On Weak concircular Symmetries of Lorentzian Concircular Structure Manifolds

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**ABSTRACT**

The object of the present paper is to study weakly concircular symmetric, weakly concircular Ricci symmetric and special weakly concircular Ricci symmetric Lorentzian concircular structure manifolds.

**RESUMEN**

El objetivo del presente artículo es estudiar las variedades de estructura simétricas concirculares débiles, las simétricas Ricci concirculares débiles y concirculares Lorentzianas simétricas Ricci concirculares débiles especiales.

**Keywords and Phrases:** Weakly concircular symmetric manifold, weakly concircular Ricci symmetric manifold, concircular Ricci tensor, special weakly concircular Ricci symmetric and Lorentzian concircular structure manifold.

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1 Introduction

The notion of weakly symmetric manifolds was introduced by Tamassy and Binh [8]. A non-flat Riemannian manifold \((M^n, g)\) \((n > 2)\) is called weakly symmetric manifold if its curvature tensor \(R\) of type \((0,4)\) satisfies the condition

\[
\begin{align*}
\end{align*}
\]

for all vector fields \(X, Y, Z, U, V \in \chi(M^n)\), \(\chi(M)\) being the Lie-algebra of the smooth vector fields of \(M\), where \(A, B, H, D\) and \(E\) are \(1\)-forms (not simultaneously zero) and \(\nabla\) denote the operator of the covariant differentiations with respect to Riemannian metric \(g\). The \(1\)-forms are called the associated \(1\)-forms of the manifold and \(n\)-dimensional manifold of this kind is denoted by \((WS)_n\).

In 1999, De and Bandyopadhyay [2] studied a \((WS)_n\) and prove that in such a manifold the associated \(1\)-forms \(B = H\) and \(D = E\). Hence from (1.1) reduces to the following:

\[
\begin{align*}
\end{align*}
\]

A transformation of \(n\)-dimensional Riemannian manifold \(M\), which transform every geodesic circle of \(M\) into a geodesic circle, is called a concircular transformation [11]. The intersecting invariant of a concircular transformation is the concircular curvature tensor \(\tilde{C}\) which is defined by [11].

\[
\tilde{C}(Y, Z, U, V) = R(Y, Z, U, V) - \frac{k}{n(n-1)} \left[ g(Y, U)g(Y, V) - g(Y, V)g(Z, V) \right],
\]

where \(k\) is the scalar curvature of the manifold.

Recently Shaikh and Hui [5] introduced the notion of weakly concircular symmetric manifolds. A Riemannian manifold is called weakly concircular symmetric manifold if its concircular curvature tensor \(\tilde{C}\) of type \((0,4)\) is not identically zero and satisfies the condition

\[
\begin{align*}
(\nabla_X \tilde{C})(Y, Z, U, V) &= A(X)\tilde{C}(Y, Z, U, V) + B(Y)\tilde{C}(X, Z, U, V) \\
\end{align*}
\]

for all vector fields \(X, Y, Z, U, V \in \chi(M^n)\) where \(A, B, H, D\) and \(E\) are \(1\)-form (not simultaneously zero) an \(n\)-dimensional manifold of this kind is denoted by \((WC\tilde{S})_n\). Also it is known that in a \((WC\tilde{S})_n\) the associated \(1\)-forms \(B = H\) and \(D = E\), and hence the defining the condition (1.4) of a \((WC\tilde{S})_n\) reduces to the following form:

\[
\begin{align*}
(\nabla_X \tilde{C})(Y, Z, U, V) &= A(X)\tilde{C}(Y, Z, U, V) + B(Y)\tilde{C}(X, Z, U, V) \\
\end{align*}
\]
where \( A, B \) and \( D \) are 1-forms (not simultaneously zero).

Again Tamassy and Binh [9] introduced the notion of weakly Ricci symmetric manifolds. A Riemannian manifold \((\mathcal{M}^n, g), (n > 2)\) is called weakly Ricci symmetric manifold if its Ricci tensor \( S \) of type \((0,2)\) is not identically zero and satisfies the condition:

\[
\left( \nabla_X S \right)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + D(Z)S(Y, X),
\]

where \( A, B \) and \( D \) are three non-zero 1-forms called the associate 1-forms of the manifold, and \( \nabla \) is the operator of covariant differentiation with respect to metric \( g \). Such \( n \)-dimensional manifold is denoted by \((\text{WRS})_n\). If \( A = B = D \) then is called pseudo Ricci symmetric.

Let \( \{e_i : i = 1, 2, \ldots, n\} \) be an orthonormal basis of the tangent space at each point of the manifold and let

\[
\tilde{S}(Y, V) = \sum_{i=1}^{n} \tilde{C}(Y, e_i, e_i, Y).
\]

Then from (1.3), we have

\[
\tilde{S}(Y, V) = S(Y, V) - \frac{k}{n} g(Y, V),
\]

(1.7)

The tensor \( \tilde{S} \) is called the concircular Ricci symmetric tensor which is symmetric tensor of type \((0,2)\). In [1] De and Ghose introduced the notion of weakly concircular Ricci symmetric manifolds. A Riemannian manifold \((\mathcal{M}^n, g), (n > 2)\) is called weakly concircular Ricci symmetric manifolds [1] if its concircular Ricci tensor \( \tilde{S} \) of type \((0,2)\) is not identically zero satisfies the condition:

\[
\left( \nabla_X \tilde{S} \right)(Y, Z) = A(X)\tilde{S}(Y, Z) + B(Y)\tilde{S}(X, Z) + D(Z)\tilde{S}(Y, X),
\]

(1.8)

where \( A, B \) and \( D \) are three 1-form (not simultaneously zero). If \( A = B = D \) then \( \mathcal{M}^n \) is called pseudo concircular Ricci symmetric. A Riemannian manifold is called special weakly Ricci symmetric manifold if

\[
\left( \nabla_X S \right)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X),
\]

(1.9)

where \( A \) is a 1-form and is defined by

\[
A(X) = g(X, \rho).
\]

(1.10)

where \( \rho \) is the associated vector field.

Motivated by above studied we define and study special weakly concircular Ricci symmetric manifold. An \( n \)-dimensional Riemannian manifold is called special weakly concircular Ricci symmetric manifolds. If

\[
\left( \nabla_X \tilde{S} \right)(Y, Z) = 2A(X)\tilde{S}(Y, Z) + A(Y)\tilde{S}(X, Z) + A(Z)\tilde{S}(Y, X).
\]

(1.11)
An \((2n + 1)\)-dimensional Lorentzian manifold \(M\) is smooth connected Para contact Hausdorff manifold with Lorentzian metric \(g\), that is, \(M\) admits a smooth symmetric tensor field \(g\) of type \((0, 2)\) such that for each point \(p \in M\), the tensor \(g_p : T_pM \times T_pM \to \mathbb{R}\) is a non degenerate inner product of signature \((-1, +, \ldots, +)\) where \(T_pM\) denotes the tangent space of \(M\) at \(p\) and \(\mathbb{R}\) is the real number space. In a Lorentzian manifold \((M, g)\) a vector field \(\rho\) defined by
\[
g(X, \rho) = A(X)
\]
for any vector field \(X \in \chi(M)\) is said to be concircular vector field [5] if
\[
(\nabla_X A)(Y) = \alpha [g(X, Y) + \omega(X)A(Y)]
\]
where \(\alpha\) is a non zero scalar function, \(A\) is a 1-form and \(\omega\) is a closed 1-form.

Let \(M\) be a Lorentzian manifold admitting a unit time like concircular vector field \(\xi\), called the characteristic vector field of the manifold. Then we have
\[
g(\xi, \xi) = -1,
\]
the equation (1.13) of the following form holds:
\[
(\nabla_X \eta)(Y) = \alpha [g(X, Y) + \eta(X)\eta(Y)] \quad (\alpha \neq 0),
\]
for all vector field \(X, Y\), where \(\nabla\) denotes the operator of covariant differentiation with respect to Lorentzian metric \(g\) and \(\alpha\) is a non zero scalar function satisfying
\[
(\nabla_X \alpha)(Y) = (X\alpha) = \rho \eta(X),
\]
where \(\rho\) being a scalar function. If we put
\[
\phi X = \frac{1}{\alpha} \nabla_X \xi,
\]
Then from (1.14) and (1.16), we have
\[
\phi^2 X = X + \eta(X)\xi,
\]
from which it follows that \(\phi\) is a symmetric \((1, 1)\)-tensor. Thus the Lorentzian manifold \(M\) together with unit time like concircular vector field \(\xi\), it’s associate 1-form \(\eta\) and \((1, 1)\)-tensor field \(\phi\) is said to be Lorentzian concircular structure manifolds (briefly \(\text{LCS}_{2n+1}\)-manifold) [6]. In particular if \(\alpha = 1\), then the manifold becomes LP-Sasakian structure of Matsumoto [3].
2 Lorentzian Concircular Structure manifolds

A differentiable manifold $M$ of dimension $(2n+1)$ is called $(\text{LCS})_{2n+1}$-manifold if it admits a $(1,1)$-tensor $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and a Lorentzian metric $g$ which satisfy the following

$$\eta(\xi) = -1, \quad (2.1)$$

$$\phi^2 = 1 + \eta \otimes \xi, \quad (2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.3)$$

$$g(X, \xi) = \eta(X), \quad (2.4)$$

$$\phi \xi = 0, \quad \eta(\phi X) = 0, \quad (2.5)$$

for all $X, Y \in TM$. Also in a $(\text{LCS})_{2n+1}$-manifold the following relations are satisfied [7].

$$\eta(R(X, Y)Z) = (\alpha^2 - \rho) [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \quad (2.6)$$

$$R(X, Y)\xi = (\alpha^2 - \rho) [\eta(Y)X - \eta(X)Y], \quad (2.7)$$

$$R(\xi, X)Y = (\alpha^2 - \rho) [g(X, Y)\xi - \eta(Y)X], \quad (2.8)$$

$$R(\xi, X)\xi = (\alpha^2 - \rho) [\eta(X)\xi + X], \quad (2.9)$$

$$\nabla_X \phi)(Y) = \alpha [g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \quad (2.10)$$

$$S(X, \xi) = 2n(\alpha^2 - \rho)\eta(X), \quad (2.11)$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n(\alpha^2 - \rho)\eta(X)\eta(Y), \quad (2.12)$$

**Definition 2.1** A Lorentzian concircular structure manifold is said to be $\eta$-Einstein if the Ricci operator $Q$ satisfies

$$Q = aI d + b\eta \otimes \xi,$$

where $a$ and $b$ are smooth functions on the manifolds, In particular if $b = 0$, then $M$ is an Einstein manifold.
3 Main Results

Definition 3.1 A Lorentzian concircular structure manifold $(M^{2n+1}, g)$ $(n > 1)$ is said to be weakly concircular symmetric if its concircular curvature tensor $\tilde{\nabla}$ of type $(0,4)$ satisfies (1.5).

Substituting $Y = V = e_i$ in (1.5) and taking summation over $i$, $1 \leq i \leq 2n + 1$, we get

$$
(\nabla_X S)(Z, U) - \frac{d\kappa(X)}{n} g(Z, U) = A(X) [S(Z, U) - \frac{\kappa}{n} g(Z, U)] + B(Z) [S(X, U) - \frac{\kappa}{n} g(X, U) + D(U)] [S(X, Z) - \frac{\kappa}{n} g(X, Z)] + B(R(X, Z)U) + D(R(X, U)Z)
$$

(3.1)

Again setting $X = Z = U = \xi$, in (3.1) and using (2.7) and (2.11), we have

$$
A(\xi) + B(\xi) + D(\xi) = \frac{dk(\xi)}{k - 2n^2(\alpha^2 - \rho)}
$$

(3.2)

This leads to the following result.

Theorem 3.1 In a weakly concircular symmetric Lorentzian concircular structure manifold $(M^{2n+1}, g)$ $(n > 1)$ the relation (3.2) holds.

Corollary 3.1 In a weakly concircular symmetric Lorentzian concircular structure manifold $(M^{2n+1}, g)$ $(n > 1)$ the sum of 1-forms $A, B$ and $D$ is zero everywhere if and only if the scalar curvature $\kappa$ of the manifold is constant.

Next, putting $X$ and $Z$ by $\xi$, in (3.1) and using (2.4), (2.7) and (2.11) we obtain

$$
(\nabla_X S)(\xi, U) - \frac{d\kappa(\xi)}{n} \eta(U) = [A(\xi) + B(\xi)] \left\{ 2n(\alpha^2 - \rho) - \frac{\kappa}{n} \right\} \eta(U) + \left[ \frac{\kappa}{n} - 2n(\alpha^2 - \rho) - \frac{\kappa}{n(n-1)} + 1 \right] D(U) + D(\xi) \left[ (\alpha^2 - \rho) - \frac{\kappa}{n(n-1)} \right] \eta(U).
$$

(3.3)

Also from (2.11), we have

$$
(\nabla_\xi S)(\xi, U) = 0.
$$

(3.4)

In view of (3.2) and (3.4), equation (3.3) reduces to

$$
D(U) = \left[ k + (n-1) \left\{ k - 2n^2(\alpha^2 - \rho) \right\} \right] D(\xi) \eta(U).
$$

(3.5)

Next setting $X = U = \xi$ in (3.1) and proceeding in the similar manner as above, we have

$$
B(Z) = \left[ k - (n-1) \left\{ k - 2n^2(\alpha^2 - \rho) \right\} \right] B(\xi) \eta(Z),
$$

(3.6)
Again, substituting $Z = U = \xi$, (3.1), we obtain

\[
\begin{align*}
(\nabla_X S)(\xi, \xi) + \frac{d(\xi)}{\kappa} &= A(X) \left[ \frac{\kappa}{n} - 2n(\alpha^2 - \rho) \right] + \left[ \frac{\kappa}{n(n-1)} - (\alpha^2 - \rho) \right] \\
B(X) + D(X) + (B(\xi) + D(\xi))\eta(X) + [2n(\alpha^2 - \rho) - \frac{\kappa}{n}] [B(\xi) + D(\xi)]\eta(X)
\end{align*}
\] (3.7)

On the other hand we have

\[
(\nabla_\xi S)(\xi, \xi) = \nabla_X S(\xi, \xi) - 2S(\nabla_\xi \xi, \xi),
\]
which yield by using (1.16) and (2.1) that.

\[
(\nabla_\xi S)(\xi, \xi) = -2n(\alpha^2 - \rho)\xi,
\] (3.8)

In view of (3.7) and (3.8), we get

\[
A(X) = \left[ \frac{d(\xi)}{\kappa} - \frac{2n(\alpha^2 - \rho)\xi}{\kappa - 2n(\alpha^2 - \rho)} \right] - \left[ \frac{\kappa}{n(n-1)} \frac{d(\xi)}{\kappa - 2n(\alpha^2 - \rho)} \right] \\
\{B(X) + D(X)\} - [D(\xi) + B(\xi)]\eta(X) - \left\{ \frac{d(\xi)}{\kappa - 2n(\alpha^2 - \rho)} - A(\xi) \right\} \eta(X)
\] (3.9)

This leads to the following result.

**Theorem 3.2.** In a weakly concircular symmetric Lorentzian concircular structure manifold $(M^{2n+1}, g)$ $(n > 1)$ the associated 1-forms $D, B$ and $A$ are given by (3.5) (3.6) and (3.9) respectively.

**Definition 3.2** A Lorentzian concircular structure manifold $(M^{2n+1}, g)$ $(n > 1)$ is said to be weakly concircular Ricci symmetric if its concircular Ricci tensor $S$ of type $(0, 2)$ satisfies (1.8).

In view of (1.8) and (1.9) yield

\[
\begin{align*}
(\nabla_X S)(Y, Z) - \frac{d(\xi)}{n} g(\xi, Z) &= A(X) \left[ S(Y, Z) - \frac{\kappa}{n} g(Y, Z) \right] + B(Y) \\
&= S(X, Z) - \frac{\kappa}{n} g(X, Z) + D(Z) \left[ S(X, Y) - \frac{\kappa}{n} g(X, Y) \right]
\end{align*}
\] (3.10)

Setting $X = Y = Z = \xi$ in above we get the relation (3.2). Hence we can state the following

**Theorem 3.3.** In a weakly concircular Ricci symmetric Lorentzian concircular structure manifold $(M^{2n+1}, g)$ $(n > 1)$ the relations (3.2) holds

**Corollary 3.2.** In a weakly concircular Ricci symmetric Lorentzian concircular structure manifold $(M^{2n+1}, g)$ $(n > 1)$ the sum of 1-forms $A, B$ and $D$ is zero everywhere and only if the scalar curvature $k$ of the manifold is constant.

Now, taking $X$ and $Y$ by $\xi$, in (3.10), we have
\[(\nabla_{\xi} S)(\xi, Z) - \frac{dk(X)}{n} \eta(Z) = (A(\xi) + B(\xi)) + D(Z) \left[ S(\xi, \xi) + \frac{k}{n} \right] \] (3.11)

In view of (3.2) and (3.4), equation (3.11) yields.

\[D(Z) = -\frac{dk(\xi)}{k - 2n^2(\alpha^2 - \rho)} \eta(Z) + \left[ \frac{dk(\xi)}{k - 2n^2(\alpha^2 - \rho)} - D(\xi) \right] \eta(Z),\] (3.12)

Again putting \(X = Z = \xi\) in (3.11) and proceeding in a similar manner as above, we get

\[B(Y) = -\frac{dk(\xi)}{k - 2n^2(\alpha^2 - \rho)} \eta(Y) + \left[ \frac{dk(\xi)}{k - 2n^2(\alpha^2 - \rho)} - B(\xi) \right] \eta(Y),\] (3.13)

\[A(X) = \frac{-2n^2(\alpha^2 - \rho)\xi \eta(X)}{k - 2n^2(\alpha^2 - \rho)} + \frac{dk(X)}{k - 2n^2(\alpha^2 - \rho)} + \left[ \frac{dk(\xi)}{k - 2n^2(\alpha^2 - \rho)} - A(\xi) \right] \eta(X),\] (3.14)

This leads to the following result.

**Theorem 3.4.** In a weakly concircular Ricci symmetric Lorentzian concircular structure manifold \((M^{2n+1}, g)\) \((n > 1)\) the associated 1-form \(D, B\) and \(A\) are given by (3.12) (3.13) and (3.14) respectively.

Adding equations (3.12) (3.13) and (3.14), using (3.3) we obtain

\[A(X) + B(X) + D(X) = \frac{dk(X) - 2n^2(\alpha^2 - \rho)\xi}{k - 2n^2(\alpha^2 - \rho)}\] (3.15)

for any vector field \(X\).

This leads to the following result.

**Theorem 3.5.** In a weakly concircular Ricci symmetric Lorentzian concircular structure manifold \((M^{2n+1}, g)\) \((n > 1)\) the sum of the associated 1-form \(A, B\) and \(D\) is given by (3.15).

**Corollary 3.3** There is no weakly concircular Ricci symmetric Lorentzian concircular structure manifold \((M^{2n+1}, g)\) \((n > 1)\) unless the sum of the 1-forms is everywhere zero if \(dk(X) = 2n^2(\alpha^2 - \rho)\xi\).

Also taking cyclic sum of (1.11), we get

\[\left( \nabla_X \tilde{S} \right)(Y, Z) + \left( \nabla_Y \tilde{S} \right)(Z, Y) + \left( \nabla_Z \tilde{S} \right)(X, Y) = 4 \ A(X) \tilde{S}(Y, Z) + A(Y) \tilde{S}(X, Z) + A(Z) \tilde{S}(Y, X),\] (3.16)
\[ A(X)\tilde{S}(Y,Z) + A(Y)\tilde{S}(X,Z) + A(Z)\tilde{S}(Y,X) = 0. \]

Taking \( Z = \xi \) in above and then using (1.7), (1.10) and (2.11), we obtain

\[ \left[ 2n^2(\alpha^2 - \rho) - \frac{K}{n} \right] \{ A(X)\eta(Y) + A(Y)\eta(X) \} + \eta(\rho)S(X,Y) = 0. \quad (3.17) \]

Again taking \( Z = \xi \) in (3.17), we get

\[ 2\eta(\rho)\eta(X) = A(X) \quad (3.18) \]

Taking \( X = \xi \) in (3.18) and using (1.7), we yields

\[ \eta(\rho) = 0. \quad (3.19) \]

In view of (3.18) and (3.19), we get \( A(X) = 0, \forall X. \)

This leads to the following result.

**Theorem 3.6.** If a special weakly concircular Ricci symmetric Lorentzian concircular structure manifold \((M^{2n+1}, g) \quad (n > 1)\) admits Cyclic Ricci tensor then the 1-form \( A \) must vanishes.

Finally for Einstein manifold \((\nabla X)S(Y,Z) = 0\) and \( S(Y,Z) = ag(Y,Z). \) Then (1.7) and (1.11), we get

\[ -\frac{d\kappa(X)}{n}g(Y,Z) = 2A(X) \left[ \left( a - \frac{K}{n} \right) g(Y,Z) \right] + A(Y) \left[ \left( a - \frac{K}{n} \right) g(X,Z) \right] + A(Z) \left[ \left( a - \frac{K}{n} \right) g(X,Y) \right], \quad (3.20) \]

Plugging \( Z = X = Y = \xi, \) in (3.20), we obtain that

\[ 4\eta(\rho)(an - \kappa) = d\kappa(\xi) \]

which implies that if \( \kappa \) is constant then \( \eta(\rho) = 0, \) that is \( A(Y) = 0, \forall Y. \) Therefore we state the results

**Theorem 3.7.** A special weakly concircular Ricci symmetric Lorentzian concircular structure manifold \((M^{2n+1}, g) \quad (n > 1)\) can not Einstein manifold if the scalar curvature of the manifold is constant.

**Corollary 3.4.** In a special weakly concircular Ricci symmetric Lorentzian concircular structure manifold \((M^{2n+1}, g) \quad (n > 1)\) the 1-form \( A \) is given by

\[ A(\xi) = \frac{d\kappa(\xi)}{2} - \frac{2n^2(\alpha^2 - \rho)}{2 \{ k - 2n^2(\alpha^2 - \rho) + n \}} \]
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