On centralizers of standard operator algebras with involution

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ABSTRACT
The purpose of this paper is to prove the following result. Let $X$ be a complex Hilbert space, let $L(X)$ be the algebra of all bounded linear operators on $X$ and let $A(X) \subseteq L(X)$ be a standard operator algebra, which is closed under the adjoint operation. Let $T : A(X) \to L(X)$ be a linear mapping satisfying the relation $2T( AA^* A ) = T(A) A^* A + AA^* T(A)$ for all $A \in A(X)$. In this case $T$ is of the form $T(A) = \lambda A$ for all $A \in A(X)$, where $\lambda$ is some fixed complex number.

RESUMEN
El propósito de este artículo es probar el siguiente resultado. Sea $X$ un espacio de Hilbert complejo, sea $L(X)$ el álgebra de todos los operadores lineales acotados sobre $X$ y sea $A(X) \subseteq L(X)$ la álgebra de operadores clásica, la cual es cerrada bajo la operación adjunta. Se define $T : A(X) \to L(X)$ una aplicación lineal satisfaciendo la relación $2T( AA^* A ) = T(A) A^* A + AA^* T(A)$ para todo $A \in A(X)$. En este caso, $T$ es de la forma $T(A) = \lambda A$ para todo $A \in A(X)$, donde $\lambda$ es un número complejo fijo.

Keywords and Phrases: ring, ring with involution, prime ring, semiprime ring, Banach space, Hilbert space, standard operator algebra, $H^*$-algebra, left (right) centralizer, two-sided centralizer.

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This research has been motivated by the work of Vukman, Kosi-Ubl[5] and Zalar [13]. Throughout, R will represent an associative ring with center \( Z(R) \). Given an integer \( n \geq 2 \), a ring \( R \) is said to be \( n \)-torsion free if for \( x \in R \), \( nx = 0 \) implies \( x = 0 \). An additive mapping \( x \mapsto x^* \) on a ring \( R \) is called involution if \((xy)^* = y^*x^* \) and \( x^{**} = x \) hold for all pairs \( x, y \in R \). A ring equipped with an involution is called a ring with involution or \( * \)-ring. Recall that a ring \( R \) is prime if for \( a, b \in R \), \( ab = 0 \) implies that either \( a = 0 \) or \( b = 0 \), and is semiprime in case \( aRa = 0 \) implies \( a = 0 \). We denote by \( Q_r \) and \( C \) the Martindale right ring of quotients and the extended centroid of a semiprime ring \( R \), respectively. For the explanation of \( Q_r \) and \( C \) we refer the reader to [2].

An additive mapping \( T : R \to R \) is called a left centralizer in case \( T(xy) = T(x)y \) holds for all pairs \( x, y \in R \). In case \( R \) has the identity element, \( T : R \to R \) is a left centralizer iff \( T \) is of the form \( T(x) = ax \) for all \( x \in R \), where \( a \) is some fixed element of \( R \). For a semiprime ring \( R \) all left centralizers are of the form \( T(x) = qx \) for all \( x \in R \), where \( q \in Q_r \) is some fixed element (see Chapter 2 in [2]). An additive mapping \( T : R \to R \) is called a left Jordan centralizer in case \( T(x^2) = T(x)x \) holds for all \( x \in R \). The definition of right centralizer and right Jordan centralizer should be self-explanatory. We call \( T : R \to R \) a two-sided centralizer in case \( T \) is both a left and a right centralizer. In case \( T : R \to R \) is a two-sided centralizer, where \( R \) is a semiprime ring with extended centroid \( C \), then \( T \) is of the form \( T(x) = \lambda x \) for all \( x \in R \), where \( \lambda \in C \) is some fixed element (see Theorem 2.3.2 in [2]). Zalar [13] has proved that any left (right) Jordan centralizer on a semiprime ring is a left (right) centralizer.

Let us recall that a semisimple \( H \)-algebra is a complex semisimple Banach*-algebra whose norm is a Hilbert space norm such that \((xy^*) = (xz, y) = (z, x^*y) \) is fulfilled for all \( x, y, z \in A \). For basic facts concerning \( H \)-algebras we refer to [1]. Vukman [10] has proved that in case there exists an additive mapping \( T : R \to R \), where \( R \) is a 2-torsion free semiprime ring satisfying the relation \( 2T(x^2) = T(x)x + xT(x) \) for all \( x \in R \), then \( T \) is a two-sided centralizer. Kosi-Ubl and Vukman [2] have proved the following result. Let \( A \) be a semisimple \( H \)-algebra and let \( T : A \to A \) be an additive mapping such that \( 2T(x^{n+1}) = T(x)x^n + x^nT(x) \) holds for all \( x \in R \) and some fixed integer \( n \geq 1 \). In this case \( T \) is a two-sided centralizer. Recently, Benkovič, Eremita and Vukman [3] have considered the relation we have just mentioned above in prime rings with suitable characteristic restrictions. Kosi-Ubl and Vukman [2] have proved that in case there exists an additive mapping \( T : R \to R \), where \( R \) is a 2-torsion free semiprime \( * \)-ring, satisfying the relation \( T(xx^*) = T(x)x^* + xT(x^*) \) for all \( x \in R \), then \( T \) is a left (right) centralizer. For results concerning centralizers on rings and algebras we refer to [4][13], where further references can be found.

Let \( X \) be a real or complex Banach space and let \( \mathcal{L}(X) \) and \( \mathcal{F}(X) \) denote the algebra of all bounded linear operators on \( X \) and the ideal of all finite rank operators in \( \mathcal{L}(X) \), respectively. An algebra \( A(X) \subset \mathcal{L}(X) \) is said to be standard in case \( \mathcal{F}(X) \subset A(X) \). Let us point out that any standard operator algebra is prime, which is a consequence of a Hahn-Banach theorem. In case \( X \) is a real or complex Hilbert space, we denote by \( A^+ \) the adjoint operator of \( A \in \mathcal{L}(X) \). We denote
by \( X^* \) the dual space of a real or complex Banach space \( X \).

Vukman and Kosi-Ulbl \cite{VUKMAN2013} have proved the following result.

**Theorem 0.1.** Let \( R \) be a 2-torsion free semiprime ring and let \( T : R \to R \) be an additive mapping. Suppose that

\[ 2T(xy) = T(x)yx + xyT(x) \quad (1) \]

holds for all \( x, y \in R \). In this case \( T \) is a two-sided centralizer.

In case we have a \(*\)-ring, we obtain, after putting \( y = x^* \) in the relation (1), the relation

\[ 2T(xx^*) = T(x)x^*x + xx^*T(x). \]

It is our aim in this paper to prove the following result, which is related to the above relation.

**Theorem 0.2.** Let \( X \) be a complex Hilbert space and let \( A(X) \) be a standard operator algebra, which is closed under the adjoint operation. Suppose \( T : A(X) \to \mathcal{L}(X) \) is a linear mapping satisfying the relation

\[ 2T(AA^*A) = T(A\lambda A^*A + AA^*T(A)) \quad (2) \]

for all \( A \in A(X) \). In this case \( T \) is of the form \( T(A) = \lambda A \), where \( \lambda \) is a fixed complex number.

**Proof.** Let us first consider the restriction of \( T \) on \( F(X) \). Let \( A \) be from \( F(X) \) (in this case we have \( A^* \in F(X) \)). Let \( P \in F(X) \) be a self-adjoint projection with the property \( AP = PA = A \) (we also have \( A^*P = PA^* = A^* \)). Putting \( P \) for \( A \) in (2) we obtain

\[ 2T(P) = T(P)P + PT(P). \]

Left multiplication by \( P \) in the above relation gives \( PT(P) = PT(P)P \). Similarly, right multiplication by \( P \) in the above relation leads to \( T(P)P = PT(P)P \). Therefore

\[ T(P) = T(P)P = PT(P) = PT(P)P. \quad (3) \]

Putting \( A + P \) for \( A \) in the relation (2) we obtain

\[ 2T(A^2) + 2T(AA^* + A^*A) + 4T(A) + 2T(A^*) = 
\]

\[ = T(A)(A + A^*) + T(P)A^*A + T(P)(A + A^*) + 
\]

\[ + (A + A^*)T(A) + PT(A) + AA^*T(P) + (A + A^*)T(P). \]

Putting \(-A\) for \( A \) in the above relation and comparing the relation so obtained with the above relation gives

\[ 2T(A^2) + 2T(AA^* + A^*A) = 
\]

\[ = T(A)(A + A^*) + T(P)A^*A + (A + A^*)T(A) + AA^*T(P) \quad (4) \]
and

\[
4T(A) + 2T(A^*) = \\
= T(A)P + PT(A) + T(P)(A + A^*) + (A + A^*)T(P).
\]

\[\text{(5)}\]

So far we have not used the assumption that $X$ is a complex Hilbert space. Putting $iA$ for $A$ in the relations 4 and 5 and comparing the relations so obtained with the above relations, respectively, we obtain

\[
2T(A^2) = T(A)A + AT(A),
\]

\[\text{(6)}\]

\[
4T(A) = T(A)P + PT(A) + T(P)A + AT(P).
\]

\[\text{(7)}\]

Putting $A^*$ for $A$ in the relation 5 gives

\[
4T(A^*) + 2T(A) = \\
= T(A^*)P + PT(A^*) + T(P)(A + A^*) + (A + A^*)T(P).
\]

Putting $iA$ for $A$ in the above relation and comparing the relation so obtained with the above relation leads to

\[
2T(A) = T(P)A + AT(P).
\]

Comparing the above relation and 7, we obtain

\[
2T(A) = T(A)P + PT(A).
\]

\[\text{(8)}\]

Right (left) multiplication by $P$ in the above relation gives $T(A)P = PT(A)P$ and $PT(A) = PT(A)P$, respectively. Hence, $PT(A) = T(A)P$, which reduces the relation 5 to

\[
T(A) = T(A)P.
\]

From the above relation one can conclude that $T$ maps $\mathcal{F}(X)$ into itself. We therefore have a linear mapping $T : \mathcal{F}(X) \to \mathcal{F}(X)$ satisfying the relation 6 for all $A \in \mathcal{F}(X)$. Since $\mathcal{F}(X)$ is prime, one can conclude, according to Theorem 1 in 10 that $T$ is a two-sided centralizer on $\mathcal{F}(X)$. We intend to prove that there exists an operator $C \in \mathcal{L}(X)$, such that

\[
T(A) = CA
\]

\[\text{(9)}\]

for all $A \in \mathcal{F}(X)$. For any fixed $x \in X$ and $f \in X^*$ we denote by $x \otimes f$ an operator from $\mathcal{F}(X)$ defined by $(x \otimes f)y = f(y)x, y \in X$. For any $A \in \mathcal{L}(X)$ we have $A(x \otimes f) = (Ax) \otimes f$. Now let us choose such $f$ and $y$ that $f(y) = 1$ and define $Cx = T(x \otimes f)y$. Obviously, $C$ is linear and applying the fact that $T$ is a left centralizer on $\mathcal{F}(X)$, we obtain

\[
(CAx) = C(AX) = T((Ax) \otimes f)y = T(A(x \otimes f))y = T(A(x \otimes f)y) = T(A)x
\]

for any $x \in X$. We therefore have $T(A) = CA$ for any $A \in \mathcal{F}(X)$. As $T$ is a right centralizer on $\mathcal{F}(X)$, we obtain $C(AB) = T(AB) = AT(B) = ACB$. We therefore have $[A, C]B = 0$ for any
A, B ∈ ℱ(Ẋ), whence it follows that [A, C] = 0 for any A ∈ ℱ(Ẋ). Using closed graph theorem one can easily prove that C is continuous. Since C commutes with all operators from ℱ(Ẋ), we can conclude that Cx = λx holds for any x ∈ Ẋ and some fixed complex number λ, which gives together with the relation (6) that T is of the form
\[ T(A) = λA \]  
for any A ∈ ℱ(Ẋ) and some fixed complex number λ. It remains to prove that the relation (10) holds on ℬ(Ẋ) as well. Let us introduce T₁ : ℬ(Ẋ) → ℬ(Ẋ) by T₁(A) = λA and consider T₀ = T − T₁. The mapping T₀ is, obviously, additive and satisfies the relation (2). Besides, T₀ vanishes on ℱ(Ẋ). It is our aim to show that T₀ vanishes on ℬ(Ẋ) as well. Let A ∈ ℬ(Ẋ), let P ∈ ℱ(Ẋ) be a one-dimensional self-adjoint projection and S = A + PAP − (AP + PA). Such S can also be written in the form S = (I − P)A(I − P), where I denotes the identity operator on Ẋ. Since S − A ∈ ℱ(Ẋ), we have T₀(S) = T₀(A). It is easy to see that SP = PS = 0. By the relation (2) we have
\[
T₀(S)S^*S + SS^*T₀(S) = \\
= 2T₀(SS^*S) \\
= 2T₀((S + P)(S + P)^*(S + P)) \\
= T₀(S + P)(S + P)^*(S + P) + (S + P)(S + P)^*T₀(S + P) \\
= T₀(S)S^*S + T₀(S)P + SS^*T₀(S) + PT₀(S).
\]
We therefore have
\[
T₀(S)P + PT₀(S) = 0.
\]
Considering T₀(S) = T₀(A) in the above relation, we obtain
\[
T₀(A)P + PT₀(A) = 0.
\]  
(11)
Multiplication from both sides by P in the above relation leads to
\[
PT₀(A)P = 0.
\]
Right multiplication by P in the relation (11) and considering the above relation gives
\[
T₀(A)P = 0.
\]
Since P is an arbitrary one-dimensional self-adjoint projection, it follows from the above relation that T₀(A) = 0 for all A ∈ ℬ(Ẋ), which completes the proof of the theorem. \(\square\)

We conclude the paper with the following conjecture.

**Conjecture 0.3.** Let ℛ be a semiprime *-ring with suitable torsion restrictions and let \( T : ℛ \to ℛ \) be an additive mapping satisfying the relation
\[
2T(xx^*x) = T(x)x^*x + xx^*T(x)
\]
for all \( x \in ℛ \). In this case \( T \) is a two-sided centralizer.
References


