The Semigroup and the Inverse of the Laplacian
on the Heisenberg Group

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ABSTRACT

By decomposing the Laplacian on the Heisenberg group into a family of parametrized partial differential operators $L_\tau, \tau \in \mathbb{R} \setminus \{0\}$, and using parametrized Fourier-Wigner transforms, we give formulas and estimates for the strongly continuous one-parameter semigroup generated by $L_\tau$, and the inverse of $L_\tau$. Using these formulas and estimates, we obtain Sobolev estimates for the one-parameter semigroup and the inverse of the Laplacian.

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RESUMEN

Mediante descomposición del Laplaceano sobre el grupo de Heisenberg en una familia de operadores diferenciales parciales parametrizados $L_\tau, \tau \in \mathbb{R} \setminus \{0\}$, y usando transformada de Fourier-Wigner parametrizada, damos fórmulas y estimativas para la continuidad fuerte del semigrupo generado por $L_\tau$, y la inversa de $L_\tau$. Usando esas fórmulas y estimativas obtenemos estimativas de Sobolev para el semigrupo a un parámetro y la inversa del Laplaceano.

**Key words and phrases:** Heisenberg group, Laplacian, parametrized partial differential operators, Hermite functions, Fourier-Wigner transforms, heat equation, one parameter semigroup, inverse of Laplacian, Sobolev spaces.

**Math. Subj. Class.:** 47F05, 47G30, 35J70.

1 The Laplacian on the Heisenberg Group

If we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ via

$$\mathbb{R}^2 \ni (x, y) \mapsto z = x + iy \in \mathbb{C}$$

and let

$$\mathbb{H} = \mathbb{C} \times \mathbb{R},$$

then $\mathbb{H}$ becomes a non-commutative group when equipped with the multiplication $\cdot$ given by

$$(z,t) \cdot (w,s) = \left[ z + w, t + s + \frac{1}{4}[z, w] \right], \quad (z,t), (w,s) \in \mathbb{H},$$

where $[z, w]$ is the symplectic form of $z$ and $w$ defined by

$$[z, w] = 2 \text{Im}(z\overline{w}).$$

In fact, $\mathbb{H}$ is a unimodular Lie group on which the Haar measure is just the ordinary Lebesgue measure $dz \, dt$.

Let $\mathfrak{h}$ be the Lie algebra of left-invariant vector fields on $\mathbb{H}$. A basis for $\mathfrak{h}$ is then given by $X$, $Y$ and $T$, where

$$X = \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial}{\partial t},$$

$$Y = \frac{\partial}{\partial y} - \frac{1}{2} \frac{\partial}{\partial t}.$$
and \[ T = \frac{\partial}{\partial t}. \]

The Laplacian \( \Delta_H \) on \( \mathbb{H} \) is defined by

\[ \Delta_H = -(X^2 + Y^2 + T^2). \]

A simple computation gives

\[ \Delta_H = -\Delta - \frac{1}{4}(x^2 + y^2) \frac{\partial^2}{\partial t^2} + \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2}, \]

where

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \]

Let \( g \) be the Riemannian metric on \( \mathbb{R}^3 \) given by

\[
g(x,y,t) = \begin{pmatrix} 1 & 0 & y/2 \\ 0 & 1 & -x/2 \\ y/2 & -x/2 & \frac{1}{4}(x^2 + y^2) \end{pmatrix} \]

for all \((x,y,t) \in \mathbb{R}^3\). Then \( \Delta_H \) is also given by

\[ -\Delta_H = \frac{1}{\sqrt{\det g}} \sum_{1 \leq j, k \leq 3} \partial_j (\sqrt{\det g} g_{j,k} \partial_k), \]

where \( \partial_1 = \partial/\partial x, \partial_2 = \partial/\partial y, \partial_3 = \partial/\partial t \). Since the symbol \( \sigma(\Delta_H) \) of \( \Delta_H \) is given by

\[
\sigma(\Delta_H)(x,y,t;\xi,\eta,\tau) = \left( \xi + \frac{1}{2} y \tau \right)^2 + \left( \eta - \frac{1}{2} x \tau \right)^2 + \tau^2
\]

for all \((x,y,t) \in \mathbb{R}^3\), it is easy to see that \( \Delta_H \) is an elliptic partial differential operator on \( \mathbb{R}^3 \) but not globally elliptic in the sense of Shubin [11]. Let us recall that \( \Delta_H \) is globally elliptic if there exist positive constants \( C \) and \( R \) such that

\[
|\sigma(\Delta_H)(x,y,t;\xi,\eta,\tau)| \geq C(1 + |x| + |y| + |t| + |\xi| + |\eta| + |\tau|)^2
\]

whenever

\[
|x| + |y| + |t| + |\xi| + |\eta| + |\tau| \geq R.
\]

The aim of this paper is to give new estimates for the strongly continuous one-parameter semigroup \( e^{-u\Delta_H}, u > 0 \), generated by \( \Delta_H \) and the inverse \( \Delta_H^{-1} \) of \( \Delta_H \). More precisely, we use the Sobolev spaces \( L^2_s(\mathbb{H}) \), \( s \in \mathbb{R} \), as in [1, 2] to estimate \( \|e^{-u\Delta_H}f\|_{L^2(\mathbb{H})}, u > 0 \), in terms of \( \|f\|_{L^2(\mathbb{H})} \) for all \( f \) in \( L^2(\mathbb{H}) \), and to give an estimate for \( \|e^{-u\Delta_H}f\|_{L^2(\mathbb{H})} \) in terms of \( \|f\|_{L^2(\mathbb{H})} \). These Sobolev spaces are also used to estimate \( \|\Delta_H^{-1}f\|_{L^2_s(\mathbb{H})} \) in terms of \( \|f\|_{L^2(\mathbb{H})} \) for all \( f \) in \( L^2_s(\mathbb{H}) \).
The function $F$ on $\mathbb{H} \times (0, \infty)$ given by

$$F(z, t, u) = (e^{-u\Delta_{\mathbb{H}}}f)(z, t), \quad (z, t) \in \mathbb{H}, \, u > 0,$$

is in fact the solution of the initial value problem

$$\begin{cases}
\frac{\partial F}{\partial u}(z, t, u) = - (\Delta_{\mathbb{H}})F(z, t, u), & (z, t) \in \mathbb{H}, \, u > 0, \\
F(z, t, 0) = f(z, t), & (z, t) \in \mathbb{H},
\end{cases}$$

for the Laplacian $\Delta_{\mathbb{H}}$.

Using the same techniques as in [1], we get for all $f \in L^2(\mathbb{H})$ and $u > 0$,

$$(e^{-u\Delta_{\mathbb{H}}}f)(z, t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-it\tau}(e^{-u\tilde{L}_{\tau}}f^\tau)(z) \, d\tau, \quad (z, t) \in \mathbb{H},$$

(1.1)

where $\tilde{L}_{\tau}, \tau \in \mathbb{R} \setminus \{0\}$, is given by

$$\tilde{L}_{\tau} = -\Delta + \frac{1}{4}(x^2 + y^2)\tau^2 - i \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \tau + \tau^2$$

and $f^\tau$ is the function on $\mathbb{C}$ given by

$$f^\tau(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{it\tau}f(z, t) \, dt, \quad z \in \mathbb{C},$$

provided that the integral exists. In fact, $f^\tau(z)$ is the inverse Fourier transform of $f(z, t)$ with respect to $t$ evaluated at $\tau$. In this paper, the nonzero parameter $\tau$ can be looked at as Planck’s constant.

To obtain the estimates in this paper, we use formulas for $e^{-u\tilde{L}_{\tau}}$ and $\tilde{L}_{\tau}^{-1}$ in terms of the $\tau$-Weyl transforms and the $\tau$-Fourier–Wigner transforms of Hermite functions, $\tau \in \mathbb{R} \setminus \{0\}$, which we recall in, respectively, Section 2 and Section 3. The $L^2$-boundedness and the Hilbert–Schmidt property of $\tau$-Weyl transforms are instrumental in obtaining the estimates.

Basic information on the classical Fourier–Wigner transforms, Wigner transforms and Weyl transforms can be found in [13] among others.

In Section 2, we introduce the $\tau$-Weyl transforms and prove results on the $L^2$-boundedness and the Hilbert–Schmidt property of the $\tau$-Weyl transforms. The $\tau$-Fourier–Wigner transforms of Hermite functions are recalled in Section 3. A formula for $e^{-u\tilde{L}_{\tau}}f$, $u > 0$, for every function $f$ in $L^2(\mathbb{C})$ and an estimate for $\|e^{-u\tilde{L}_{\tau}}f\|_{L^2(\mathbb{C})}$, $u > 0$, in terms of $\|f\|_{L^p(\mathbb{C})}$, $1 \leq p \leq 2$, are given in Section 4. This formula gives a formula for $e^{-u\Delta_{\mathbb{H}}}$, $u > 0$, immediately using the inverse Fourier transform as indicated by (1.1). In Section 5, we use the family $L^s_{L^2}(\mathbb{H})$, $s \in \mathbb{R}$, of Sobolev spaces with respect to the center of the Heisenberg group as in [1, 2] to obtain Sobolev estimates for $e^{-u\Delta_{\mathbb{H}}}f$, $u > 0$, in terms of $\|f\|_{L^2(\mathbb{H})}$, and Sobolev estimates for
∥e^{-u\Delta_H}f\|_{L^2(\mathbb{R})}, \quad u > 0, \quad \text{in terms of the Sobolev norms } \|f\|_{L^2_H(H)} \text{ of } f \text{ in } L^2_H(H). \quad \text{In Section } 6, \quad \text{we obtain a formula for } \hat{L}_\tau^{-1} \text{ and estimates for } \hat{L}_\tau^{-1} \text{ which are then used to estimate } \Delta_H^{-1}. \quad \text{In Section } 7, \quad \text{estimates for } \|\Delta_H^{-1}f\|_{L^2_{s,2}(H)} \text{ in terms of } \|f\|_{L^2_{s,2}(H)} \text{ for all } f \in L^2_{s,2}(H) \text{ are given.}

We end this section by putting in perspectives the results in this paper. While the semigroup and the inverse can be studied in the framework of functional analysis as explained in [3, 4, 5, 8, 9, 16], the results and methods in this paper are based on explicit formulas in hard analysis and are related to the works in [1, 2, 6, 7, 10, 12, 14, 15].

\section{\(\tau\)-Weyl Transforms}

Let \(f\) and \(g\) be functions in \(L^2(\mathbb{R})\). Then for \(\tau \in \mathbb{R} \setminus \{0\}\), the \(\tau\)-Fourier–Wigner transform \(V_\tau(f, g)\) is defined by

\[
V_\tau(f, g)(q, p) = (2\pi)^{-1/2} |\tau|^{1/2} \int_{-\infty}^{\infty} e^{i\tau q y} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} \, dy
\]

for all \(q, p \in \mathbb{R}\). In fact,

\[
V_\tau(f, g)(q, p) = |\tau|^{1/2} V(f, g)(\tau q, p), \quad q, p \in \mathbb{R},
\]

where \(V(f, g)\) is the classical Fourier–Wigner transform of \(f\) and \(g\). A proof can be found in [1].

It can be proved that \(V_\tau(f, g)\) is a function in \(L^2(\mathbb{C})\) and we have the Moyal identity stating that

\[
\|V_\tau(f, g)\|_{L^2(\mathbb{C})} = \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}, \quad \tau \in \mathbb{R} \setminus \{0\}.
\]  \hfill (2.1)

We define the \(\tau\)-Wigner transform \(W_\tau(f, g)\) of \(f\) and \(g\) by

\[
W_\tau(f, g) = V_\tau(f, g)^\wedge.
\]  \hfill (2.2)

Then we have the following connection of the \(\tau\)-Wigner transform with the usual Wigner transform.

**Theorem 2.1.** Let \(\tau \in \mathbb{R} \setminus \{0\}\). Then for all functions \(f\) and \(g\) in \(L^2(\mathbb{R})\),

\[
W_\tau(f, g)(x, \xi) = |\tau|^{-1/2} W(f, g)(x/\tau, \xi), \quad x, \xi \in \mathbb{R},
\]

where \(W(f, g)\) is the classical Wigner transform of \(f\) and \(g\).

It is obvious that

\[
W_\tau(f, g) = \overline{W(\tau^{-1}g, f)}, \quad f, g \in L^2(\mathbb{R}).
\]  \hfill (2.3)
Let $\sigma \in L^p(\mathbb{C}), 1 \leq p \leq \infty$. Then for all $r \in \mathbb{R}\setminus\{0\}$ and all functions $f$ in the Schwartz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$, we define $W^r_\sigma f$ to be the tempered distribution on $\mathbb{R}$ by

$$(W^r_\sigma f, g) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x, \xi) W_r(f, g)(x, \xi) \, dx \, d\xi$$

for all $g$ in $\mathcal{S}(\mathbb{R})$, where $(F, G)$ is defined by

$$(F, G) = \int_{\mathbb{R}^n} F(z) \overline{G(z)} \, dz$$

for all measurable functions $F$ and $G$ on $\mathbb{R}^n$, provided that the integral exists. We call $W^r_\sigma f$ the $r$-Weyl transform associated to the symbol $\sigma$. It is easy to see that if $\sigma$ is a symbol in the Schwartz space $\mathcal{S}(\mathbb{C})$ on $\mathbb{C}$, then $W^r_\sigma f$ is a function in $\mathcal{S}(\mathbb{R})$ for all $f$ in $\mathcal{S}(\mathbb{R})$.

We have the following estimate for the norm of the Weyl transform $W^r_\sigma$ in terms of the $L^p$ norm of the symbol $\sigma$ when $\sigma \in L^p(\mathbb{C}), 1 \leq p \leq 2$.

**Theorem 2.2.** Let $\sigma \in L^p(\mathbb{C}), 1 \leq p \leq 2$. Then $W^r_\sigma : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is a bounded linear operator and

$$\|W^r_\sigma\|_* \leq (2\pi)^{-1/p} |\tau|^{-1/2} \|\sigma\|_{L^p(\mathbb{C})},$$

where $\|W^r_\sigma\|_*$ is the operator norm of $W^r_\sigma : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$.

**Proof** Let $f$ and $g$ be functions in $\mathcal{S}(\mathbb{R})$. Then

$$(W^r_\sigma f, g) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x, \xi) W_r(f, g)(x, \xi) \, dx \, d\xi$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x, \xi) W(f, g)(x/\tau, \xi) \, dx \, d\xi$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(tx, \xi) W(f, g)(x, \xi) \, dx \, d\xi.$$

But

$$\sigma(tx, \xi) = |\tau|^{-1} \sigma_1/\tau(x, \xi), \quad x, \xi \in \mathbb{R},$$

where $\sigma_1/\tau$ is the dilation of $\sigma$ with respect to the first variable by the amount $1/\tau$. More precisely,

$$\sigma_1/\tau(q, p) = \sigma(q/\tau, p), \quad q, p \in \mathbb{R}.$$ 

So,

$$(W^r_\sigma f, g) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma_1/\tau(x, \xi) W(f, g)(x, \xi) \, dx \, d\xi$$

$$= |\tau|^{-1/2} (W^{1/\tau}_\sigma f, g),$$
where $W_{\sigma_{1/r}}$ is the classical Weyl transform with symbol $\sigma_{1/r}$. Thus, it follows from Theorem 21.1 in [14] that $W_{\sigma}^f : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is a bounded linear operator and
\[
\|W_{\sigma}^f\|_* \leq |r|^{-1/2}(2\pi)^{-1/p}\|\sigma_{1/r}\|_{L^p(\mathbb{C})} = (2\pi)^{-1/p}|r|^{-(1/2)+(1/p)}\|\sigma\|_{L^p(\mathbb{C})}.
\]

We have the following result for the Hilbert–Schmidt norm of the Weyl transform $W_{\sigma}^f$ in terms of the $L^2$ norm of the symbol $\sigma$ when $\sigma \in L^2(\mathbb{C})$.

**Theorem 2.3.** Let $\sigma \in L^2(\mathbb{C})$. Then $W_{\sigma}^f : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is a Hilbert–Schmidt operator and
\[
\|W_{\sigma}^f\|_{HS} = (2\pi)^{-1/2}\|\sigma\|_{L^2(\mathbb{C})},
\]
where $\|W_{\sigma}^f\|_{HS}$ is the Hilbert–Schmidt norm of $W_{\sigma}^f : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$.

**Proof** Let $f$ and $g$ be functions in $\mathcal{S}(\mathbb{R})$. Then
\[
(W_{\sigma}^f, g) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, \xi)W_{\sigma}^f(x, \xi)dx d\xi = (2\pi)^{-1/2}|r|^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, \xi)W(f, g)(x/r, \xi)dx d\xi = (2\pi)^{-1/2}|r|^{1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\tau x, \xi)W(f, g)(x, \xi)dx d\xi.
\]
But
\[
\delta(\tau x, \xi) = |r|^{-1/2}\sigma_{1/r}(x, \xi), \quad x, \xi \in \mathbb{R},
\]
where $\sigma_{1/r}$ is the dilation of $\sigma$ with respect to the first variable by the amount $1/r$, i.e.,
\[
\sigma_{1/r}(q, p) = \sigma(q/r, p), \quad q, p \in \mathbb{R}.
\]
So,
\[
(W_{\sigma}^f, g) = (2\pi)^{-1/2}|r|^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_{1/r}(x, \xi)W(f, g)(x, \xi)dx d\xi = |r|^{-1/2}(W_{\sigma_{1/r}}f, g),
\]
where $W_{\sigma_{1/r}}$ is the classical Weyl transform with symbol $\sigma_{1/r}$. Thus, it follows from Theorem 7.5 in [13] that $W_{\sigma}^f : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is a Hilbert–Schmidt operator and
\[
\|W_{\sigma}^f\|_{HS} = |r|^{-1/2}\|W_{\sigma_{1/r}}\|_{HS} = (2\pi)^{-1/2}|r|^{-1/2}\|\sigma_{1/r}\|_{L^2(\mathbb{C})} = (2\pi)^{-1/2}\|\sigma\|_{L^2(\mathbb{C})}.
\]
3 Fourier–Wigner Transforms of Hermite Functions

For $\tau \in \mathbb{R} \setminus \{0\}$ and for $k = 0, 1, 2, \ldots$, we define $e^\tau_k$ to be the function on $\mathbb{R}$ by

$$e^\tau_k(x) = |\tau|^{1/4} e_k(\sqrt{|\tau|} x), \quad x \in \mathbb{R}.$$ 

Here, $e_k$ is the Hermite function of order $k$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R},$$

where $H_k$ is the Hermite polynomial of degree $k$ given by

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} (e^{-x^2}), \quad x \in \mathbb{R}.$$ 

For $j, k = 0, 1, 2, \ldots$, we define $e^\tau_{j,k}$ on $\mathbb{R}^2$ by

$$e^\tau_{j,k} = V(\sqrt{|\tau|} q, \sqrt{|\tau|} p) e^\tau_j e^\tau_k.$$ 

The following theorem gives the connection of $\{e^\tau_{j,k} : j, k = 0, 1, 2, \ldots\}$ with $\{e_{j,k} : j, k = 0, 1, 2, \ldots\}$, where

$$e_{j,k} = V(e_j, e_k), \quad j, k = 0, 1, 2, \ldots.$$ 

A proof can be found in [1].

**Theorem 3.1.** For $\tau \in \mathbb{R} \setminus \{0\}$ and for $j, k = 0, 1, 2, \ldots$,

$$e^\tau_{j,k} (q, p) = |\tau|^{1/2} e_{j,k} \left( \frac{\tau}{\sqrt{|\tau|}} q, \sqrt{|\tau|} p \right), \quad q, p \in \mathbb{R}.$$ 

**Theorem 3.2.** $\{e^\tau_{j,k} : j, k = 0, 1, 2, \ldots\}$ forms an orthonormal basis for $L^2(\mathbb{R}^2)$.

Theorem 3.2 follows from Theorem 3.1 and Theorem 21.2 in [13] to the effect that $\{e_{j,k} : j, k = 0, 1, 2, \ldots\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$.

**Theorem 3.3.** For $j, k = 0, 1, 2, \ldots$,

$$\tilde{L}_\tau e^\tau_{j,k} = (2k + 1 + |\tau|) |\tau| e^\tau_{j,k}.$$ 

Theorem 3.3 can be proved using Theorem 3.1, Theorem 3.3 in [2] and Theorem 22.2 in [13] telling us that for $j, k = 0, 1, 2, \ldots$, $e_{j,k}$ is an eigenfunction of $L_1$ corresponding to the eigenvalue $2k + 1$ and the fact that, $\tilde{L}_\tau = L_\tau + \tau^2$. 
4 A Formula and an Estimate for $e^{-uL_t}$, $u > 0$

Let $\tau \in \mathbb{R} \setminus \{0\}$. Then a formula for $e^{-uL_t}$, $u > 0$, is given by the following theorem.

**Theorem 4.1.** Let $f \in L^2(\mathbb{C})$. Then for $u > 0$,

$$e^{-uL_t} f = (2\pi)^{1/2} \sum_{k=0}^{\infty} e^{-(2k+1+|\tau|)|u|} W_j e^i \hat{V}_{j^k} e^i,$$

where the convergence of the series is understood to be in $L^2(\mathbb{C})$.

**Proof** Let $f \in L^2(\mathbb{C})$. Then from Theorem 3.3 we have for $u > 0$

$$e^{-uL_t} f = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} e^{-(2k+1+|\tau|)|u|} e^{ij} \hat{V}_{j^k} e^i,$$

where the series is convergent in $L^2(\mathbb{C})$. Now, using the formula for $e^{-uL_t} f$ in [2] and (4.1), we get

$$e^{-uL_t} f = (2\pi)^{1/2} \sum_{k=0}^{\infty} e^{-(2k+1+|\tau|)|u|} V_k e^i h_k,$$

for all $f \in L^2(\mathbb{C})$ and $u > 0$. $\square$

For all $\tau \in \mathbb{R} \setminus \{0\}$, we have the following estimate for the $L^2$ norm of $e^{-uL_t} f$, $u > 0$, in terms of the $L^p$ norm of $f$.

**Theorem 4.2.** Let $\tau \in \mathbb{R} \setminus \{0\}$. Then for all functions $f \in L^p(\mathbb{C})$, $1 \leq p \leq 2$,

$$\|e^{-uL_t} f\|_{L^2(\mathbb{C})} \leq (2\pi)^{-1/2} \sum_{k=0}^{\infty} e^{-2k|\tau|} \|W_k^i e^i\|_{L^2(\mathbb{C})},$$

where the convergence of the series is understood to be in $L^2(\mathbb{C})$.

**Proof** By Theorem 4.1, the Moyal identity (2.1) and the fact that

$$\|e^i\|_{L^2(\mathbb{C})} = 1, \quad k = 0, 1, 2, \ldots,$$

we get

$$\|e^{-uL_t} f\|_{L^2(\mathbb{C})} \leq (2\pi)^{1/2} \sum_{k=0}^{\infty} e^{-2k|\tau|} \|W_k^i e^i\|_{L^2(\mathbb{C})}, \quad u > 0.$$ (4.2)

Applying Theorem 2.2 to (4.2), we get

$$\|e^{-uL_t} f\|_{L^2(\mathbb{C})} \leq (2\pi)^{-1/2} \sum_{k=0}^{\infty} e^{-2k|\tau|} \|f\|_{L^p(\mathbb{C})}$$

$$= (2\pi)^{-1/2} \sum_{k=0}^{\infty} e^{-|\tau|} \|f\|_{L^p(\mathbb{C})} \frac{1}{2\sinh(|\tau|)} \|f\|_{L^p(\mathbb{C})},$$

as asserted. $\square$
5 Sobolev Estimates for $e^{-\Delta_H}$, $u > 0$

Let $s \in \mathbb{R}$. Then we define $L^2_s(\mathbb{H})$ to be the set of all tempered distributions $f$ in $\mathcal{S}(\mathbb{H})$ such that $f^\tau(z)$ is a measurable function and

$$\int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} |f^\tau(z)|^2 d\tau dz < \infty.$$ 

For every $f$ in $L^2_s(\mathbb{H})$, we define the norm $\|f\|_{L^2_s(\mathbb{H})}$ by

$$\|f\|_{L^2_s(\mathbb{H})}^2 = \int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} |f^\tau(z)|^2 d\tau dz.$$ 

Then it can be shown easily that $L^2_s(\mathbb{H})$ is an inner product space in which the inner product $(\cdot, \cdot)_{L^2_s(\mathbb{H})}$ is given by

$$(f, g)_{L^2_s(\mathbb{H})} = \int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} f^\tau(z) \overline{g^\tau(z)} d\tau dz$$ 

for all $f$ and $g$ in $L^2_s(\mathbb{H})$.

**Theorem 5.1.** Let $s \geq 1$. Then for $u > 0$, $e^{-u\Delta_H} : L^2(\mathbb{H}) \to L^2_s(\mathbb{H})$ is a bounded linear operator and

$$\|e^{-u\Delta_H}f\|_{L^2_s(\mathbb{H})} \leq \frac{c_s}{2u} \|f\|_{L^2(\mathbb{H})}, \quad f \in L^2(\mathbb{H}),$$

where

$$c_s = \sup_{t \in \mathbb{R} \setminus \{0\} \} (|t|^s / \sinh |t|).$$

**Proof** Let $u > 0$ and $f \in L^2(\mathbb{H})$. Then by (1.1), Fubini’s theorem, Plancherel’s theorem and Theorem 4.2 with $p = 2$,

$$\|e^{-u\Delta_H}f\|_{L^2_s(\mathbb{H})}^2 = \int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} |(e^{-u\Delta_H}f)^\tau(z)|^2 d\tau dz = \int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} \left( \int_{\mathbb{C}} |(e^{-u\Delta_H}f)^\tau(z)|^2 d\tau \right) dz = \int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} \left( \int_{\mathbb{C}} |(e^{-u\Delta_H}f)^\tau(z)|^2 d\tau \right) dz = \int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} \left( \int_{\mathbb{C}} |(e^{-u\Delta_H}f)^\tau(z)|^2 d\tau \right) dz = \int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} \left( \int_{\mathbb{C}} |(e^{-u\Delta_H}f)^\tau(z)|^2 d\tau \right) dz \leq \frac{1}{4} \int_{-\infty}^{\infty} e^{-2t^2u} \left( \int_{\mathbb{C}} |(e^{-u\Delta_H}f)^\tau(z)|^2 d\tau \right) \sinh^2(|t|u) \left( \int_{\mathbb{C}} |f^\tau(z)|^2 d\tau \right) dt \leq \frac{1}{4} \int_{-\infty}^{\infty} \frac{|t|^{2s}}{\sinh^2(|t|u)} \left( \int_{\mathbb{C}} |f^\tau(z)|^2 d\tau \right) dt.$$
\[
= \frac{1}{4u^{2x+1}} \int_{-\infty}^{\infty} \frac{|r|^{2s}}{\sinh^{2}(|r|u)} \left( \int_{C} \left| \hat{f}(z,\tau) \right|^{2} dz \right) d\tau,
\]
where \( \hat{f} \) is the inverse Fourier transform of \( f \) with respect to \( t \). So, using a simple change of variable and letting
\[
C_{s} = \sup_{r \in \mathbb{R} \setminus \{0\}} \left( |r|^{2s}/\sinh^{2}|r| \right),
\]
we get
\[
\| e^{-u\Delta_{H}} f \|_{L_{s}^{2}(\mathcal{H})}^{2} \leq \frac{C_{s}}{4u^{2x}} \int_{-\infty}^{\infty} \left( \int_{C} \left| \hat{f}(z,\tau) \right|^{2} dz \right) d\tau = \frac{C_{s}}{4u^{2x}} \| f \|_{L_{s}^{2}(\mathcal{H})}^{2}
\]
and this completes the proof. \( \Box \)

The following result complements Theorem 5.1.

**Theorem 5.2.** Let \( s \leq -1 \). Then for \( u > 0 \), \( e^{-u\Delta_{H}} : L_{s}^{2}(\mathcal{H}) \to L_{s}^{2}(\mathcal{H}) \) is a bounded linear operator and
\[
\| e^{-u\Delta_{H}} f \|_{L_{s}^{2}(\mathcal{H})} \leq \frac{c_{-s}}{2u^{s}} \| f \|_{L_{s}^{2}(\mathcal{H})}, \quad f \in L_{s}^{2}(\mathcal{H}),
\]
where
\[
c_{-s} = \sup_{r \in \mathbb{R}} (|r|^{-s} \sinh|r|).
\]

The proof of Theorem 5.2 is very similar to that of Theorem 5.1 and is hence omitted.

### 6 Two Formulas and an Estimate for \( \tilde{L}_{\tau}^{-1} \)

Let \( \tau \in \mathbb{R} \setminus \{0\} \). Then a formula for \( \tilde{L}_{\tau}^{-1} \) is given by the following theorem.

**Theorem 6.1.** Let \( f \in L^{2}(\mathcal{C}) \). Then
\[
\tilde{L}_{\tau}^{-1} f = (2\pi)^{1/2} \sum_{k=0}^{\infty} \frac{1}{(2k+1+|\tau|)|\tau|} V_{t}(W_{f}^{T} e_{k}^{T}, e_{k}^{T}),
\]
where the convergence of the series is understood to be in \( L^{2}(\mathcal{C}) \).

**Proof** Let \( f \in L^{2}(\mathcal{C}) \). Then
\[
\tilde{L}_{\tau}^{-1} f = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(2k+1+|\tau|)|\tau|} (f_{j,k}) e_{j,k}^{T},
\]
where the series is convergent in \( L^{2}(\mathcal{C}) \). Now, by Plancherel’s theorem and (2.2)–(2.4),
\[
(f_{j,k}) = \int_{\mathcal{C}} f(z) \overline{V_{r}(e_{j}^{T}, e_{k}^{T})(z)} dz = \int_{\mathcal{C}} \hat{f}(\zeta) \overline{V_{r}(e_{j}^{T}, e_{k}^{T})(\zeta)} d\zeta
\]
for $j, k = 0, 1, 2, \ldots$. Similarly, for $j, k = 0, 1, 2, \ldots$, and $g$ in $L^2(\mathbb{C})$, we get

$$
(e^r_{j, k}, g) = (g, e^r_{j, k}) = (2\pi)^{1/2} W^r g W e^r_k,
$$

(6.3)

So, by (6.1)–(6.3), Fubini's theorem and Parseval's identity,

$$
(L^{-1}_r f, g) = 2\pi \sum_{k=0}^{\infty} \frac{1}{(2k + 1 + |\tau|)|\tau|} \sum_{j=0}^{\infty} (W^r f e^r_k, e^r_j)(e^r_j, W^r g e^r_k)
$$

= $2\pi \sum_{k=0}^{\infty} \frac{1}{(2k + 1 + |\tau|)|\tau|} (W^r f e^r_k, W^r g e^r_k).
$$

(6.4)

By Plancherel's theorem and (2.2)–(2.4),

$$
(W^r f e^r_k, W^r g e^r_k) = (2\pi)^{-1/2} \int_{\mathbb{C}} \hat{g}(z) W_r(e^r_k, W^r f e^r_k)(z) \, dz
$$

= $2\pi^{1/2} \int_{\mathbb{C}} W_r(W^r f e^r_k, e^r_j)(z) \, dz
$$

= $2\pi^{1/2} \int_{\mathbb{C}} V_r(W^r f e^r_k, e^r_j)(z) \, dz
$$

(6.5)

for $k = 0, 1, 2, \ldots$. Thus, by (6.4), (6.5) and Fubini's theorem,

$$
(L^{-1}_r f, g) = (2\pi)^{1/2} \sum_{k=0}^{\infty} \frac{1}{(2k + 1 + |\tau|)|\tau|} (V_r(W^r f e^r_k, e^r_j), g)
$$

= $2\pi^{1/2} \left( \sum_{k=0}^{\infty} \frac{1}{(2k + 1 + |\tau|)|\tau|} V_r(W^r f e^r_k, e^r_j), g \right)
$$

(6.6)

for all $f$ and $g$ in $L^2(\mathbb{C})$. Thus, by (6.6),

$$
L^{-1}_r f = (2\pi)^{1/2} \sum_{k=0}^{\infty} \frac{1}{(2k + 1 + |\tau|)|\tau|} V_r(W^r f e^r_k, e^r_j)
$$

for all $f$ in $L^2(\mathbb{C})$. □

The formula (6.4) is an important formula in its own right and we upgrade it to the status of a theorem.

**Theorem 6.2.** For all $\tau \in \mathbb{R} \setminus \{0\}$, the inverse $L^{-1}_r$ of the parametrized partial differential operators $L_r$ is given by

$$
(L^{-1}_r f, g) = 2\pi \sum_{k=0}^{\infty} \frac{1}{(2k + 1 + |\tau|)|\tau|} (W^r f e^r_k, W^r g e^r_k), \quad f, g \in L^2(\mathbb{C}).
$$

(6.7)
For all $\tau$ in $\mathbb{R} \setminus \{0\}$, we have the following estimate for the $L^2$ norm of $\tilde{L}_\tau^{-1}f$ in terms of the $L^2$ norm of $f$.

**Theorem 6.3.** Let $\tau \in \mathbb{R} \setminus \{0\}$. Then for all functions $f$ in $L^2(\mathbb{C})$,

$$\|\tilde{L}_\tau^{-1}f\|_{L^2(\mathbb{C})} \leq |\tau|^{-2} \|f\|_{L^2(\mathbb{C})}.$$

**Proof** Let $f$ and $g$ be functions in $L^2(\mathbb{R})$. Then by Theorems 2.3 and 6.2,

$$|(\tilde{L}_\tau^{-1}f, g)| \leq 2\pi \frac{1}{|\tau|^2} \sum_{k=0}^{\infty} |(W_{\tau}^f e_k^t, W_{\tau}^g e_k^t)| \leq 2\pi \frac{1}{|\tau|^2} \|W_{\tau}^f\|_{HS} \|W_{\tau}^g\|_{HS} \leq \frac{1}{|\tau|^2} \|f\|_{L^2(\mathbb{C})} \|g\|_{L^2(\mathbb{C})}$$

and this completes the proof. $\square$

## 7 Sobolev Estimates for $\Delta_{\mathbb{H}}^{-1}$

We have the following simple result giving the connection of $\Delta_{\mathbb{H}}^{-1}$ with $\tilde{L}_\tau^{-1}$, $\tau \in \mathbb{R} \setminus \{0\}$, which can be proved easily using the elementary properties of the Fourier transform and the Fourier inversion formula.

**Theorem 7.1.** Let $f \in L^2(\mathbb{H})$. Then

$$(\Delta_{\mathbb{H}}^{-1}f)(z, t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-it\tau} (\tilde{L}_\tau^{-1}f^t)(z) d\tau, \quad (z, t) \in \mathbb{H}.$$  

We can now give the following theorem, which can be seen as another manifestation of the ellipticity of $\Delta_{\mathbb{H}}$.

**Theorem 7.2.** Let $s \in \mathbb{R}$. Then $\Delta_{\mathbb{H}}^{-1} : L^2_s(\mathbb{H}) \to L^2_{s+2}(\mathbb{H})$ and

$$\|\Delta_{\mathbb{H}}^{-1}f\|_{L^2_{s+2}(\mathbb{H})} \leq \|f\|_{L^2_s(\mathbb{H})}, \quad f \in L^2_s(\mathbb{H}).$$

**Proof** By Fubini’s theorem, Plancherel’s theorem, Theorems 6.3 and 7.1,

$$\|\Delta_{\mathbb{H}}^{-1}f\|_{L^2_{s+2}(\mathbb{H})}^2 = \int_{\mathbb{H}} \left( \int_{-\infty}^{\infty} |\tau|^{2(s+2)} |(\Delta_{\mathbb{H}}^{-1}f^t)(z)|^2 d\tau dz \right) dz$$

$$= \int_{-\infty}^{\infty} |\tau|^{2(s+2)} \left( \int_{\mathbb{H}} |(\Delta_{\mathbb{H}}^{-1}f^t)(z)|^2 dz \right) d\tau$$
\begin{align*}
&= \int_{-\infty}^{\infty} |\tau|^{2(s+2)} \left( \int_{C} |(\tilde{L}_t^{-1} f^*)(z)|^2 \, dz \right) \, d\tau \\
&= \int_{-\infty}^{\infty} |\tau|^{2(s+2)} \| \tilde{L}_t^{-1} f^* \|_{L^2(C)}^2 \, d\tau \\
&\leq \int_{-\infty}^{\infty} |\tau|^{2s} \| f^* \|_{L^2(C)}^2 \, d\tau \\
&= \int_{C} \int_{-\infty}^{\infty} |\tau|^{2s} |f^*(z)|^2 \, d\tau \, dz \\
&= \| f \|_{L^2_\mathbb{F}(0)}^2,
\end{align*}

as asserted. \hfill \Box

\section*{References}


