Strong convergence of an implicit iteration process for a finite family of strictly asymptotically pseudocontractive mappings

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ABSTRACT

In this paper, we establish the strong convergence theorems for a finite family of $k$-strictly asymptotically pseudo-contractive mappings in the framework of Hilbert spaces. Our results improve and extend the corresponding results of Liu [5] and many others.

RESUMEN

Keywords: Strictly asymptotically pseudo-contractive mapping, implicit iteration scheme, common fixed point, strong convergence, Hilbert space.

AMS Subject Classification: 47H09, 47H10.

1 Introduction

Let $H$ be a real Hilbert space with the scalar product and norm denoted by the symbols $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively, and $C$ be a closed convex subset of $H$. Let $T$ be a (possibly) nonlinear mapping from $C$ into $C$. We now consider the following classes:

1. $T$ is contractive, i.e., there exists a constant $k < 1$ such that
   \[ \| T x - T y \| \leq k \| x - y \|, \] (1.1)
   for all $x, y \in C$.

2. $T$ is nonexpansive, i.e.,
   \[ \| T x - T y \| \leq \| x - y \|, \] (1.2)
   for all $x, y \in C$.

3. $T$ is uniformly $L$-Lipschitzian, i.e., if there exists a constant $L > 0$ such that
   \[ \| T^n x - T^n y \| \leq L \| x - y \|, \] (1.3)
   for all $x, y \in C$ and $n \in \mathbb{N}$.

4. $T$ is pseudo-contractive, i.e.,
   \[ \langle T x - T y, j(x - y) \rangle \leq \| x - y \|^2, \] (1.4)
   for all $x, y \in C$.

5. $T$ is strictly pseudo-contractive, i.e., there exists a constant $k \in [0, 1)$ such that
   \[ \| T x - T y \|^2 \leq \| x - y \|^2 + k \| (x - T x) - (y - T y) \|^2, \] (1.5)
for all \( x, y \in C \).

(6) \( T \) is asymptotically nonexpansive \([3]\), i.e., if there exists a sequence \( \{r_n\} \subset [0, \infty) \) with \( \lim_{n \to \infty} r_n = 0 \) such that

\[
\|T^n x - T^n y\| \leq (1 + r_n) \|x - y\|, \tag{1.6}
\]

for all \( x, y \in C \) and \( n \in \mathbb{N} \).

(7) \( T \) is \( k \)-strictly asymptotically pseudo-contractive \([6]\), i.e., if there exists a sequence \( \{r_n\} \subset [0, \infty) \) with \( \lim_{n \to \infty} r_n = 0 \) such that

\[
\|T^n x - T^n y\|^2 \leq (1 + r_n)^2 \|x - y\|^2 + k \|(x - T^n x) - (y - T^n y)\|^2, \tag{1.7}
\]

for some \( k \in [0, 1) \) for all \( x, y \in C \) and \( n \in \mathbb{N} \).

**Remark 1.1** [6]: If \( T \) is \( k \)-strictly asymptotically pseudo-contractive mapping, then it is uniformly \( L \)-Lipschitzian, but the converse does not hold.

Concerning the convergence problem of iterative sequences for strictly pseudocontractive mappings has been studied by several authors (see, e.g., \([2, 4, 7, 11, 12]\)). Concerning the class of strictly asymptotically pseudocontractive mappings, Liu \([5]\) proved the following result in Hilbert space:

**Theorem 1.1** (Liu \([5]\)): Let \( H \) be a real Hilbert space, let \( C \) be a nonempty closed convex and bounded subset of \( H \), and let \( T: C \to C \) be a completely continuous uniformly \( L \)-Lipschitzian \((\lambda, \{k_n\})\)-strictly asymptotically pseudocontractive mapping such that \( \sum_{n=1}^{\infty} (k_n^2 - 1) < \infty \). Let \( \{\alpha_n\} \subset (0, 1) \) be a sequence satisfying the following condition:

\[
0 < \epsilon \leq \alpha_n \leq 1 - \lambda \epsilon \quad \forall \ n \geq 1 \text{ and some } \epsilon > 0.
\]

Then, the sequence \( \{x_n\} \) generated from an arbitrary \( x_1 \in C \) by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall \ n \geq 1 \tag{1.8}
\]

converges strongly to a fixed point of \( T \).
In 2001, Xu and Ori [12] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space $H$. Let $C$ be a nonempty subset of $H$. Let $T_1, T_2, \ldots, T_N$ be self-mappings of $C$ and suppose that $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i, i = 1, 2, \ldots, N$. An implicit iteration process for a finite family of nonexpansive mappings is defined as follows, with $\{t_n\}$ a real sequence in $(0, 1)$, $x_0 \in C$:

$$x_1 = t_1 x_0 + (1 - t_1) T_1 x_1,$$

$$x_2 = t_2 x_1 + (1 - t_2) T_2 x_2,$$

\vdots

$$x_N = t_N x_{N-1} + (1 - t_N) T_N x_N,$$

$$x_{N+1} = t_{N+1} x_N + (1 - t_{N+1}) T_1 x_{N+1},$$

\vdots

which can be written in the following compact form:

$$x_n = t_n x_{n-1} + (1 - t_n) T_n x_n, \quad n \geq 1 \tag{1.9}$$

where $T_k = T_k \bmod N$. (Here the mod N function takes values in $\{1, 2, \ldots, N\}$). And they proved the weak convergence of the process (1.9).

Very recently, Acedo and Xu [1] still in the framework of Hilbert spaces introduced the following cyclic algorithm.

Let $C$ be a closed convex subset of a Hilbert space $H$ and let $\{T_i\}_{i=0}^{N-1}$ be $N$ $k$-strict pseudocontractions on $C$ such that $\mathcal{F} = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset$. Let $x_0 \in C$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$. The cyclic algorithm generates a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$x_1 = \alpha_0 x_0 + (1 - \alpha_0) T_0 x_0,$$

$$x_2 = \alpha_1 x_1 + (1 - \alpha_1) T_1 x_1,$$

\vdots

$$x_N = \alpha_{N-1} x_{N-1} + (1 - \alpha_{N-1}) T_{N-1} x_{N-1},$$

$$x_{N+1} = \alpha_N x_N + (1 - \alpha_N) T_0 x_N,$$

\vdots

In general, $\{x_{n+1}\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{[n]} x_n, \quad (1.10)$$
where $T[n] = T_i$ with $i = n \pmod{N}$, $0 \leq i \leq N - 1$. They also proved a weak convergence theorem for $k$-strict pseudo-contractions in Hilbert spaces by cyclic algorithm (1.10). More precisely, they obtained the following theorem:

**Theorem AX [1]:** Let $C$ be a closed convex subset of a Hilbert space $H$. Let $N \geq 1$ be an integer. Let for each $0 \leq i \leq N - 1$, $T_i : C \to C$ be a $k_i$-strict pseudo-contraction for some $0 \leq k_i < 1$. Let $k = \max\{k_i : 1 \leq i \leq N\}$. Assume the common fixed point set $\bigcap_{i=0}^{N-1} F(T_i)$ of $\{T_i\}_{i=0}^{N-1}$ is nonempty. Given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the cyclic algorithm (1.10). Assume that the control sequence $\{\alpha_n\}$ is chosen so that $k + \epsilon < \alpha_n < 1 - \epsilon$ for all $n$ and for some $\epsilon \in (0, 1)$. Then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.

Motivated by Xu and Ori [12], Acedo and Xu [1] and some others we introduce and study the following:

Let $C$ be a closed convex subset of a Hilbert space $H$ and let $\{T_i\}_{i=0}^{N-1}$ be $N$ $k$-strictly asymptotically pseudo-contractions on $C$ such that $F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset$. Let $x_0 \in C$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$. The implicit iteration scheme generates a sequence $\{x_n\}_{n=0}^{\infty}$ in the following way:

$$
x_1 = \alpha_0 x_0 + (1 - \alpha_0) T_0 x_0,
$$
$$
x_2 = \alpha_1 x_1 + (1 - \alpha_1) T_1 x_1,
$$
$$
\vdots
$$
$$
x_N = \alpha_{N-1} x_{N-1} + (1 - \alpha_{N-1}) T_{N-1} x_{N-1},
$$
$$
x_{N+1} = \alpha_N x_N + (1 - \alpha_N) T_0^2 x_0,
$$
$$
\vdots
$$
$$
x_{2N} = \alpha_{2N-1} x_{2N-1} + (1 - \alpha_{2N-1}) T_{N-1}^2 x_{2N-1},
$$
$$
x_{2N+1} = \alpha_{2N} x_{2N} + (1 - \alpha_{2N}) T_0^3 x_0,
$$
$$
\vdots
$$

In general, $\{x_n\}$ is defined by

$$
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{[n]}^s x_n, \quad (1.11)
$$

where $T_{[n]}^s = T_{n+(s-1)N}^{n+i}$ with $n = s - 1)N + i$ and $i \in I = \{0, 1, \ldots, N - 1\}$.

The aim of this paper is to establish strong convergence theorems of implicit iteration process (1.11) for a finite family of $k$-strictly asymptotically pseudo-contraction mappings in Hilbert
In the sequel, we will need the following lemmas.

**Lemma 1.1**: Let $H$ be a real Hilbert space. There holds the following identities:

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle \quad \forall \, x, y \in H$.

(ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2,$

$\forall \, t \in [0, 1], \, \forall \, x, y \in H$.

(iii) If $\{x_n\}$ be a sequence in $H$ weakly converges to $z$, then

$$\limsup_{n \to \infty} \|x_n - y\|^2 = \limsup_{n \to \infty} \|x_n - z\|^2 + \|z - y\|^2 \quad \forall y \in H.$$

**Lemma 1.2** [9]: Let $\{a_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + r_n)a_n + \beta_n, \quad n \geq 1.$$  

If $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\lim_{n \to \infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n \to \infty} a_n = 0$.

### 2 Main Results

**Theorem 2.1**: Let $C$ be a closed convex subset of a Hilbert space $H$. Let $N \geq 1$ be an integer. Let for each $0 \leq i \leq N - 1$, $T_i : C \to C$ be an $N$-strictly asymptotically pseudo-contraction mappings for some $0 \leq k_i < 1$ and $\sum_{n=1}^{\infty} r_n < \infty$. Let $k = \max\{k_i : 0 \leq i \leq N - 1\}$ and $r_n = \max\{r_n : 0 \leq i \leq N - 1\}$. Assume that $F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset$. Given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by an implicit iteration scheme (1.11). Assume that the control sequence $\{\alpha_n\}$ is chosen so that $k < \alpha_n < 1$ for all $n$ and $\sum_{n=0}^{\infty}(\alpha_n - k)(1 - \alpha_n) = \infty$. Then the iterative sequence $\{x_n\}$ has the following properties:
(1) \( \lim_{n \to \infty} \|x_n - p\| \) exists for each \( p \in F \),

(2) \( \lim_{n \to \infty} d(x_n, F) \) exists,

(3) \( \liminf_{n \to \infty} \|x_n - T_{[n]}^p x_n\| = 0 \),

(4) the sequence \( \{x_n\}_{n=0}^\infty \) converges strongly to a common fixed point \( p \in F \) if and only if

\[
\liminf_{n \to \infty} d(x_n, F) = 0.
\]

**Proof:** We divide the proof of Theorem 2.1 into three steps.

(1) First, we proof the conclusions (1) and (2).

For any \( p \in F \), it follows from (1.11) and Lemma 1.1(ii), we note that

\[
\|x_{n+1} - p\|^2 = \|\alpha_n x_n + (1 - \alpha_n)T_{[n]}^p x_n - p\|^2 
= \|\alpha_n (x_n - p) + (1 - \alpha_n)(T_{[n]}^p x_n - p)\|^2 
\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|T_{[n]}^p x_n - p\|^2 
- \alpha_n(1 - \alpha_n) \|x_n - T_{[n]}^p x_n\|^2 
\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\{(1 + r_n)^2 \|x_n - p\|^2 + k \|x_n - T_{[n]}^p x_n\|^2 - \alpha_n(1 - \alpha_n) \|x_n - T_{[n]}^p x_n\|^2 \}
\leq \alpha_n(1 + r_n)^2 \|x_n - p\|^2 - \alpha_n(1 - \alpha_n) \|x_n - T_{[n]}^p x_n\|^2 
\leq (1 + d_n) \|x_n - p\|^2 - \alpha_n(1 - \alpha_n) \|x_n - T_{[n]}^p x_n\|^2 
\leq (1 + d_n) \|x_n - p\|^2 - (\alpha_n - k)(1 - \alpha_n) \|x_n - T_{[n]}^p x_n\|^2
\]

where \( d_n = r_n^2 + 2r_n \), since \( \sum_{n=1}^\infty r_n < \infty \) thus \( \sum_{n=1}^\infty d_n < \infty \) and since \( k < \alpha_n < 1 \), we get

\[
\|x_{n+1} - p\|^2 \leq (1 + d_n) \|x_n - p\|^2
\]

and therefore
\[ \|x_{n+1} - p\| \leq (1 + d_n)^{1/2} \|x_n - p\|. \]  
\quad (2.3)

Since \( \sum_{n=1}^{\infty} d_n < \infty \), it follows from Lemma 1.2, we know that \( \lim_{n \to \infty} \|x_n - p\| \) exists for each \( p \in F \). So that there exists \( K > 0 \) such that \( \|x_n - p\| \leq K \) for all \( n \geq 1 \). Consequently, we obtain from (2.3) that

\[ \|x_{n+1} - p\| \leq (1 + d_n)^{1/2} \|x_n - p\| \leq (1 + d_n) \|x_n - p\| \leq \|x_n - p\| + Kd_n. \]  
\quad (2.4)

It follows from (2.4) that

\[ d(x_{n+1}, F) \leq (1 + d_n)d(x_n, F), \quad \forall \ n \geq 1 \]  
\quad (2.5)

so that it again follows from Lemma 1.2 that \( \lim_{n \to \infty} d(x_n, F) \) exists.

The conclusions (1) and (2) are proved.

(II) The proof of conclusion (3).

It follows from (2.1) that

\[ \|x_{n+1} - p\|^2 \leq (1 + d_n) \|x_n - p\|^2 - (\alpha_n - k)(1 - \alpha_n) \left\| x_n - T_{[n]}^n x_n \right\|^2 \]  
\quad (2.6)

where \( d_n = r_n^2 + 2r_n \), since \( \sum_{n=1}^{\infty} r_n < \infty \) thus \( \sum_{n=1}^{\infty} d_n < \infty \) and since \( k < \alpha_n < 1 \), we get

\[ \|x_{n+1} - p\|^2 \leq (1 + d_n) \|x_n - p\|^2 \]  
\quad (2.7)

that means the sequence \( \{\|x_n - p\|\} \) is decreasing. Now, since \( \sum_{n=1}^{\infty} d_n < \infty \) it follows that \( \prod_{i=1}^{\infty} (1 + d_i) < \infty \), from (2.6), we have

\[ \sum_{n=0}^{\infty} (\alpha_n - k)(1 - \alpha_n) \left\| x_n - T_{[n]}^n x_n \right\|^2 \leq \prod_{i=1}^{\infty} (1 + d_i) \|x_0 - p\|^2 \]  
\quad (2.8)

\[ < \infty. \]
Since $\sum_{n=0}^{\infty}(\alpha_n - k)(1 - \alpha_n) = \infty$, (2.8) implies that

$$\liminf_{n \to \infty} \left\| x_n - T_{[n]}^k x_n \right\| = 0. \quad (2.9)$$

(IV) Next, we prove the conclusion (4).

**Necessity**

If $\{x_n\}$ converges strongly to some point $p \in F$, then from $0 \leq d(x_n, F) \leq \|x_n - p\| \to 0$ as $n \to \infty$, we have

$$\liminf_{n \to \infty} d(x_n, F) = 0. \quad (2.10)$$

**Sufficiency**

If $\liminf_{n \to \infty} d(x_n, F) = 0$, it follows from the conclusion (2) that $\lim_{n \to \infty} d(x_n, F) = 0$. Next, we prove that $\{x_n\}$ is a Cauchy sequence in $C$. In fact, since for any $x > 0$, $1 + x \leq \exp(x)$, therefore, for any $m, n \geq 1$ and for given $p \in F$, from (2.4), we have

$$\|x_{n+m} - p\| \leq (1 + d_{n+m-1}) \|x_{n+m-1} - p\|$$
$$\leq e^{d_{n+m-1}} \|x_{n+m-1} - p\|$$
$$\leq e^{d_{n+m-1}} [e^{d_{n+m-2}} \|x_{n+m-2} - p\|]$$
$$\leq e^{(d_{n+m-1} + d_{n+m-2})} \|x_{n+m-2} - p\|$$
$$\leq \ldots$$
$$\leq e^{\sum_{j=n}^{n+m-1} d_j} \|x_n - p\|$$
$$\leq K' \|x_n - p\| < \infty \quad (2.11)$$

where $K' = e^{\sum_{j=1}^{\infty} d_j} < \infty$. Since

$$\lim_{n \to \infty} d(x_n, F) = 0, \quad (2.12)$$

for any given $\epsilon > 0$, there exists a positive integer $n_1$ such that
\[ d(x_n, F) < \frac{\epsilon}{2(K' + 1)} , \forall \ n \geq n_1. \] (2.13)

Hence, there exists \( p_1 \in F \) such that

\[ \|x_n - p_1\| < \frac{\epsilon}{(K' + 1)} , \forall \ n \geq n_1. \] (2.14)

Consequently, for any \( n \geq n_1 \) and \( m \geq 1 \), from (2.11), we have

\[
\|x_{n+m} - x_n\| \leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\
\leq K' \|x_n - p_1\| + \|x_n - p_1\| \\
\leq (K' + 1) \|x_n - p_1\| \\
< (K' + 1) \frac{\epsilon}{(K' + 1)} = \epsilon.
\]

This implies that \( \{x_n\} \) is a Cauchy sequence in \( C \). Let \( x_n \to x^* \in C \). Since \( \liminf_{n \to \infty} d(x_n, F) = 0 \), and so \( d(x^*, F) = 0 \). Again since \( \{T_i\}_{i=0}^{N-1} \) is a finite family of \( k \)-strictly asymptotically pseudo-contractive mappings, by Remark 1.1 of [6], it is a finite family of uniformly Lipschitzian mappings. Hence, the set \( F \) of common fixed points of \( \{T_i\}_{i=0}^{N-1} \) is closed and so \( x^* \in F \). Thus the sequence \( \{x_n\} \) converges strongly to a common fixed point of the family \( \{T_i\}_{i=0}^{N-1} \). This completes the proof.

**Theorem 2.2:** Let \( C \) be a closed convex compact subset of a Hilbert space \( H \). Let \( N \geq 1 \) be an integer. Let for each \( 0 \leq i \leq N - 1 \), \( T_i : C \to C \) be \( N k_i \)-strictly asymptotically pseudo-contractive mappings for some \( 0 \leq k_i < 1 \) and \( \sum_{i=1}^{\infty} r_n < \infty \) and \( r_n = \max\{r_n : 0 \leq i \leq N - 1\} \). Assume that \( F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset \). Given \( x_0 \in C \), let \( \{x_n\}_{n=0}^{\infty} \) be the sequence generated by an implicit iteration scheme (1.11). Assume that the control sequence \( \{\alpha_n\} \) is chosen so that \( k < \alpha_n < 1 \) for all \( n \). Then \( \{x_n\} \) converges strongly to a common fixed point of the family \( \{T_i\}_{i=0}^{N-1} \).

**Proof:** We only conclude the difference. By compactness of \( C \) this immediately implies that there is a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) which converges to a common fixed point of \( \{T_i\}_{i=0}^{N-1} \), say, \( p \). Combining (2.3) with Lemma 1.2, we have \( \lim_{n \to \infty} \|x_n - p\| = 0 \). Thus \( \{x_n\} \) converges strongly to a common fixed point of the family \( \{T_i\}_{i=0}^{N-1} \). This completes the proof.

**Remark 2.1** Our results extend and improve the corresponding results of Liu [5] and we also extend the iteration process (1.8) of [5] to an implicit iteration process for a finite family of
mappings.

Received: June 2009. Revised: November 2009.

References


