} : Y \rightarrow [-\infty, \infty)$ such that $\phi = \hat{\phi}$ almost everywhere.

Lemma 5.1. (Compare [9, Theorem L8]) *A mapping $\phi : Y \rightarrow [-\infty, \infty)$ is presubharmonic (respectively, preplurisubharmonic) if and only if the upper envelop*

$$\phi^{[*]}(z) := \max(\phi(z), \phi^{(*)}(z)), \quad \forall z \in Y, \tag{5.1}$$

is subharmonic (respectively, plurisubharmonic), with $\phi^{[]} = \phi$ almost everywhere, in Y ; hence $\phi^{[*]}$ is the least such majorant of ϕ .*

Proof. The sufficiency part of the lemma is trivial. Suppose that ϕ admits a subharmonic majorant $\hat{\phi}$ which agrees with ϕ almost everywhere. Then the same argument as in the proof of [9, Theorem L8] shows that $\phi(z) \leq \phi^{[*]}(z) \leq \hat{\phi}^{[*]}(z) = \hat{\phi}(z)$ for all $z \in Y$ and hence $\phi(z) = \phi^{[*]}(z)$ almost everywhere. On the other hand, if $\phi^{[*]}(a) < \hat{\phi}(a)$ at some $a \in Y$, then by the definition (5.1), there are constants $\varepsilon > 0, \delta > 0$, such that $\sup_{z \in \Delta_a(\varepsilon)} \phi(z) = \hat{\phi}(a) - \varepsilon$. But since $\hat{\phi}$ is subharmonic and $\hat{\phi} = \phi$ almost everywhere in Y , it follows from the regularity of $\hat{\phi}$ (Proposition 3.2) that, for some standard domain Δ at a ,

$$\nu_{p_\Delta}(a) \hat{\phi}(a) = \lim_{r \rightarrow 0} \langle \phi | \Delta \rangle_{a,r} \leq \nu_{p_\Delta}(a) (\hat{\phi}(a) - \varepsilon),$$

a contradiction, thus proving the desired claim. The remaining case of preplurisubharmonicity is entirely similar. \square

Thanks to Corollary 4.1 and Lemma 5.1, the preservation of preplurisubharmonicity on a complex space can be assured in reference to: (1) the pointwise limit of a monotonically decreasing sequence in

$\mathcal{PSH}[Y]$; (2) the supremum and \limsup of a sequence in $\mathcal{PSH}[Y]$ that is locally uniformly bounded from above (as in [9, Theorems L9]); and, for a normal space Y , (3) the supremum of a family $\{\phi_t\}_{t \in T} \subset \mathcal{PSH}[Y]$ that is locally uniformly bounded from above (as in [9, Theorems L10]), and (4) the operation of forming $\psi_{t_0} := \limsup_{t \rightarrow t_0} \phi_t$, for $\{\phi_t\}_{t \in T} \subset \mathcal{PSH}[Y]$ that is locally uniformly bounded from above (where $t_0 \in T$ and T is an open (or closed) subset of $\mathbb{R} \cup \{\infty\}$ or \mathbb{C}^m). Similar assertions hold on a general space for presubharmonic functions. An application of such preservation properties is given by the following generalization of the Hartogs' lemma:

Corollary 5.1. (*Hartogs' lemma for presubharmonic functions*) *Assume that $\{\phi_t\}_{t>0}$ is a family in $\mathcal{SH}[Y]$ that is locally uniformly bounded from above. Then (1) the function $G := \limsup_{t \rightarrow \infty} \phi_t$ is presubharmonic in Y ; (2) if for some $g \in C^0(Y)$, $G \leq g$ in Y , then for every compact set $K \subset Y$ and every positive ε , there is a positive number N such that $\sup_K \phi_t < g + \varepsilon$ for all $t \geq N$.*

Proof. Let $F_t := \sup_{s \geq t} \phi_s$. Then $G = \lim_{t \rightarrow \infty} F_t$. By the preceding remark, both F_t and G are presubharmonic in Y , (alternatively, this can be proved using Theorem 3.1 and Lemma 6.2). Since $\{F_t\}$ is a decreasing sequence, so is also the sequence $\{F_t^{[*]}\}$, hence the limit function $\lim_{t \rightarrow \infty} F_t^{[*]}$ is subharmonic. It follows from Lemma 5.1 that $G^{[*]}$ is equal to $\lim_{t \rightarrow \infty} F_t^{[*]}$ outside a set of measure zero, and hence by Proposition 3.2, everywhere in Y . The second assertion follows by showing that the nested family $E_t := \{z \in K \mid F_t^{[*]}(z) - g(z) \geq \varepsilon\}$, for $t > 0$, has an empty intersection (by the same argument as in [12, p. 93]). □

In a similar vein it is of interest to see if the theorem of Lelong-Norguet-Bremermann [13, p. 54] on the representation of plurisubharmonic functions remains valid for a P -convex domain D in a general complex space, where $P = \mathcal{PSH}(D)$, that is, whether each $\psi \in \mathcal{PSH}(D)$ admits an representation

$$\psi = \left(\limsup_{k \rightarrow \infty} \frac{1}{k} \log \|f_k\| \right)^{[*]},$$

for some sequence of holomorphic functions f_k in D .

Let $p : Y \rightarrow \Omega$ be a Riemann semicovering and $G \Subset Y$ a weak Stokes domain ([24, p. 568]). The (real) *energy* of a function $\tau \in C^1(\overline{G})$ is defined by

$$\mathbb{E}_G(\tau) := \int_G d\tau \wedge d^c \bar{\tau} \wedge v_p^{m-1}$$

([25, § 3]). If $\tau \in C^{1,1}(\overline{G})$ ([24, p. 562]) is real-valued, then its energy is given by

$$\mathbb{E}_G(\tau) = \int_G \tau d^c \tau \wedge v_p^{m-1} - \int_G \tau dd^c \tau \wedge v_p^{m-1}. \tag{5.2}$$

Suppose now that $p : Y \rightarrow \Omega$ is an unramified Riemann covering. By the expression [24, (5.7)] for the normal derivatives one has

$$\int_{dG} \tau, d^c \tau \wedge v_p^{m-1} = (-1)^{\frac{m(m-1)}{2}} \frac{1}{2\|\mathbb{S}\|} \int_{dG} \tau \partial_\nu \tau d\sigma_{dG}. \tag{5.3}$$

Also, since the energy $\mathbb{E}_G(\tau)$ is never negative [24, (3.7)], the expression (5.2) (applied to the pseudoannular region $G = U_a(r) \setminus \overline{U}_a(s)$, where $0 < s < r < r_0$) and the identity (5.6) of [24] imply that the spherical mean $[\tau \mathfrak{R}_{p,a}(\tau)]_{a,r}$ of a subharmonic C^2 -function $\tau : Y \rightarrow [0, \infty)$ is nonnegative and increasing in r (for small $r > 0$).

If $\phi = (\phi_1, \dots, \phi_N) : Y \rightarrow \mathbb{C}^N$, set $\tau_\rho := \|\phi\|^{2\rho}$ for each positive constant ρ . A mapping $\phi : Y \rightarrow \mathbb{C}^N$ is called *subharmonic*, respectively, *semiharmonic*, if so are the real and imaginary parts of each component of ϕ (regarded as an \mathbb{R} -valued function); ϕ is called *weakly real-analytic* if each component ϕ_j is continuous in Y and real-analytic in Y_{reg} . Kellogg ([11, Theorem IV, p. 213]) showed that a real-valued harmonic function on a closed, regular plane region is nonconstant unless it has vanishing normal derivative at every boundary point. The relations (5.2) and (5.3) lead to a generalization of this result to subharmonic mappings:

Proposition 5.1. (1) *Assume that $G \Subset Y$ is a Stokes domain and $\phi : \overline{G} \rightarrow \mathbb{C}^N$, with components $\phi_j \in C^{1,1}(\overline{G}) \cap C^2(G)$, is subharmonic in G . If for some constant $\rho \geq \frac{1}{2}$,*

$$\int_{dG} \tau_\rho d^c \tau_\rho \wedge v_p^{m-1} = 0, \tag{5.4}$$

then $\phi = \text{constant}$ in \overline{G} . (2) Assume that Y is a connected complex space and $\phi : Y \rightarrow \mathbb{C}^N$ a weakly real-analytic subharmonic map. Then ϕ is a constant if and only if there exist a point $a \in Y_{\text{reg}}$ and $\rho \geq \frac{1}{2}$ such that the function $[\tau_\rho \mathfrak{R}_{p,a}(\tau_\rho)]_{a,r}$ is monotonically decreasing in r (for small $r > 0$) relative to an unramified Riemann covering p at a .

Proof. By Example 6.1 of the Appendix and Lemma 3.2, the function $\tau_\rho = \|\phi\|^{2\rho}$ is subharmonic in G for all $\rho \geq \frac{1}{2}$. (1) Observe that

$$dd^c \|\phi_j\|^2 = \bar{\phi}_j dd^c \phi_j + \phi_j dd^c \bar{\phi}_j + d\phi_j \wedge d^c \bar{\phi}_j - d^c \phi_j \wedge d\bar{\phi}_j \tag{5.5}$$

almost everywhere in G . Writing $\phi_j = u_j + i v_j$ (in terms of its real and imaginary parts), one has $\|\phi\|^2 = \sum_{j=1}^N (\|u_j\|^2 + \|v_j\|^2)$. Thus the mapping

$$\tilde{\phi} := (|u_1|, |v_1|, \dots, |u_N|, |v_N|) : \overline{G} \rightarrow \mathbb{R}^{2N}$$

is continuous in \overline{G} and subharmonic in G with $\|\tilde{\phi}\|^2 = \|\phi\|^2$. Therefore it can be assumed without loss of generality that each ϕ_j is real-valued. Then the expression (5.5) implies that

$$dd^c \tau_1 \wedge v_p^{m-1} \geq 2 \sum_{j=1}^N (du_j \wedge d^c u_j + dv_j \wedge d^c v_j) \wedge v_p^{m-1} \geq 0 \tag{5.6}$$

almost everywhere in G . To prove the assertion (1), observe that by the condition (5.4) and Proposition 3.4, the energy (5.2) (for τ_ρ) is nonpositive. Since the integral $\mathbb{E}_G(\tau_\rho)$ is always nonnegative, the norm of the gradient, $\|\nabla \tau_\rho\|$, must vanish locally almost everywhere in G . Consequently τ_ρ , hence also τ_1 , is constant in G . Hence the relation (5.6) implies that $\phi = \text{constant}$ in G . For the assertion (2), assume that relative to an unramified Riemann covering p at a point $a \in Y_{\text{reg}}$, the function $[\tau_\rho \mathfrak{R}_{p,a}(\tau_\rho)]_{a,r}$ is decreasing in r for some $\rho \geq \frac{1}{2}$. Then by the expression (5.2), the energy $\mathbb{E}_D(\tau_\rho)$

for the pseudoannular region $G := U_a(r) \setminus \overline{U}_a(s)$, $0 < s < r < r_0$, is nonpositive. It follows that $\|\nabla \tau_\rho\|^2 = 0$ in G . Therefore τ_ρ is constant in every component of $G \setminus Z$, where Z is the zero set of $f = \|\phi\|^2$. Since each component ϕ_j of ϕ is real analytic in Y off a thin analytic subset, the same is true for the function f ; hence (choosing a to be a point where f is real analytic and r_0 sufficiently small) the closure of every component of $G \setminus Z$ must contain a ([16, Lemma 1, p. 96]). This implies that τ_ρ is constant in G ; in particular, so is τ_1 . It follows as above from the relation (5.6) that $\phi = \text{constant}$ in G . Hence by the identity theorem for real-analytic functions, $\phi = \text{constant}$ in Y . \square

Corollary 5.2. *Let Y be a connected complex space and $\phi = (\phi_1, \dots, \phi_N) : Y \rightarrow \mathbb{C}^N$ a weakly real-analytic and subharmonic map. If $\|\phi\|$ attains a local maximum at some point of Y , then $\phi = \text{constant}$.*

Proof. Observe that the function $\tau_1 = \|\phi\|^2 : Y \rightarrow \mathbb{R}$ is solidly submedian in Y . Assume that $\|\phi\|$ attains a local maximum at a point $z_0 \in Y$. Hence Proposition 3.1 (applied to a Riemann covering p_U of a small open, connected neighborhood U of z_0) implies that τ_1 is constant in U . Thus τ_1 has vanishing mean spherical radial derivative at some point of U , and by Proposition 5.1-(2), $\phi = \text{constant}$ in U . Since each ϕ_j is real analytic in Y_{reg} , $\phi_j = \text{constant}$ in Y_{reg} by the identity theorem for real-analytic functions, whence the conclusion follows. \square

By means of the solid submean-value property of subharmonic functions on a general space, two results of Bochner and Montgomery on matrix-valued mappings [2, Theorems 10 and 11, p. 155] can be extended:

Proposition 5.2. *Assume that Y is a connected complex space. (1) If $\phi = (\phi_1, \dots, \phi_{kN}) : Y \rightarrow \mathbb{R}^{kN}$ is continuous, subharmonic such that the image of each map $g_j := (\phi_{(j-1)N+1}, \dots, \phi_{jN}) : Y \rightarrow \mathbb{R}^N$, $1 \leq j \leq k$, is contained in the unit sphere in \mathbb{R}^N , then $\phi = \text{constant}$. (2) Every continuous mapping $\phi : Y \rightarrow U(n)$ (a unitary group) with semiharmonic components is a constant.*

6 Appendix

Lemma 6.1. *If $\{\phi_n\}$ is a monotonically decreasing sequence of subharmonic (respectively, plurisubharmonic) functions in a complex space Y , then the limit function $\lim \phi_n$ is subharmonic (respectively, plurisubharmonic) in Y .*

Lemma 6.2. *Assume that (i) Y is locally irreducible; (ii) $\psi : Y \rightarrow [-\infty, \infty)$ is locally integrable and locally bounded from above; and (iii) ψ is solidly submedian with respect to every pseudoball in Y . Then ψ is presubharmonic in Y .*

Proof. Since ψ is locally bounded from above, it can be proved in the same way as in [9, Lemma L6-(c)] that $\psi^{[*]}$ is upper semicontinuous in Y . Let Δ be a pseudoball at at point $z \in Y$. Then either $\psi^{[*]}(z) = \psi(z)$, hence the solid submean-value inequality holds at z , or there exists a sequence $\{\zeta_n\}$ in $\Delta \setminus \{z\}$ tending to z such that $\psi^{[*]}(z) = \lim_{n \rightarrow \infty} \psi(\zeta_n)$; in the latter case one has

$$\psi^{[*]}(z) \leq \liminf_{n \rightarrow \infty} \langle \psi \mid \Delta \rangle_{\zeta_n, r} \leq \liminf_{n \rightarrow \infty} \langle \psi^{[*]} \mid \Delta \rangle_{\zeta_n, r} = \langle \psi^{[*]} \mid \Delta \rangle_{z, r},$$

for sufficiently small $r > 0$. Thus $\psi^{[*]}$ is subharmonic in Y by Theorem 3.1. Also,

$$\psi^{[*]}(z) \leq \liminf_{n \rightarrow \infty} \langle \psi \rfloor \Delta \rangle_{\zeta_n, r} = \langle \psi \rfloor \Delta \rangle_{z, r}.$$

It follows then from the regularity of $\psi^{[*]}$ that

$$\lim_{r \rightarrow 0} \langle \psi^{[*]} \rfloor \Delta \rangle_{z, r} = \lim_{r \rightarrow 0} \langle \psi \rfloor \Delta \rangle_{z, r}.$$

Consequently $\psi^{[*]} = \psi$ almost everywhere in Y . □

Lemma 6.3. (Compare [17, 3.13]) *Let $g : I \rightarrow \mathbb{R}$ be an increasing function on a finite (or infinite) interval $I \subseteq [-\infty, -\infty)$ such that $g \rfloor I \cap \mathbb{R}$ is convex. Assume that $\phi \in SH(Y)$ (respectively, $\phi \in PSH(Y)$) with $\phi(Y) \subset I$. Then $g(\phi) \in SH(Y)$ (respectively, $g(\phi) \in PSH(Y)$).*

Proof. Suppose that $\phi \in SH(Y)$. Let Δ be a pseudoball at a point $a \in Y$ of radius r_0 . By Theorem 3.1, one has $\phi(a) \leq \langle \phi \rfloor \Delta \rangle_{a, r}$ for every $r \in (0, r_0)$. Note that, since $g(\phi) = g^+(\phi) - g^-(\phi)$, the function $g(\phi)$ is locally integrable on Y . Using the supporting line argument of [19, p. 115], it can be shown that the Jensen's inequality for convex functions holds for the pair (g, ϕ) :

$$g(\langle \phi \rfloor \Delta \rangle_{a, r}) \leq \langle g \circ \phi \rfloor \Delta \rangle_{a, r}, \quad \forall r \in (0, r_0).$$

Therefore, by Theorem 3.1, the composition $g \circ \phi$ is subharmonic in Y . The corresponding assertion for $\phi \in PSH(Y)$ follows then from the definition of plurisubharmonicity. □

Example 6.1. *Assume that $\phi = (\phi_1, \dots, \phi_N) : Y \rightarrow \mathbb{C}^N$ is a subharmonic map with components $\phi_j \in C^2(Y \setminus A)$ for some thin analytic subset A of Y . Then for all constants c, ρ with $c > 1, c\rho \geq 1$, the function $(\lfloor \phi \rfloor^{\hat{c}})^\rho \in SH(Y)$ (where $\hat{c} = (c, \dots, c)$).*

Proof. It is a consequence of [17, 3.23, p. 19], Proposition 3.4 and Lemma 3.2 that the function $g := (\lfloor \phi \rfloor^{\hat{c}})^{1/c}$ is subharmonic in Y , hence so is $g^{c\rho} = (\lfloor \phi \rfloor^{\hat{c}})^\rho$ by Lemma 6.3, provided $c\rho \geq 1$. □

Lemma 6.4. (Compare [17, 2.13]) *Assume that $\phi : Y \rightarrow [0, \infty)$ is upper semicontinuous and $\phi \not\equiv 0$. Then ϕ is logarithmically subharmonic in Y if and only if $\phi e^h \in SH(D)$ for every subdomain $D \Subset Y$ and every continuous function $h : \overline{D} \rightarrow \mathbb{R}$ which is semiharmonic in D . In particular, if ϕ is logarithmically subharmonic in Y , then $\phi \in SH(Y)$. The converse is false.*

Proof. By virtue of Proposition 3.1 and Lemmas 3.2 and 6.3, the desired conclusion follows from the same argument as in [17, 2.13]. □

Example 6.2. *If $\phi_k : Y \rightarrow [0, \infty)$ is upper semicontinuous and logarithmically subharmonic for $1 \leq k \leq l$ then so are $\sum_{k=1}^l \phi_k$ and $\max_{1 \leq k \leq l} \phi_k$.*

Example 6.3. (Compare [4, p. 119]) *Let $F_j = (f_{j1}, \dots, f_{jN}) : Y \rightarrow \mathbb{C}^N$ be a holomorphic map for each $1 \leq j \leq s$. Then for all constants $\rho > 0$ and $c_{(j)} = (c_{j1}, \dots, c_{jN})$ with $c_{jk} > 0$, the functions $(\lfloor F_j \rfloor^{c_{(j)}})^\rho$ and $\max_{1 \leq j \leq s} \lfloor F_j \rfloor^{c_{(j)}}$ are logarithmically plurisubharmonic in Y .*

Remark 6.1. If $\phi : Y \rightarrow [0, \infty)$ is logarithmically plurisubharmonic, it can be shown as in Lemma 6.4 that, for every subdomain $D \Subset Y$ and every continuous function $h : \overline{D} \rightarrow \mathbb{R}$ which is semi-harmonic in D , one has $\phi e^h \in PSH(D)$ (in particular, $\phi \in PSH(Y)$). In the converse direction, using Theorem 4.1 the following Lemma can be proved (as in [12, Theorem 2.6.1]):

Lemma 6.5. (Compare also [3, Proposition VI.4.9] and, respectively, [17, 3.12]) Let (X, p) be a normal semi-Riemann domain. Assume that $\phi : X \rightarrow [0, \infty)$ is upper semicontinuous and locally $\phi \not\equiv 0$. Then ϕ is logarithmically plurisubharmonic (respectively, logarithmically subharmonic) in X if and only if $\phi |e^{\sum c_k p_k}| \in PSH(X)$ (respectively, $\phi |e^{\sum c_k p_k}| \in SH(X)$) for all constants $(c_1, \dots, c_m) \in \mathbb{C}^m$.

Received: January 2009. Revised: May 2009.

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