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Composition operators in hyperbolic general Besov-type spaces

A. El-Sayed Ahmed[1,2]
[1] *Sohag University, Faculty of Science, Department of Mathematics, 82524 Sohag, Egypt.*
[2] *Taif University, Faculty of Science, Mathematics, Department, box 888 El-Hawiyah, El-Taif 5700, Saudi Arabia.*
ahsayed80@hotmail.com

M. A. Bakhit
Department of Mathematics, Faculty of Science, Assiut Branch, Al-Azhar University, Assiut 32861, Egypt.
mabakhit2007@hotmail.com

ABSTRACT

In this paper we introduce natural metrics in the hyperbolic α-Bloch and hyperbolic general Besov-type classes $F^*(p, q, s)$. These classes are shown to be complete metric spaces with respect to the corresponding metrics. Moreover, compact composition operators C_ϕ acting from the hyperbolic α-Bloch class to the class $F^*(p, q, s)$ are characterized by conditions depending on an analytic self-map $\phi : \mathbb{D} \to \mathbb{D}$.

RESUMEN

En este artículo introducimos una métrica natural en las clases hiperbólicas α-Bloch y tipo Besov generales. Estas clases se muestra que son espacios métricos completos respecto de las métricas correspondientes. Además se caracterizan los operadores de composición compactos C_ϕ que actúan desde las clases hiperbólicas α-Bloch en la clase $F^*(p, q, s)$ por condiciones que dependen de la autoaplicación analítica $\phi : \mathbb{D} \to \mathbb{D}$.

Keywords and Phrases: Hyperbolic classes, composition operators, Lipschitz continuous, α-Bloch space, $F^*(p, q, s)$ class.

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1 Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc of the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ it's boundary. Let $\mathcal{H}(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$ and let $B(\mathbb{D})$ be the subset of $\mathcal{H}(\mathbb{D})$ consisting of those $f \in \mathcal{H}(\mathbb{D})$ for which $|f(z)| < 1$ for all $z \in \mathbb{D}$. Also, $dA(z)$ be the normalized area measure on $\mathbb{D}$ so that $A(\mathbb{D}) \equiv 1$.

Let the Green's function of $\mathbb{D}$ be defined as $g(z,a) = \log\frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, for $z, a \in \mathbb{D}$ is the Möbius transformation related to the point $a \in \mathbb{D}$.

If (X, d) is a metric space, we denote the open and closed balls with center x and radius $r > 0$ by $B(x,r) := \{y \in X : d(y,x) < r\}$ and $\bar{B}(x,r) := \{y \in X : d(x,y) \leq r\}$, respectively.

Hyperbolic function classes are usually defined by using either the hyperbolic derivative $f^*(z) = \frac{|f'(z)|}{1-|f(z)|^2}$ of $f \in B(\mathbb{D})$, or the hyperbolic distance $\rho(f(z),0) := \frac{1}{2}\log\left(\frac{1+|f(z)|}{1-|f(z)|}\right)$ between $f(z)$ and zero.

A function $f \in B(\mathbb{D})$ is said to belong to the hyperbolic α-Bloch class $\mathcal{B}^*_\alpha$ if

$$\|f\|_{\mathcal{B}^*_\alpha} = \sup_{z\in\mathbb{D}} f^*(z)(1-|z|^2)^\alpha < \infty,$$

The little hyperbolic Bloch-type class $\mathcal{B}^*_{\alpha,0}$ consists of all $f \in \mathcal{B}^*_\alpha$ such that

$$\lim_{|z|\to 1} f^*(z)(1-|z|^2)^\alpha = 0.$$

The usual α-Bloch spaces $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha,0}$ are defined as the sets of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\alpha < \infty,$$

and

$$\lim_{|z|\to 1} |f'(z)|(1-|z|^2)^\alpha = 0,$$

respectively.

It is obvious that $\mathcal{B}^*_\alpha$ is not a linear space since the sum of two functions in $B(\mathbb{D})$ does not necessarily belong to $B(\mathbb{D})$.

We now turn to consider hyperbolic $F(p,q,s)$ type classes, which will be called $F^*(p,q,s)$. For $0 < p, s < \infty$, $-2 < q < \infty$, the hyperbolic class $F^*(p,q,s)$ consists of those functions $f \in B(\mathbb{D})$ for which (see [7])

$$\|f\|^p_{F^*(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^*(z))^p (1-|z|^2)^q g^s(z,a)\, dA(z) < \infty.$$

Moreover, we say that $f \in F^*(p,q,s)$ belongs to the class $F^*_0(p,q,s)$ if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} (f^*(z))^p (1-|z|^2)^q g^s(z,a)\, dA(z) = 0.$$

The usual general Besov-type spaces $F(p, q, s)$ (defined using the conventional derivative f' instead of f^*) and their norms are denoted by the same symbols but with f'.

Yamashita was probably the first one considered systematically hyperbolic function classes. He introduced and studied hyperbolic Hardy, BMOA and Dirichlet classes in [14, 15, 16] and others. More recently, Smith studied inner functions in the hyperbolic little Bloch-class [11], and the hyperbolic counterparts of the Q_p spaces were studied by Li in [7] and Li et. al. in [8]. Further, hyperbolic Q_p classes and composition operators studied by Pérez-González et. al. in [10]. Very recently the first author in [1], gave some characterizations of hyperbolic $Q(p, \alpha)$ classes and the hyperbolic (p, α)-Bloch classes by composition operators.

In this paper we will study the hyperbolic α-Bloch classes $\mathcal{B}^*_\alpha$ and the general hyperbolic $F^*(p, q, s)$ type classes. We will also give some results to characterize Lipschitz continuous and compact composition operators mapping from the hyperbolic α-Bloch class $\mathcal{B}^*_\alpha$ to $F^*(p, q, s)$ class by conditions depending on the symbol ϕ only.

Note that the general hyperbolic $F^*(p, q, s)$ type classes include the class of so-called Q^*_p classes and the class of (hyperbolic) Besov class B^*_p. Thus, the results are generalizations of the recent results of Pérez-González, Rättyä and Taskinen [10].

For any holomorphic self-mapping ϕ of $\mathbb{D}$. The symbol ϕ induces a linear composition operator $C_\phi(f) = f \circ \phi$ from $\mathcal{H}(\mathbb{D})$ or $B(\mathbb{D})$ into itself. The study of composition operator C_ϕ acting on spaces of analytic functions has engaged many analysts for many years (see e.g. [2, 3, 4, 5, 8, 9, 17] and others).

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that $\|T(f)\|_Y \le C\|f\|_X$ for all maps $f \in X$. By elementary functional analysis, a linear operator between normed spaces is bounded if and only if it is continuous, and the boundedness is trivially also equivalent to the Lipschitz-continuity. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y contained in $B(\mathbb{D})$ or $\mathcal{H}(\mathbb{D})$, $T : X \to Y$ is compact if and only if for each bounded sequence $\{x_n\} \in X$, the sequence $\{Tx_n\} \in Y$ contains a subsequence converging to a function $f \in Y$.

Definition 1.1. *A composition operator* $C_\phi : \mathcal{B}^*_\alpha \to F^*(p, q, s)$ *is said to be bounded, if there is a positive constant* C *such that* $\|C_\phi f\|_{F^*(p,q,s)} \le C\|f\|_{\mathcal{B}^*_\alpha}$ *for all* $f \in \mathcal{B}^*_\alpha$.

Definition 1.2. *A composition operator* $C_\phi : \mathcal{B}^*_\alpha \to F^*(p, q, s)$ *is said to be compact, if it maps any ball in* $\mathcal{B}^*_\alpha$ *onto a precompact set in* $F^*(p, q, s)$.

The following lemma follows by standard arguments similar to those outline in Lemma 3.8 of [12]. Hence we omit the proof.

Lemma 1.3. *Assume* ϕ *is a holomorphic mapping from* $\mathbb{D}$ *into itself. Let* $0 < p, s < \infty$, $-1 <$

$q < \infty$ *and* $0 < \alpha < \infty$. *Then* $C_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ *is compact if and only if for any bounded sequence* $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{B}^*_\alpha$ *which converges to zero uniformly on compact subsets of* $\mathbb{D}$ *as* $n \to \infty$, *we have* $\lim_{n\to\infty} \|C_\phi f_n\|_{F^*(p,q,s)} = 0$.

The following lemma can be found in [6], Theorem 2.1.1.

Lemma 1.4. *Let* $0 < \alpha < \infty$, *then there exist two holomorphic maps* $f, g : \mathbb{D} \to \mathbb{C}$ *such that for some constant* C,

$$\big(f'(z) + g'(z)\big)(1-|z|^2)^\alpha \geq C > 0, \qquad \textit{for each}\; z \in \mathbb{D}.$$

2 Hyperbolic classes and natural metrics

In this section we introduce natural metrics on the hyperbolic α-Bloch classes $\mathcal{B}^*_\alpha$ and the classes $F^*(p,q,s)$.

Let $0 < p, s < \infty, -2 < q < \infty$ and $0 < \alpha < 1$. First, we can find a natural metric in $\mathcal{B}^*_\alpha$ (see [10]) by defining

$$d(f,g;\mathcal{B}^*_\alpha) := d_{\mathcal{B}^*_\alpha}(f,g) + \|f-g\|_{\mathcal{B}_\alpha} + |f(0)-g(0)|, \tag{1}$$

where

$$d_{\mathcal{B}^*_\alpha}(f,g) := \sup_{z\in\mathbb{D}} \left| \frac{f'(z)}{1-|f(z)|^2} - \frac{g'(z)}{1-|g(z)|^2} \right| (1-|z|^2)^\alpha,$$

for $f, g \in \mathcal{B}^*_\alpha$. The presence of the conventional α-Bloch-norm here perhaps unexpected. It is motivated by example (see [10], Example in Section 7). It shows the phenomenon that, though trivially $d_{\mathcal{B}^*_\alpha}(f,0) \geq \|f\|_{\mathcal{B}_\alpha}$ for all $f \in \mathcal{B}^*_\alpha$, the same does no more hold for the differences of two functions: there does not even exist a constant $C > 0$ such that

$$\sup_{z\in\mathbb{D}} \left| \frac{f'(z)}{1-|f(z)|^2} - \frac{g'(z)}{1-|g(z)|^2} \right| (1-|z|^2)^\alpha \geq C\|f-g\|_{\mathcal{B}_\alpha}$$

would hold for all $f, g \in \mathcal{B}^*_\alpha, 0 < \alpha < 1$.

For $f, g \in F^*(p,q,s)$, define their distance by

$$d(f,g;F^*(p,q,s)) := d_{F^*}(f,g) + \|f-g\|_{F(p,q,s)} + |f(0)-g(0)|,$$

where

$$d_{F^*}(f,g) := \left(\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \left| \frac{f'(z)}{1-|f(z)|^2} - \frac{g'(z)}{1-|g(z)|^2} \right|^p (1-|z|^2)^q g^s(z,a)\, dA(z) \right)^{\frac{1}{p}}.$$

The following characterization of complete metric space $d(.,.;\mathcal{B}^*_\mu)$ can be proved as in Proposition 2.1 of [10].

Proposition 2.1. *The class* $\mathcal{B}^*_\alpha$ *equipped with the metric* $d(.,.;\mathcal{B}^*_\alpha)$ *is a complete metric space. Moreover,* $\mathcal{B}^*_{\alpha,0}$ *is a closed (and therefore complete) subspace of* $\mathcal{B}^*_\alpha$.

Now we prove the following proposition

Proposition 2.2. *The class* $F^*(p,q,s)$ *equipped with the metric* $d(.,.;F^*(p,q,s))$ *is a complete metric space. Moreover,* $F^*_0(p,q,s)$ *is a closed (and therefore complete) subspace of* $F^*(p,q,s)$.

Proof. For $f, g, h \in F^*(p,q,s)$, then clearly

- $d(f,g;F^*(p,q,s)) \geq 0$,
- $d(f,f;F^*(p,q,s)) = 0$,
- $d(f,g;F^*(p,q,s)) = 0$ implies $f = g$.
- $d(f,g;F^*(p,q,s)) = d(g,f;F^*(p,q,s))$,
- $d(f,h;F^*(p,q,s)) \leq d(f,g;F^*(p,q,s)) + d(g,h;F^*(p,q,s))$.

Hence, d is metric on $F^*(p,q,s)$.

For the completeness proof, let $(f_n)_{n=0}^{\infty}$ be a Cauchy sequence in the metric space $F^*(p,q,s)$, that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ such that $d(f_n, f_m) < \varepsilon$, for all $n, m > N$. Since $f_n \in B(\mathbb{D})$ such that f_n converges to f uniformly on compact subsets of $\mathbb{D}$. Let $m > N$ and $0 < r < 1$. Then Fatou's lemma yields

$$\begin{aligned}
&\int_{\mathbb{D}(0,r)} \left| \frac{f'(z)}{1-|f(z)|^2} - \frac{f'_m(z)}{1-|f_m(z)|^2} \right|^p (1-|z|^2)^q g^s(z,a)\,dA(z) \\
=\ & \int_{\mathbb{D}(0,r)} \lim_{n\to\infty} \left| \frac{f'_n(z)}{1-|f_n(z)|^2} - \frac{f'_m(z)}{1-|f_m(z)|^2} \right|^p (1-|z|^2)^q g^s(z,a)\,dA(z) \\
\leq\ & \lim_{n\to\infty} \int_{\mathbb{D}} \left| \frac{f'_n(z)}{1-|f_n(z)|^2} - \frac{f'_m(z)}{1-|f_m(z)|^2} \right|^p (1-|z|^2)^q g^s(z,a)\,dA(z) \leq \varepsilon^p.
\end{aligned} \tag{2}$$

By letting $r \to 1^-$, it follows from inequalities (2) and $(a+b)^p \leq 2^p(a^p + b^p)$ that

$$\int_{\mathbb{D}} (f^*(z))^p (1-|z|^2)^q g^s(z,a)\,dA(z) \leq 2^p \varepsilon^p + 2^p \int_{\mathbb{D}} (f^*_m(z))^p (1-|z|^2)^q g^s(z,a)\,dA(z). \tag{3}$$

This yields

$$\|f\|^p_{F^*(p,q,s)} \leq 2^p \varepsilon^p + 2^p \|f_m\|^p_{F^*(p,q,s)},$$

and thus $f \in F^*(p,q,s)$. We also find that $f_n \to f$ with respect to the metric of $F^*(p,q,s)$.

The second part of the assertion follows by (3).

3 Compactness of C_ϕ in hyperbolic classes

For $0 < p, s < \infty, -2 < q < \infty$ and $0 < \alpha < \infty$. We define the following notations:

$$\Phi_\phi(p,q,s,a) = \int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}}(1-|z|^2)^q g^s(z,a)dA(z)$$

and

$$\Omega_{\phi,r}(p,q,s,a) = \int_{|\phi|\geq r} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}}(1-|z|^2)^q g^s(z,a)dA(z).$$

Theorem 3.1. *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself. Let $0 \leq p < \infty$, $0 \leq s \leq 1$, $-1 < q < \infty$ and $0 < \alpha \leq 1$. Then the following are equivalent:*

(i) $C_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ *is bounded;*

(ii) $C_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ *is Lipschitz continuous;*

(iii) $\sup_{a\in\mathbb{D}} \Phi_\phi(p,q,s,a) < \infty.$

Proof. First, assume that (i) holds, then there exists a constant C such that

$$\|C_\phi f\|_{F^*(p,q,s)} \leq C\|f\|_{\mathcal{B}^*_\alpha}, \quad \text{for all } f \in \mathcal{B}^*_\alpha.$$

For given $f \in \mathcal{B}^*_\alpha$, the function $f_t(z) = f(tz)$, where $0 < t < 1$, belongs to $\mathcal{B}^*_\alpha$ with the property $\|f_t\|_{\mathcal{B}^*_\alpha} \leq \|f\|_{\mathcal{B}^*_\alpha}$. Let f, g be the functions from Lemma 1.4, such that

$$\frac{1}{(1-|z|^2)^\alpha} \leq f^*(z) + g^*(z),$$

for all $z \in \mathbb{D}$, so that

$$\frac{|\phi'(z)|}{(1-|\phi(z)|)^\alpha} \leq (f\circ\phi)^*(z) + (g\circ\phi)^*(z).$$

Thus, the inequalities

$$\begin{aligned} &\int_{\mathbb{D}} \frac{|t\phi'(z)|^p}{(1-|t\phi(z)|^2)^{\alpha p}}(1-|z|^2)^q g^s(z,a)dA(z) \\ \leq\ & 2^p \int_{\mathbb{D}} \Big[\big((f\circ t\phi)^*(z)\big)^p + \big((g\circ t\phi)^*(z)\big)^p\Big](1-|z|^2)^q g^s(z,a)dA(z) \\ \leq\ & 2^p \|C_\phi\|^p \big(\|f\|^p_{\mathcal{B}^*_\alpha} + \|g\|^p_{\mathcal{B}^*_\alpha}\big). \end{aligned}$$

This estimate together with the Fatou's lemma implies (iii).

Conversely, assuming that (iii) holds and that $f \in \mathcal{B}_\alpha^*$, we see that

$$\begin{aligned}
& \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \big((f\circ\phi)^*(z)\big)^p (1-|z|^2)^q g^s(z,a)\,dA(z)) \\
= & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \big(f^*(\phi(z))\big)^p |\phi'(z)|^p (1-|z|^2)^q g^s(z,a)\,dA(z) \\
\le & \|f\|_{\mathcal{B}_\alpha^*}^p \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}} (1-|z|^2)^q g^s(z,a)\,dA(z).
\end{aligned}$$

Hence, it follows that (i) holds.

(ii)$\Longleftrightarrow$(iii). Assume first that $C_\phi : \mathcal{B}_\alpha^* \to F^*(p,q,s)$ is Lipschitz continuous, that is, there exists a positive constant C such that

$$d(f\circ\phi, g\circ\phi; F^*(p,q,s)) \le C d(f,g;\mathcal{B}_\alpha^*), \quad \text{for all } f,g \in \mathcal{B}_\alpha^*.$$

Taking $g = 0$, this implies

$$\|f\circ\phi\|_{F^*(p,q,s)} \le C\big(\|f\|_{\mathcal{B}_\alpha^*} + \|f\|_{\mathcal{B}_\alpha} + |f(0)|\big), \quad \text{for all } f \in \mathcal{B}_\alpha^*. \tag{4}$$

The assertion (iii) for $\alpha = 1$ follows by choosing $f(z) = z$ in (4). If $0 < \alpha < 1$, then

$$\begin{aligned}
|f(z)| &= \left|\int_0^z f'(s)\,ds + f(0)\right| \le \|f\|_{\mathcal{B}_\alpha} \int_0^{|z|} \frac{dx}{(1-x^2)^\alpha} + |f(0)| \\
&\le \frac{\|f\|_{\mathcal{B}_\alpha}}{(1-\alpha)} + |f(0)|,
\end{aligned}$$

and $|f(z)| \le \tanh^{-1}(|z|)\|f\|_{\mathcal{B}_1} + |f(0)|$, where $\tanh^{-1}(.)$ stands for inverse hyperbolic tangent function. Then, for $0 < \alpha < 1$, we deduce that

$$\big|f(\phi(0)) - g(\phi(0))\big| \le \frac{\|f-g\|_{\mathcal{B}_\alpha}}{(1-\alpha)} + |f(0) - g(0)|. \tag{5}$$

Moreover, Lemma 1.4 implies the existence of $f, g \in \mathcal{B}_\alpha^*$ such that

$$\big(f'(z) + g'(z)\big)(1-|z|^2)^\alpha \ge C > 0, \quad \text{for all } z \in \mathbb{D}. \tag{6}$$

Combining (4) and (6) we obtain

$$\begin{aligned}
& \|f\|_{\mathcal{B}_\alpha^*} + \|g\|_{\mathcal{B}_\alpha^*} + \|f\|_{\mathcal{B}_\alpha} + \|g\|_{\mathcal{B}_\alpha} + |f(0)| + |g(0)| \\
\ge & C \int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}} (1-|z|^2)^q g^s(z,a)\,dA(z) \\
\ge & C\, \Phi_\phi(\alpha,p,q,s,a),
\end{aligned}$$

for which the assertion (iii) follows.

Assume now that (iii) is satisfied, we have from (5) that

$$\begin{aligned}
d(f\circ\phi, g\circ\phi; F^*(p,q,s)) &= d_{F^*}(f\circ\phi, g\circ\phi) + \|f\circ\phi - g\circ\phi\|_{F(p,q,s)} \\
&\quad + \big|f(\phi(0)) - g(\phi(0))\big| \\
&\le d_{\mathcal{B}^*_\alpha}(f,g)\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}}(1-|z|^2)^q g^s(z,a)\,dA(z)\right)^{\frac{1}{p}} \\
&\quad + \|f-g\|_{\mathcal{B}_\alpha}\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}}(1-|z|^2)^q g^s(z,a)\,dA(z)\right)^{\frac{1}{p}} \\
&\quad + \frac{\|f-g\|_{\mathcal{B}_\alpha}}{(1-\alpha)} + |f(0)-g(0)| \\
&\le C' d(f,g;\mathcal{B}^*_\alpha).
\end{aligned}$$

Thus $C_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ is Lipschitz continuous and the proof is completed.

Remark 3.2. *Theorem 3.1 shows that* $C_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ *is bounded if and only if it is Lipschitz-continuous, that is, if there exists a positive constant* C *such that*

$$d(f\circ\phi, g\circ\phi; F^*(p,q,s)) \le C d(f,g;\mathcal{B}^*_\alpha), \quad \textit{for all}\, f,g\in\mathcal{B}^*_\alpha.$$

By elementary functional analysis, a linear operator between normed spaces is bounded if and only if it is continuous, and the boundedness is trivially also equivalent to the Lipschitz-continuity. So, our result for composition operators in hyperbolic spaces is the correct and natural generalization of the linear operator theory.

The following observation is sometimes useful.

Proposition 3.3. *Assume* ϕ *is a holomorphic mapping from* $\mathbb{D}$ *into itself. Let* $0 < p, s < \infty$, $-1 < q < \infty$ *and* $0 < \alpha < \infty$. *If* $C_\phi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ *is compact, it maps closed balls onto compact sets.*

Proof. If $B \subset \mathcal{B}^*_\alpha$ is a closed ball and $g \in F^*(p,q,s)$ belongs to the closure of $C_\phi(B)$, we can find a sequence $(f_n)_{n=1}^\infty \subset B$ such that $f_n \circ \phi$ converges to $g \in F^*(p,q,s)$ as $n \to \infty$. But $(f_n)_{n=1}^\infty$ is a normal family, hence it has a subsequence $(f_{n_j})_{j=1}^\infty$ converging uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. As in earlier arguments of Proposition 2.1 in [10], we get a positive estimate which shows that f must belong to the closed ball B. On the other hand, also the sequence $(f_{n_j} \circ \phi)_{j=1}^\infty$ converges uniformly on compact subsets to an analytic function, which is $g \in F^*(p,q,s)$. We get $g = f\circ\phi$, i.e. g belongs to $C_\phi(B)$. Thus, this set is closed and also compact.

Compactness of composition operators can be characterized in full analogy with the linear case.

Theorem 3.4. *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself. Let $0 < p < \infty$, $-1 < q < \infty$, $0 \leq s \leq 1$ and $0 < \alpha \leq 1$. Then the following are equivalent:*

(i) $C_\phi : \mathcal{B}_\alpha^* \to F^*(p, q, s)$ *is compact;*

(ii) $\lim_{r \to 1^-} \sup_{a \in \mathbb{D}} \Omega_{\phi,r}(p, q, s, a) = 0.$

Proof. We first assume that (ii) holds. Let $B := \bar{B}(g, \delta) \subset \mathcal{B}_\alpha^*$, where $g \in \mathcal{B}_\alpha^*$ and $\delta > 0$, be a closed ball, and let $(f_n)_{n=1}^\infty \subset B$ be any sequence. We show that its image has a convergent subsequence in $F^*(p, q, s)$, which proves the compactness of C_ϕ by definition.

Again, $(f_n)_{n=1}^\infty \subset B(\mathbb{D})$ implies that, there is a subsequence $(f_{n_j})_{j=1}^\infty$ which converges uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. By the Cauchy formula for the derivative of an analytic function, also the sequence $(f'_{n_j})_{j=1}^\infty$ converges uniformly on compact subsets of $\mathbb{D}$ to f'. It follows that also the sequences $(f_{n_j} \circ \phi)_{j=1}^\infty$ and $(f'_{n_j} \circ \phi)_{j=1}^\infty$ converge uniformly on compact subsets of $\mathbb{D}$ to $f \circ \phi$ and $f' \circ \phi$, respectively. Moreover, $f \in B \subset \mathcal{B}_\alpha^*$ since for any fixed $R, 0 < R < 1$, the uniform convergence yield $d(f, g; \mathcal{B}_\alpha^*) \leq \delta$ (see [10] pp.130).

Let $\varepsilon > 0$. Since (ii) is satisfied, we may fix $r, 0 < r < 1$, such that

$$\sup_{a \in \mathbb{D}} \int_{|\phi(z)| \geq r} \frac{|\phi(z)|^p}{(1 - |\phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q g^s(z, a) dA(z) \leq \varepsilon.$$

By the uniform convergence, we may fix $N_1 \in \mathbb{N}$ such that

$$|f_{n_j} \circ \phi(0) - f \circ \phi(0)| \leq \varepsilon, \qquad \text{for all } j \geq N_1. \tag{7}$$

The condition (ii) is known to imply the compactness of $C_\phi : \mathcal{B}_\alpha \to F(p, q, s)$, hence, possibly to passing once more to a subsequence and adjusting the notations, we may assume that

$$\|f_{n_j} \circ \phi - f \circ \phi\|_{F(p,q,s)} \leq \varepsilon, \qquad \text{for all } j \geq N_2, \text{ for some } N_2 \in \mathbb{N}. \tag{8}$$

Now let

$$I_1(a, r) = \sup_{a \in \mathbb{D}} \int_{|\phi(z)| \geq r} \left[(f_{n_j} \circ \phi)^*(z) - (g \circ \phi)^*(z)\right]^p (1 - |z|^2)^q g^s(z, a) dA(z),$$

and

$$I_2(a, r) = \sup_{a \in \mathbb{D}} \int_{|\phi(z)| \leq r} \left[(f_{n_j} \circ \phi)^*(z) - (g \circ \phi)^*(z)\right]^p (1 - |z|^2)^q g^s(z, a) dA(z).$$

Since $(f_{n_j})_{j=1}^{\infty} \subset B$ and $f \in B$, it follows from (1) that

$$\begin{aligned} I_1(a,r) &= \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\geq r} \left[(f_{n_j}\circ\phi)^*(z) - (g\circ\phi)^*(z)\right]^p (1-|z|^2)^q g^s(z,a)dA(z) \\ &\leq \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\geq r} \left|\frac{(f_{n_j}\circ\phi)'(z)}{1-|(f_{n_j}\circ\phi)(z)|^2} - \frac{(g\circ\phi)'(z)}{1-|(g\circ\phi)(z)|^2}\right|^p (1-|z|^2)^q g^s(z,a)dA(z) \\ &= \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\geq r} M(f_{n_j}, g, \phi; \alpha, p)(1-|z|^2)^q g^s(z,a)dA(z) \\ &\leq d_{\mathcal{B}^*_\alpha}(f_{n_j}, f) \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\geq r} \frac{|\phi(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}} (1-|z|^2)^q g^s(z,a)dA(z), \end{aligned}$$

where

$$M(f_{n_j}, g, \phi; \alpha, p) = \left|\left(\frac{f'_{n_j}(\phi(z))}{1-|f_{n_j}(\phi(z))|^2} - \frac{g'(\phi(z))}{1-|g((\phi(z)|^2}\right)(1-|\phi(z)|^2)^\alpha\right|^p \left|\frac{\phi'(z)}{(1-|\phi(z)|^2)^\alpha}\right|^p.$$

Hence,

$$I_1(a,r) \leq 2\delta\ \varepsilon. \tag{9}$$

On the other hand, by the uniform convergence on compact subsets of $\mathbb{D}$, we can find an $N_3 \in \mathbb{N}$ such that for all $j \geq N_3$,

$$\left|\frac{f'_{n_j}(\phi(z))}{1-|f_{n_j}(\phi(z))|^2} - \frac{f'(\phi(z))}{1-|f(\phi(z))|^2}\right| \leq \varepsilon$$

for all z with $|\phi(z)| \leq r$. Hence, for such j,

$$\begin{aligned} I_2(a,r) &= \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\leq r} \left[(f_{n_j}\circ\phi)^*(z) - (g\circ\phi)^*(z)\right]^p (1-|z|^2)^q g^s(z,a)dA(z) \\ &\leq \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\leq r} \left|\frac{(f_{n_j}\circ\phi)'(z)}{1-|(f_{n_j}\circ\phi)(z)|^2} - \frac{(g\circ\phi)'(z)}{1-|(g\circ\phi)(z)|^2}\right|^p (1-|z|^2)^q g^s(z,a)dA(z) \\ &\leq \varepsilon\left(\sup_{a\in\mathbb{D}} \int_{|\phi(z)|\leq r} \frac{|\phi(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}} (1-|z|^2)^q g^s(z,a)dA(z)\right)^{\frac{1}{p}} \leq C\varepsilon, \end{aligned}$$

hence,

$$I_2(a,r) \leq C\ \varepsilon. \tag{10}$$

where C is the bounded obtained from (iii) of Theorem 3.1. Combining (7), (8), (9) and (10) we deduce that $f_{n_j} \to f$ in $F^*(p,q,s)$.

As for the converse direction, let $f_n(z) := \frac{1}{2}n^{\alpha-1}z^n$ for all $n \in N, n \geq 2$. Then the sequence $(f_n)_{n=1}^{\infty}$ belongs to the ball $\bar{B}(0,3) \subset \mathcal{B}^*_\alpha$(see [10] pp.131).

We are assuming that C_ϕ maps the closed ball $\bar{B}(0,3) \subset \mathcal{B}^*_\alpha$ into a compact subset of $F^*(p,q,s)$, hence, there exists an unbounded increasing subsequence $(f_{n_j})_{j=1}^{\infty}$ such that the image subsequence $(C_\phi f_{n_j})_{j=1}^{\infty}$ converges with respect to the norm. Since, both $(f_n)_{n=1}^{\infty}$ and $(C_\phi f_{n_j})_{j=1}^{\infty}$ converge to the zero function uniformly on compact subsets of $\mathbb{D}$, the limit of the latter sequence must be 0. Hence,

$$\|n_j^{\alpha-1}\phi^{n_j}\|_{F^*(p,q,s)} \to 0, \quad \text{as } j \to \infty. \tag{11}$$

Now let $r_j = 1 - \frac{1}{n_j}$. For all numbers $a, r_j \le a < 1$, we have the estimate (see [10])

$$\frac{n_j^{\alpha} a^{n_j-1}}{1-a^{n_j}} \ge \frac{1}{e(1-a)^{\alpha}} \tag{12}$$

Using (12) we obtain

$$\begin{aligned}\|n_j^{\alpha-1}\phi^{n_j}\|^p_{F^*(p,q,s)} &\ge \sup_{a\in\mathbb{D}}\int_{|\phi|\ge r_j}\left|\frac{n_j{}^{\alpha}(\phi(z))^{n_j-1}\phi'(z)}{1-|\phi^{n_j}(z)|^2}\right|^p(1-|z|^2)^q g^s(z,a)dA(z)\\ &\ge \frac{1}{(2e)^p}\sup_{a\in\mathbb{D}}\int_{|\phi|\ge r_j}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}}(1-|z|^2)^q g^s(z,a)dA(z).\end{aligned}$$

Hence, the condition (ii) follows.

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