A unique common coupled fixed point theorem for four maps under $\psi$ - $\phi$ contractive condition in partial metric spaces

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ABSTRACT

In this paper, we obtain a unique coupled common fixed point theorem for four maps in partial metric spaces.

RESUMEN

En este artículo obtenemos un teorema del punto fijo clásico acoplado único para cuatro aplicaciones en espacios métricos parciales.

Keywords and Phrases: Partial metric, weakly compatible maps, complete space.

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1 Introduction and Preliminaries

The notion of partial metric space was introduced by S.G.Matthews [13] as a part of the study of denotational semantics of data flow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation ([6-10, 14-16], etc).


Regarding the concept of coupled fixed points introduced by Bhaskar and Lakshmikantham [17], in [4], Aydi proved some coupled fixed point theorems for the mappings satisfying contractive conditions in partial metric spaces. In this paper, we obtain a unique common coupled fixed point theorem for four self mappings satisfying a $\psi - \phi$ contractive condition in partial metric spaces. Our result is inspired by the results of Luong and Thuan [18].

First we recall some definitions and lemmas of partial metric spaces.

**Definition 1.1.** [13]. A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

1. $x = y \iff p(x, x) = p(x, y) = p(y, y)$,
2. $p(x, x) \leq p(x, y), p(y, y) \leq p(x, y)$,
3. $p(x, y) = p(y, x)$,
4. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z) + p(x, x)$.

$(X, p)$ is called a partial metric space.

It is clear that $p(x, y) = 0$ implies $x = y$ from (p1) and (p2).

But if $x = y$, $p(x, y)$ may not be zero. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$.

Each partial metric $p$ on $X$ generates $\tau_0$ topology $\tau_p$ on $X$ which has a base the family of open $p$-balls $\{B_p(x, \varepsilon)|x \in X, \varepsilon > 0\}$ for all $x \in X$ and $\varepsilon > 0$, where $B_p(x, \varepsilon) = \{y \in X|p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If $p$ is a partial metric on $X$, then the function $p^*: X \times X \to \mathbb{R}^+$ given by $p^*(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on $X$.

**Definition 1.2.** [13]. Let $(X, p)$ be a partial metric space.

(i) A sequence $\{x_n\}$ in $(X, p)$ is said to converge to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to \infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in $(X, p)$ is said to be Cauchy sequence if $\lim_{n, m \to \infty} p(x_n, x_m)$ exists and is finite.

(iii) $(X, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, w.r.t $\tau_p$, to a point $x \in X$ such that $p(x, x) = \lim_{n, m \to \infty} p(x_n, x_m)$. 

Lemma 1.1. [13]. Let \((X, p)\) be a partial metric space.
(a) \([x_n]\) is a Cauchy sequence in \((X, p)\) if and only if it is a Cauchy sequence in the metric space \((X, p^s)\).
(b) \((X, p)\) is complete if and only if the metric space \((X, p^s)\) is complete.
Furthermore, \(\lim_{n \to \infty} p^s(x_n, x) = 0\) if and only if
\[ p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n, m \to \infty} p(x_n, x_m). \]

Remark 1.2. Let \((X, p)\) be a partial metric space. If \([x_n]\) converges to \(x\) in \((X, p)\), then
\[ \lim_{n \to \infty} p(x_n, y) \leq p(x, y), \ \forall \ y \in X. \]

Proof. Since \([x_n]\) converges to \(x\) we have \(p(x, x) = \lim_{n \to \infty} p(x_n, x)\).
Now \(p(x_n, y) \leq p(x_n, x) + p(x, y) - p(x, x)\).
Letting \(n \to \infty\),
\[ \lim_{n \to \infty} p(x_n, y) \leq \lim_{n \to \infty} p(x_n, x) + p(x, y) - p(x, x). \]
Thus \(\lim_{n \to \infty} p(x_n, y) \leq p(x, y)\).

Definition 1.3. [17]. An element \((x, y)\) \(\in X \times X\) is called a coupled fixed point of mapping \(F : X \times X \to X\) if \(x = F(x, y)\) and \(y = F(y, x)\).

Definition 1.4. [2]. An element \((x, y)\) \(\in X \times X\) is called
\((g_1)\) a coupled coincident point of mappings \(F : X \times X \to X\) and \(f : X \to X\) if \(fx = F(x, y)\) and \(fy = F(y, x)\).
\((g_2)\) a common coupled fixed point of mappings \(F : X \times X \to X\) and \(f : X \to X\) if \(x = f(x) = F(x, y)\) and \(y = fy = F(y, x)\).

Definition 1.5. [2]. The mappings \(F : X \times X \to X\) and \(f : X \to X\) are called \(w\)-compatible if
\(f(F(x, y)) = F(fx, fy)\) and \(f(F(y, x)) = F(fy, fx)\) whenever \(fx = F(x, y)\) and \(fy = F(y, x)\).

Using concept of coupled fixed point, Luong and Thuan in [18] proved some coupled fixed point theorems for a mapping \(F : X \times X \to X\) satisfying the following contractive condition in the partially ordered metric spaces \((X, d, \leq)\)
\[ \psi (d(F(x, y), F(u, v))) \leq \frac{1}{2} \psi (d(x, u) + d(y, v)) - \phi \left( \frac{d(x, u) + d(y, v)}{2} \right) \]
for all \(x, y, u, v \in X\) with \(x \geq u\) and \(y \leq v\), with \(\psi \in \Phi\) and \(\psi \in \Psi\), where \(\Psi\) denotes the set of all functions \(\psi : [0, \infty) \to [0, \infty)\) satisfying
\((\psi_1)\) \(\psi\) is continuous and non-decreasing,
\((\psi_2)\) \(\psi(t) = 0\) if and only if \(t = 0\),
\((\psi_3)\) \(\psi(t + s) \leq \psi(t) + \psi(s)\), for all \(t, s \in [0, \infty)\),
while \(\Phi\) denotes the set of all functions \(\phi : [0, \infty) \to [0, \infty)\) satisfying
\((\phi_1)\) \(\lim_{t \to r, r > 0} \phi(t) > 0\) for all \(r > 0\).
(ϕ₂) \lim_{t \to 0^+} ϕ(t) = 0.

From (ϕ₁), it is clear that ϕ(t) > 0 for all t > 0.

Now we prove our main result.

2 Main Result

Theorem 1. Let (X, p) be a partial metric space and let f, g : X → X and F, G : X × X → X be such that
(i) For all x, y, u, v ∈ X,
\[ \psi(p(F(x, y), G(u, v))) \leq \frac{1}{2} \psi(p(fx, gu) + p(fy, gv)) - ϕ(p(fx, gu) + p(fy, gv)), \]
where ψ ∈ Ψ and ϕ ∈ Φ,
(ii) F(X × X) ⊆ g(X), G(X × X) ⊆ f(X),
(iii) either f(X) or g(X) is a complete subspace of X and
(iv) the pairs (F, f) and (G, g) are w-compatible.

Then F, G, f and g have a unique common coupled fixed point in X × X. Moreover, the common coupled fixed point of F, G, f and g have the form (u, u).

Proof. Let x₀, y₀ be arbitrary points in X.
From(ii), there exist sequences \{xₙ\}, \{yₙ\}, \{zₙ\} and \{wₙ\} in X such that
\[ F(x₂n, y₂n) = gx₂n₊₁ = z₂n, \]
\[ F(y₂n, x₂n) = gy₂n₊₁ = w₂n, \]
\[ G(x₂n₊₁, y₂n₊₁) = fx₂n₊₂ = z₂n₊₁ \]
and
\[ G(y₂n₊₁, x₂n₊₁) = fy₂n₊₄ = w₂n₊₄, \]
\[ n = 0, 1, 2, ....... \]

We have
\[ \psi(p(z₂n₊₁, z₂n)) = \psi(p(F(x₂n, y₂n), G(x₂n₊₁, y₂n₊₁)) \]
\[ \leq \frac{1}{2} \psi(p(z₂n, z₂n₋₁) + p(w₂n, w₂n₋₁)) - ϕ(p(z₂n, z₂n₋₁) + p(w₂n, w₂n₋₁)) \] (2.1)

Similarly,
\[ \psi(p(w₂n₊₁, w₂n)) \leq \frac{1}{2} \psi(p(z₂n, z₂n₋₁) + p(w₂n, w₂n₋₁)) - ϕ(p(z₂n, z₂n₋₁) + p(w₂n, w₂n₋₁)) \] (2.2)
From (2.1), (2.2) and \((\psi_3)\), we have
\[
\psi(p(z_{2n+1}, z_{2n}) + p(w_{2n+1}, w_{2n})) \leq \psi(p(z_{2n}, z_{2n} - 1) + p(w_{2n}, w_{2n} - 1)) - 2\phi(p(z_{2n}, z_{2n} - 1) + p(w_{2n}, w_{2n} - 1)) \tag{2.3}
\]
Since \(\psi\) is non-decreasing, we have
\[
p(z_{2n+1}, z_{2n}) + p(w_{2n+1}, w_{2n}) \leq p(z_{2n}, z_{2n} - 1) + p(w_{2n}, w_{2n} - 1).
\]
Similarly, we can show that
\[
p(z_{2n}, z_{2n} - 1) + p(w_{2n}, w_{2n} - 1) \leq p(z_{2n-1}, z_{2n-2}) + p(w_{2n-1}, w_{2n-2}).
\]
Thus
\[
p(z_{n+1}, z_n) + p(w_{n+1}, w_n) \leq p(z_{n-1}, z_{n-2}) + p(w_{n-1}, w_{n-2}).
\]
Put \(\delta_n = p(z_{n+1}, z_n) + p(w_{n+1}, w_n)\). Then we have
\[
\delta_n \leq \delta_{n-1}, n = 1, 2, 3, ...
\]
Thus \(\{\delta_n\}\) is a non-increasing sequence of non-negative real numbers and must converge to a real number, say, \(\delta \geq 0\).
Suppose \(\delta > 0\).
Letting \(n \to \infty\) in (2.3) and using the properties of \(\psi\) and \(\phi\), we get
\[
\psi(\delta) \leq \psi(\delta) - 2 \lim_{\delta_{2n} \to \delta} \phi(\delta_{2n}) < \psi(\delta)
\]
which is a contradiction. Hence \(\delta = 0\).
Thus
\[
\lim_{n \to \infty} [p(z_{n+1}, z_n) + p(w_{n+1}, w_n)] = 0 \tag{2.4}
\]
Hence from (p2),
\[
\lim_{n \to \infty} [p(z_n, z_n) + p(w_n, w_n)] = 0 \tag{2.5}
\]
From (2.4) and (2.5) we have that
\[
\lim_{n \to \infty} p^s(z_{n+1}, z_n) = 0 \tag{2.6}
\]
and
\[
\lim_{n \to \infty} p^s(w_{n+1}, w_n) = 0 \tag{2.7}
\]
Now we prove that \(\{z_{2n}\}\) and \(\{w_{2n}\}\) are Cauchy sequences.
On contrary, suppose that \(\{z_{2n}\}\) or \(\{w_{2n}\}\) is not Cauchy. This implies that \(p^s(z_{2m}, z_{2n}) \neq 0\) or
\( p^s(w_{2m}, w_{2n}) \neq 0 \) as \( n, m \to \infty \).

Consequently

\[
\max\{p^s(z_{2m}, z_{2n}), p^s(w_{2m}, w_{2n})\} \neq 0 \quad \text{as} \quad n, m \to \infty.
\]

Then there exist an \( \epsilon > 0 \) and monotone increasing sequences of natural numbers \( \{2m_k\} \) and \( \{2n_k\} \) such that \( n_k > m_k \),

\[
\max\{p^s(z_{2m_k}, z_{2n_k}), p^s(w_{2m_k}, w_{2n_k})\} \geq \epsilon \quad (2.8)
\]

and

\[
\max\{p^s(z_{2m_k}, z_{2n_k-2}), p^s(w_{2m_k}, w_{2n_k-2})\} < \epsilon. \quad (2.9)
\]

From (2.8) and (2.9), we have

\[
\epsilon \leq \max\{p^s(z_{2m_k}, z_{2n_k}), p^s(w_{2m_k}, w_{2n_k})\}
\leq \max\{p^s(z_{2m_k}, z_{2n_k-2}), p^s(w_{2m_k}, w_{2n_k-2})\}
+ \max\{p^s(z_{2n_k-2}, z_{2n_k-1}), p^s(w_{2n_k-2}, w_{2n_k-1})\}
+ \max\{p^s(z_{2n_k-1}, z_{2n_k}), p^s(w_{2n_k-1}, w_{2n_k})\}
< \epsilon + \max\{p^s(z_{2n_k-2}, z_{2n_k-1}), p^s(w_{2n_k-2}, w_{2n_k-1})\}
+ \max\{p^s(z_{2n_k-1}, z_{2n_k}), p^s(w_{2n_k-1}, w_{2n_k})\}.
\]

Letting \( k \to \infty \) and using (2.6) and (2.7) we have

\[
\lim_{k \to \infty} \max\{p^s(z_{2m_k}, z_{2n_k}), p^s(w_{2m_k}, w_{2n_k})\} = \epsilon. \quad (2.10)
\]

Also,

\[
\epsilon \leq \max\{p^s(z_{2m_k}, z_{2n_k}), p^s(w_{2m_k}, w_{2n_k})\}
\leq \max\{p^s(z_{2m_k}, z_{2m_k-1}), p^s(w_{2m_k}, w_{2m_k-1})\}
+ \max\{p^s(z_{2m_k-1}, z_{2n_k}), p^s(w_{2m_k-1}, w_{2n_k})\}
\leq \max\{p^s(z_{2m_k}, z_{2m_k-1}), p^s(w_{2m_k}, w_{2m_k-1})\}
+ \max\{p^s(z_{2m_k-1}, z_{2m_k}), p^s(w_{2m_k-1}, w_{2m_k})\}
+ \max\{p^s(z_{2m_k}, z_{2n_k}), p^s(w_{2m_k}, w_{2n_k})\}
= 2 \max\{p^s(z_{2m_k}, z_{2m_k-1}), p^s(w_{2m_k}, w_{2m_k-1})\}
+ \max\{p^s(z_{2m_k}, z_{2n_k}), p^s(w_{2m_k}, w_{2n_k})\}.
\]

Letting \( k \to \infty \) and using (2.6), (2.7), (2.10) and (2.11), we have

\[
\lim_{k \to \infty} \max\{p^s(z_{2m_k-1}, z_{2n_k}), p^s(w_{2m_k-1}, w_{2n_k})\} = \epsilon. \quad (2.12)
\]

On other hand we have

\[
\max\{p^s(z_{2m_k}, z_{2n_k}), p^s(w_{2m_k}, w_{2n_k})\} \leq \max\{p^s(z_{2m_k}, z_{2n_k+1}), p^s(w_{2m_k}, w_{2n_k+1})\}
+ \max\{p^s(z_{2n_k+1}, z_{2n_k}), p^s(w_{2n_k+1}, w_{2n_k})\}
\]
Letting $k \to \infty$ and using (2.5), (2.6) and (2.7), we have
\[
\epsilon \leq \lim_{k \to \infty} \max \{p^s(z_{2m_k}, z_{2n_k+1}), p^s(w_{2m_k}, w_{2n_k+1}) + 0
\leq \lim_{k \to \infty} \max \left\{ 2p(z_{2m_k}, z_{2n_k+1}) - p(z_{2m_k}, z_{2m_k}) - p(z_{2n_k+1}, z_{2n_k+1}),
2p(w_{2m_k}, w_{2n_k+1}) - p(w_{2m_k}, w_{2m_k}) - p(w_{2n_k+1}, w_{2n_k+1}) \right\}
\leq 2 \lim_{k \to \infty} \max \{p(z_{2m_k}, z_{2n_k+1}), p(w_{2m_k}, w_{2n_k+1}) \}
\]
Thus,
\[
\frac{\epsilon}{2} \leq \lim_{k \to \infty} \max \{p(z_{2m_k}, z_{2n_k+1}), p(w_{2m_k}, w_{2n_k+1}) \}
\]
By the properties of $\psi$
\[
\psi \left( \frac{\epsilon}{2} \right) \leq \psi \left( \lim_{k \to \infty} \max \{p(z_{2m_k}, z_{2n_k+1}), p(w_{2m_k}, w_{2n_k+1}) \} \right)
= \lim_{k \to \infty} \max \{\psi(p(z_{2m_k}, z_{2n_k+1})), \psi(p(w_{2m_k}, w_{2n_k+1}))\}
\tag{2.13}
\]
Now
\[
\psi(p(z_{2m_k}, z_{2n_k+1})) = \psi \left( p(F(x_{2m_k}, y_{2m_k}), G(x_{2n_k+1}, y_{2n_k+1})) \right)
\leq \frac{1}{2} \psi \left( p(z_{2m_k-1}, z_{2n_k}) + p(w_{2m_k-1}, w_{2n_k}) \right)
- \phi \left( p(z_{2m_k-1}, z_{2n_k}) + p(w_{2m_k-1}, w_{2n_k}) \right)
\leq \frac{1}{2} \left[ \psi \left( \psi(p(z_{2m_k-1}, z_{2n_k})) + \psi(p(w_{2m_k-1}, w_{2n_k})) \right) \right]
- \phi \left( p(z_{2m_k-1}, z_{2n_k}) + p(w_{2m_k-1}, w_{2n_k}) \right)
\leq \max \left\{ \psi \left( p(z_{2m_k-1}, z_{2n_k}), \psi(p(w_{2m_k-1}, w_{2n_k})) \right) \right\}
- \phi \left( p(z_{2m_k-1}, z_{2n_k}) + p(w_{2m_k-1}, w_{2n_k}) \right)
= \psi \max \left\{ \psi(p(z_{2m_k-1}, z_{2n_k}), p(w_{2m_k-1}, w_{2n_k})) \right\}
- \phi \left( p(z_{2m_k-1}, z_{2n_k}) + p(w_{2m_k-1}, w_{2n_k}) \right)
\]
Similarly
\[
\psi(p(w_{2m_k}, w_{2n_k+1})) \leq \psi \left( \max \{p(z_{2m_k-1}, z_{2n_k}), p(w_{2m_k-1}, w_{2n_k}) \} \right)
- \phi \left( p(z_{2m_k-1}, z_{2n_k}) + p(w_{2m_k-1}, w_{2n_k}) \right).
\]
Hence from (2.13), (2.5) and (2.12), we have

\[
\psi \left( \frac{\epsilon}{2} \right) \leq \lim_{k \to \infty} \left\{ \psi(\max[p(z_{2m_k-1}, z_{2n_k}), p(w_{2m_k-1}, w_{2n_k})]) - \phi(p(z_{2m_k-1}, z_{2n_k}) + p(w_{2m_k-1}, w_{2n_k})) \right\}
\]

\[
\leq \lim_{k \to \infty} \psi \left( \max \left\{ \frac{1}{2} \left( p^s(z_{2m_k-1}, z_{2n_k}) + p(z_{2m_k-1}, z_{2m_k}) \right) + p(w_{2m_k-1}, w_{2n_k}) \right\} \right) - \lim_{k \to \infty} \phi(p(z_{2m_k-1}, z_{2n_k}) + p(w_{2m_k-1}, w_{2n_k}))
\]

\[
= \psi \left( \frac{\epsilon}{2} \right) - \lim_{k \to \infty} \phi(p(z_{2m_k-1}, z_{2n_k}) + p(w_{2m_k-1}, w_{2n_k}))
\]

\[
= \psi \left( \frac{\epsilon}{2} \right) - \lim_{t \to \frac{1}{2}} \phi(t),
\]

where

\[
\epsilon = \lim_{k \to \infty} \frac{1}{2} \left( p^s(z_{2m_k-1}, z_{2n_k}) + p(z_{2m_k-1}, z_{2m_k}) \right) + p(w_{2m_k-1}, w_{2n_k})
\]

\[
< \psi \left( \frac{\epsilon}{2} \right),
\]

which is a contradiction. Hence \(z_{2n}\) and \(w_{2n}\) are Cauchy sequences in the metric space \((X, p^s)\).

Letting \(n, m \to \infty\) in

\[
|p^s(z_{2n+1}, z_{2m+1}) - p^s(z_{2n}, z_{2m})| \leq p^s(z_{2n+1}, z_{2n}) + p^s(z_{2m+1}, z_{2m}).
\]

we get

\[
\lim_{n \to \infty} p^s(z_{2n+1}, z_{2m+1}) = 0.
\]

Letting \(n, m \to \infty\) in

\[
|p^s(w_{2n+1}, w_{2m+1}) - p^s(w_{2n}, w_{2m})| \leq p^s(w_{2n+1}, w_{2n}) + p^s(w_{2m+1}, w_{2m})
\]

we get

\[
\lim_{n \to \infty} p^s(w_{2n+1}, w_{2m+1}) = 0.
\]

Thus \(z_{2n+1}\) and \(w_{2n+1}\) are Cauchy sequences in the metric space \((X, p^s)\).

Hence \(z_n\) and \(w_n\) are Cauchy sequences in the metric space \((X, p^s)\).

Hence we have that

\[
\lim_{n \to \infty} p^s(z_n, z_m) = 0 = \lim_{n \to \infty} p^s(w_n, w_m).
\]

Now from definition of \(p^s\) and from (2.5) we have

\[
\lim_{n \to \infty} p(z_n, z_m) = 0 \quad (2.14)
\]
and

\[
\lim_{n \to \infty} p(w_n, w_m) = 0. \tag{2.15}
\]

Suppose \( f(X) \) is complete. Since \( \{z_{2n+1}\} \subseteq f(X) \) and \( \{w_{2n+1}\} \subseteq f(X) \) are Cauchy sequences in the complete metric space \( (f(X), p^s) \), it follows that the sequences \( \{z_{2n+1}\} \) and \( \{w_{2n+1}\} \) are convergent in \( (f(X), p^s) \). Thus

\[
\lim_{n \to \infty} p^s(z_{2n+1}, u) = 0
\]

and

\[
\lim_{n \to \infty} p^s(w_{2n+1}, v) = 0
\]

for some \( u \) and \( v \) in \( f(X) \).

Since \( u, v \in f(X) \), there exist \( s, t \in X \) such that \( u = fs \) and \( v = ft \).

Since \( \{z_n\} \) and \( \{w_n\} \) are Cauchy sequences in \( X \) and \( \{z_{2n+1}\} \to u \) and \( \{w_{2n+1}\} \to v \), it follows that \( \{z_{2n}\} \to u \) and \( \{w_{2n}\} \to v \).

From Lemma 1.1, we have

\[
p(u, u) = \lim_{n \to \infty} p(z_{2n}, u) = \lim_{n \to \infty} p(z_{2n+1}, u) = \lim_{n, m \to \infty} p(z_n, z_m) \tag{2.16}
\]

and

\[
p(v, v) = \lim_{n \to \infty} p(w_{2n}, v) = \lim_{n \to \infty} p(w_{2n+1}, v) = \lim_{n, m \to \infty} p(w_n, w_m) \tag{2.17}
\]

From (2.16), (2.17), (2.14) and (2.15) we have

\[
p(u, u) = 0 = p(v, v). \tag{2.18}
\]

Now,

\[
p(F(s,t), u) \leq p(F(s,t), z_{2n+1}) + p(z_{2n+1}, u) - p(z_{2n+1}, z_{2n+1})
\]

\[
\leq p(F(s,t), G(x_{2n+1}, y_{2n+1})) + p(z_{2n+1}, u).
\]

Therefore,

\[
\psi(p(F(s,t), u)) \leq \psi(p(F(s,t), G(x_{2n+1}, y_{2n+1})) + p(z_{2n+1}, u))
\]

\[
\leq \psi(p(F(s,t), G(x_{2n+1}, y_{2n+1}))) + \psi(p(z_{2n+1}, u)), \text{ from } (\psi_3)
\]

\[
\leq \frac{1}{2} \psi(p(u, z_{2n}) + p(v, w_{2n})) - \phi(p(u, z_{2n}) + p(v, w_{2n})) + \psi(p(z_{2n+1}, u)).
\]

Letting \( n \to \infty \) and using (2.16), (2.17), (2.18) and \( (\psi_2) \), \( (\psi_1) \) we get \( \psi(p(F(s,t), u)) \leq 0 \). Hence \( F(s,t) = u = fs \) (by \( (\psi_2) \)).

Similarly, we have \( F(t,s) = v = ft \).

Since the pair \( (f, f) \) is \( w \) - compatible, we have \( fu = F(u, v) \) and \( fv = F(v, u) \). Suppose that \( fu \neq u \) or \( fv \neq v \).

\[
p^s(fu, z_{2n}) = 2p(fu, z_{2n}) - p(fu, fu) - p(z_{2n}, z_{2n}).
\]
Letting $n \to \infty$, we get
\[ p^s(fu, u) = 2 \lim_{n \to \infty} p(fu, z_{2n}) - p(fu, fu) - 0, \text{ from (2.5)} \]
or
\[ 2p(fu, u) - p(fu, fu) - p(u, u) = 2 \lim_{n \to \infty} p(fu, z_{2n}) - p(fu, fu) \]
or
\[ p(fu, u) = \lim_{n \to \infty} p(fu, z_{2n}), \text{ from (2.18)}. \]

Similarly, we have $p(fv, v) = \lim_{n \to \infty} p(fv, w_{2n})$. Thus
\[ \lim_{n \to \infty} [p(fv, z_{2n}) + p(fv, w_{2n})] = p(fu, u) + p(fv, v) > 0 \quad (2.19) \]

We have
\[
\begin{align*}
p(fu, u) & \leq p(fu, z_{2n+1}) + p(z_{2n+1}, u) - p(z_{2n+1}, z_{2n+1}) \\
& \leq p(F(u, v), G(x_{2n+1}, y_{2n+1})) + p(z_{2n+1}, u).
\end{align*}
\]

Thus,
\[
\begin{align*}
\psi(p(fu, u)) & \leq \psi(p(F(u, v), G(x_{2n+1}, y_{2n+1})) + p(z_{2n+1}, u)), \text{ from (ψ3)} \\
& \leq \frac{1}{2} \psi(p(fu, z_{2n}) + p(fv, w_{2n})) \\
& - \phi(p(fu, z_{2n}) + p(fv, w_{2n})) + \psi(p(z_{2n+1}, u)).
\end{align*}
\]

Similarly, we have
\[
\begin{align*}
\psi(p(fv, v)) & \leq \frac{1}{2} \psi(p(fu, z_{2n}) + p(fv, w_{2n})) \\
& - \phi(p(fu, z_{2n}) + p(fv, w_{2n})) + \psi(p(w_{2n+1}, v)).
\end{align*}
\]

Hence
\[
\begin{align*}
\psi(p(fu, u) + p(fv, v)) & \leq \psi(p(fu, u)) + \psi(p(fv, v)), \text{ from (ψ3)} \\
& \leq \psi(p(fu, z_{2n}) + p(fv, w_{2n})) \\
& - 2\phi(p(fu, z_{2n}) + p(fv, w_{2n})) \\
& + \psi(p(z_{2n+1}, u)) + \psi(p(w_{2n+1}, v)).
\end{align*}
\]

Letting $n \to \infty$ and using (2.19), $(\phi_1), (2.16), (2.17)$ and $(\psi_1)$, we get
\[ \psi(p(fu, u) + p(fv, v)) < \psi(p(fu, u) + p(fv, v)). \]

It is a contradiction. Hence $fu = u$ and $fv = v$. Thus
\[ F(u, v) = fu = u \text{ and } F(v, u) = fv = v. \quad (2.20) \]
Since \( F(X \times X) \subseteq g(X) \), there exist \( a, b \in X \) such that \( u = F(u, v) = ga \) and \( v = F(v, u) = gb \).

\[
\psi(p(u, G(a, b))) = \psi(p(F(u, v), G(a, b))) \\
\leq \frac{1}{2} \psi(p(u, u) + p(v, v)) - \phi(p(u, u) + p(v, v)) \\
= \frac{1}{2} \psi(0) - \phi(0), \quad (\text{from (2.18)}) \\
\leq 0, \quad (\text{since } \psi(0) = 0 \text{ and } \phi(0) \geq 0).
\]

Hence \( \psi(p(u, G(a, b))) = 0 \), which implies that \( G(a, b) = u = ga \).

Similarly, we have \( G(b, a) = v = gb \).

Since the pair \((G, g)\) is \( w \)-compatible, we have \( gu = G(u, v) \) and \( gv = G(v, u) \). Suppose \( gu \neq u \) or \( gv \neq v \). We have

\[
\psi(p(u, gu)) = \psi(p(F(u, v), G(u, v))) \\
\leq \frac{1}{2} \psi(p(u, gu) + p(v, gv)) - \phi(p(u, gu) + p(v, gv))
\]

and

\[
\psi(p(v, gv)) = \psi(p(F(v, u), G(v, u))) \\
\leq \frac{1}{2} \psi(p(u, gu) + p(v, gv)) - \phi(p(u, gu) + p(v, gv)).
\]

Hence

\[
\psi(p(u, gu) + p(v, gv)) \leq \psi(p(u, gu)) + \psi(p(v, gv)) \\
\leq \psi(p(u, gu) + p(v, gv) - 2\phi(p(u, gu) + p(v, gv)) \\
\leq \psi(p(u, gu) + p(v, gv)) \quad (\text{since } \phi(t) > 0 \forall t > 0).
\]

Hence \( gu = u \) and \( gv = v \). Thus,

\[
u = gu = G(u, v) \quad \text{and} \quad v = gv = G(v, u) \quad (2.21)
\]

From (2.20) and (2.21), it follows that \((u, v)\) is a common coupled fixed point of \( F, G, f \) and \( g \).

Let \((u^*, v^*)\) be another common coupled fixed point of \( F, G, f \) and \( g \). We have

\[
\psi(p(u, u^*) + p(v, v^*)) \leq \psi(p(u, u^*)) + \psi(p(v, v^*)) \\
\leq \psi(p(F(u, v), G(u^*, v^*))) + \psi(p(F(v, u), G(v^*, u^*))) \\
\leq \frac{1}{2} \psi(p(u, u^*) + p(v, v^*)) - \phi(p(u, u^*) + p(v, v^*)) \\
+ \frac{1}{2} \psi(p(u, u^*) + p(v, v^*)) - \phi(p(u, u^*) + p(v, v^*)) \\
= \psi(p(u, u^*) + p(v, v^*)) - 2\phi(p(u, u^*) + p(v, v^*)) \\
< \psi(p(u, u^*) + p(v, v^*)),
\]

\[\frac{1}{2} \psi(p(u, u^*) + p(v, v^*)) - \phi(p(u, u^*) + p(v, v^*)) \leq 0, \quad (\text{since } \phi(0) \geq 0).\]

Hence \( \psi(p(u, u^*) + p(v, v^*)) \leq 0 \), which implies that \( u^* = gu \) and \( v^* = gv \). Thus, \((u^*, v^*)\) is a common coupled fixed point of \( F, G, f \) and \( g \).
which is a contradiction. Hence \((u, v)\) is the unique common coupled fixed point of \(F, G, f\) and \(g\). Now we will show that \(u = v\). Suppose \(u \neq v\).

\[
\psi(p(u, v)) = \psi(p(F(u, v), G(u, v))) \\
\leq \frac{1}{2} \psi(p(u, v) + p(v, u)) - \phi(p(u, v)) \\
\leq \psi(p(u, v)) - \phi(p(u, v)) \\
< \psi(p(u, v)).
\]

Hence \(u = v\).

Thus \(u = fu = F(u, u) = G(u, u) = gu\), that is, the common coupled fixed point of \(F, G, f\) and \(g\) has the form \((u, u)\).

\[\square\]

References


