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Viscosity approximation methods with a sequence of contractions

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ABSTRACT

The aim of this paper is to prove that, in an appropriate setting, every iterative sequence generated by the viscosity approximation method with a sequence of contractions is convergent whenever so is every iterative sequence generated by the Halpern type iterative method. Then, using our results, we show some convergence theorems for variational inequality problems, zero point problems, and fixed point problems.

RESUMEN

La meta de este artículo es probar en un marco de trabajo adecuado que cada sucesión iterativa generada por el método de aproximación de viscosidad con una sucesión cualquiera de contracciones es convergente como lo es cada sucesión iterativa generada por el método iterativo del tipo Halpern. Así, usando nuestro resultado mostramos algunos teoremas de convergencia para problemas de desigualdades variacionales, problemas de punto cero y problemas de punto fijo.

Keywords and Phrases: Viscosity approximation method, nonexpansive mapping, fixed point, hybrid steepest descent method.

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1 Introduction

Let C be a nonempty closed convex subset of a Hilbert space. This paper is devoted to the study of strong convergence of a sequence $\{y_n\}$ in C defined by an arbitrary point $y_1 \in C$ and

$$y_{n+1} = \lambda_n f_n(y_n) + (1 - \lambda_n) T_n y_n \tag{1.1}$$

for $n \in \mathbb{N}$, where λ_n is a real number in $[0, 1]$, f_n is a contraction on C, and T_n is a nonexpansive mapping on C for $n \in \mathbb{N}$. In particular, our main interest is the relationship between convergence of such a sequence $\{y_n\}$ and a sequence $\{x_n\}$ defined by an arbitrary point $x_1 \in C$ and

$$x_{n+1} = \lambda_n u + (1 - \lambda_n) T_n x_n \tag{1.2}$$

for $n \in \mathbb{N}$, where u is a point in C. In §3, using the technique developed in [21], we prove that their convergence are equivalent under some assumptions. Then, as applications of our convergence results in §3, we discuss strong convergence of the sequences generated by the hybrid steepest descent method [30] and we give another proof of Iemoto and Takahashi's theorem [17] in §4. Moreover, we show one generalization of Ceng, Petruşel, and Yao's theorem [13] in §5.

The iterative method defined by (1.1) is based on the viscosity approximation method due to Moudafi [19]. He considered the fixed point problem of a single nonexpansive mapping and proved strong convergence of sequences generated by the viscosity approximation methods; see also Xu [28] and Suzuki [21].

The iterative method defined by (1.2) is called the Halpern type iterative method; see Halpern [16], Wittmann [25], and Shioji and Takahashi [20]; see also [1, 5, 7].

2 Preliminaries

Throughout the present paper, H denotes a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, C a nonempty closed convex subset of H, I the identity mapping on H, and $\mathbb{N}$ the set of positive integers.

A mapping $S\colon C \to H$ is said to be lipschitzian if there exists a constant $\eta \geq 0$ such that $\|Sx - Sy\| \leq \eta \|x - y\|$ for all $x, y \in C$. In this case, S is called an η-lipschitzian mapping. In particular, an η-lipschitzian mapping is said to be nonexpansive if $\eta = 1$; an η-lipschitzian mapping is said to be an η-contraction if $0 \leq \eta < 1$. It is known that $\mathrm{Fix}(S)$ is closed and convex if $S\colon C \to H$ is nonexpansive, where $\mathrm{Fix}(S)$ denotes the set of fixed points of S. The metric projection of H onto C is denoted by P_C and we know that P_C is nonexpansive. We also know the following; see [22].

Lemma 2.1. *Let $x \in H$ and $z \in C$. Then $z = P_C(x)$ if and only if $\langle y - z, x - z \rangle \leq 0$ for all $y \in C$.*

Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into H. We say that $\{S_n\}$ satisfies the *condition (Z)* if every weak cluster point of $\{x_n\}$ is a common fixed point of $\{S_n\}$ whenever $\{x_n\}$

is a bounded sequence in C and $x_n - S_n x_n \to 0$; see [1, 3, 8–11]. We say that $\{S_n\}$ satisfies the *condition (R)* if

$$\lim_{n\to\infty} \sup_{y\in D} \|S_{n+1}y - S_n y\| = 0$$

for every nonempty bounded subset D of C; see [1, 5]. We say that $\{S_n\}$ is *stable* on a nonempty subset D of C if $\{S_n z : n \in \mathbb{N}\}$ is a singleton for every $z \in D$.

We need the following lemmas:

Lemma 2.2. *Let C_1 and C_2 be nonempty closed convex subsets of H, $\{S_n\}$ a sequence of nonexpansive mappings of C_1 into H, and $\{T_n\}$ a sequence of nonexpansive mappings of C_2 into H. Suppose that $\{S_n\}$ and $\{T_n\}$ satisfy the condition (R), $C_1 \supset T_n(C_2)$ for every $n \in \mathbb{N}$, and $\{T_n\}$ has a common fixed point. Then $\{S_n T_n\}$ satisfies the condition (R).*

Proof. Let D be a nonempty bounded subset of C_2. Then it is clear that each $S_n T_n$ is nonexpansive and

$$\begin{aligned} \|S_{n+1}T_{n+1}y - S_n T_n y\| &\le \|S_{n+1}T_{n+1}y - S_n T_{n+1}y\| + \|S_n T_{n+1}y - S_n T_n y\| \\ &\le \|S_{n+1}T_{n+1}y - S_n T_{n+1}y\| + \|T_{n+1}y - T_n y\| \end{aligned} \tag{2.1}$$

for all $y \in D$ and $n \in \mathbb{N}$. Let z be a common fixed point of $\{T_n\}$. Then it is obvious that

$$\|T_n y\| \le \|T_n y - T_n z\| + \|z\| \le \|y - z\| + \|z\|$$

for all $y \in D$ and $n \in \mathbb{N}$. This shows that $D' = \{T_n y : n \in \mathbb{N}, y \in D\}$ is a bounded subset of C_1. Since $\{S_n\}$ and $\{T_n\}$ satisfy the condition (R), it follows from (2.1) that

$$\sup_{y\in D} \|S_{n+1}T_{n+1}y - S_n T_n y\| \le \sup_{y'\in D'} \|S_{n+1}y' - S_n y'\| + \sup_{y\in D} \|T_{n+1}y - T_n y\| \to 0.$$

Therefore, $\{S_n T_n\}$ satisfies the condition (R). □

Lemma 2.3. *Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into H and $\{\gamma_n\}$ a sequence in $[0, 1]$ such that $\gamma_{n+1} - \gamma_n \to 0$. Suppose that $\{S_n\}$ satisfies the condition (R) and $\{S_n\}$ has a common fixed point. Then $\{\gamma_n I + (1 - \gamma_n)S_n\}$ satisfies the condition (R).*

Proof. Set $U_n = \gamma_n I + (1 - \gamma_n)S_n$ for $n \in \mathbb{N}$. Let D be a nonempty bounded subset of C. Then it is clear that each U_n is nonexpansive and

$$\begin{aligned} \|U_{n+1}y - U_n y\| &\le |\gamma_{n+1} - \gamma_n|\, \|y - S_n y\| + |1 - \gamma_{n+1}|\, \|S_{n+1}y - S_n y\| \\ &\le |\gamma_{n+1} - \gamma_n|\, \|y - S_n y\| + \|S_{n+1}y - S_n y\| \end{aligned} \tag{2.2}$$

for all $y \in D$ and $n \in \mathbb{N}$. Let z be a common fixed point of $\{S_n\}$. Then it is obvious that

$$\|y - S_n y\| \le \|y - z\| + \|S_n z - S_n y\| \le 2\,\|y - z\| \tag{2.3}$$

for all $y \in D$ and $n \in \mathbb{N}$. Since $\{S_n\}$ satisfies the condition (R) and $\gamma_{n+1} - \gamma_n \to 0$, it follows from (2.2) and (2.3) that

$$\sup_{y \in D} \|U_{n+1}y - U_n y\| \leq 2\,|\gamma_{n+1} - \gamma_n|\,\|y - z\| + \sup_{y \in D} \|S_{n+1}y - S_n y\| \to 0.$$

Therefore, $\{U_n\}$ satisfies the condition (R). □

A set-valued mapping A of H into H, which is denoted by $A \subset H \times H$, is said to be a monotone operator if $\langle x - y, x' - y' \rangle \geq 0$ for all $(x, x'), (y, y') \in A$. A monotone operator $A \subset H \times H$ is said to be maximal if $A = B$ whenever $B \subset H \times H$ is a monotone operator such that $A \subset B$. Let $A \subset H \times H$ be a maximal monotone operator. It is known that $(I + \rho A)^{-1}$ is a single-valued mapping of H onto $\mathrm{dom}(A) = \{x \in H : Ax \neq \emptyset\}$ for all $\rho > 0$. Such a mapping $(I + \rho A)^{-1}$ is called the resolvent of A and denoted by J_ρ. It is also known that the resolvent J_ρ is nonexpansive and $\mathrm{Fix}(J_\rho) = A^{-1}0 = \{x \in H : Ax \ni 0\}$; see [22] for more details.

A mapping $A \colon H \to H$ is said to be strongly monotone if there is a constant $\kappa > 0$ such that $\langle x - y, Ax - Ay \rangle \geq \kappa \|x - y\|^2$ for all $x, y \in H$. In this case, A is called a κ-strongly monotone mapping. The following lemma is well known; see, for example, [4].

Lemma 2.4. *Let κ and η be positive real numbers such that $\eta^2 < 2\kappa$. Let F be a nonempty closed convex subset of H and $A \colon H \to H$ a κ-strongly monotone and η-lipschitzian mapping. Then the following hold:*

(1) $\kappa \leq \eta$, $0 \leq 1 - 2\kappa + \eta^2 < 1$ *and $I - A$ is a θ-contraction, where $\theta = \sqrt{1 - 2\kappa + \eta^2}$.*

(2) *There exists a unique point $z \in F$ such that $\langle y - z, Az \rangle \geq 0$ for all $y \in F$, and moreover, z is the unique fixed point of $P_F(I - A)$.*

The following lemma is well known; see [7, 18, 24, 26, 27].

Lemma 2.5. *Let $\{\epsilon_n\}$ be a sequence of nonnegative real numbers, $\{\gamma_n\}$ a sequence of real numbers, and $\{\lambda_n\}$ a sequence in $[0, 1]$. Suppose that $\epsilon_{n+1} \leq (1 - \lambda_n)\epsilon_n + \lambda_n \gamma_n$ for every $n \in \mathbb{N}$, $\limsup_{n\to\infty} \gamma_n \leq 0$, and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then $\epsilon_n \to 0$.*

3 Viscosity approximation method with a sequence of contractions

In this section, we deal with the viscosity approximation method due to Moudafi [19] in order to find a common fixed point of a sequence of nonexpansive mappings. In particular, we focus on the viscosity approximation method with a sequence of contractions. We first investigate the relationship between this method and the Halpern type iterative method (Theorem 3.1). Then, by using known results (Theorems 3.3 and 3.5), we show convergence theorems (Theorems 3.4 and 3.6).

Using the technique in [21], we can prove the following:

Theorem 3.1. *Let* H *be a Hilbert space,* C *a nonempty closed convex subset of* H*,* $\{T_n\}$ *a sequence of nonexpansive self-mappings of* C*,* F *a nonempty closed convex subset of* C*,* θ *a nonnegative real number with* $\theta < 1$*, and* $\{\lambda_n\}$ *a sequence in* $[0,1]$ *such that* $\sum_{n=1}^{\infty} \lambda_n = \infty$*. Then the following are equivalent:*

(1) *For any* $(x,u) \in C \times C$*, the sequence* $\{x_n\}$ *defined by* $x_1 = x$ *and*

$$x_{n+1} = \lambda_n u + (1-\lambda_n) T_n x_n \tag{3.1}$$

for $n \in \mathbb{N}$ *converges strongly to* $P_F(u)$*.*

(2) *For any* $y \in C$ *and any sequence* $\{f_n\}$ *of* θ*-contractions on* C *which is stable on* F*, the sequence* $\{y_n\}$ *defined by* $y_1 = y$ *and*

$$y_{n+1} = \lambda_n f_n(y_n) + (1-\lambda_n) T_n y_n \tag{3.2}$$

for $n \in \mathbb{N}$ *converges strongly to* w*, where* w *is the unique fixed point of* $P_F \circ f_1$*.*

Proof. We first show that (1) implies (2). Let $\{f_n\}$ be a sequence of θ-contractions on C which is stable on F, w the fixed point of a contraction $P_F \circ f_1$, and $y \in C$. Let $\{x_n\}$ be a sequence defined by $x_1 = y$ and

$$x_{n+1} = \lambda_n f_1(w) + (1-\lambda_n) T_n x_n$$

for $n \in \mathbb{N}$. Then $x_n \to P_F\big(f_1(w)\big) = w$ by (1). Since T_n is nonexpansive and f_n is a θ-contraction, it follows from $f_1(w) = f_n(w)$ that

$$\begin{aligned}
\|x_{n+1} - y_{n+1}\| &= \big\|(1-\lambda_n)(T_n x_n - T_n y_n) + \lambda_n\big(f_1(w) - f_n(y_n)\big)\big\| \\
&\le (1-\lambda_n)\|T_n x_n - T_n y_n\| + \lambda_n \|f_n(w) - f_n(y_n)\| \\
&\le (1-\lambda_n)\|x_n - y_n\| + \lambda_n \theta \|w - y_n\| \\
&\le (1-\lambda_n)\|x_n - y_n\| + \lambda_n \theta (\|w - x_n\| + \|x_n - y_n\|) \\
&\le \big(1-(1-\theta)\lambda_n\big)\|x_n - y_n\| + (1-\theta)\lambda_n \frac{\theta}{1-\theta}\|x_n - w\|
\end{aligned}$$

for every $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} (1-\theta)\lambda_n = \infty$ and $x_n \to w$, Lemma 2.5 shows that $x_n - y_n \to 0$. Therefore, we conclude that $y_n \to w$.

We next show that (2) implies (1). Let $(x,u) \in C \times C$ be given. For each $n \in \mathbb{N}$, let f_n be a mapping defined by $f_n(z) = u$ for $z \in C$. Then, obviously, each f_n is a 0-contraction and $\{f_n\}$ is stable on F. Thus it follows from (2) that $\{x_n\}$ converges strongly to $w = P_F\big(f_1(w)\big) = P_F(u)$. □

Remark 3.2. *It is easy to check that Theorem 3.1 holds even if* H *is a Banach space under appropriate conditions.*

We know the following result; see [2, 7] and see also [3, 11].

Theorem 3.3. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{T_n\}$ a sequence of nonexpansive self-mappings of C with a common fixed point, F the set of common fixed points of $\{T_n\}$, and $\{\lambda_n\}$ a sequence in $[0,1]$ such that*

$$\lambda_n \to 0, \sum_{n=1}^{\infty} \lambda_n = \infty, \text{ and } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty. \tag{3.3}$$

Suppose that $\{T_n\}$ satisfies the condition (Z) and

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}y - T_n y\| : y \in D\} < \infty$$

for every nonempty bounded subset D of C. Let x and u be points in C and $\{x_n\}$ a sequence defined by $x_1 = x$ and (3.1) for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_F(u)$.

Using Theorems 3.1 and 3.3, we obtain the following:

Theorem 3.4. *Let H, C, $\{T_n\}$, F, and $\{\lambda_n\}$ be the same as in Theorem 3.3. Let θ be a nonnegative real number with $\theta < 1$ and $\{f_n\}$ a sequence of θ-contractions on C which is stable on F. Let y be a point in C and $\{y_n\}$ a sequence defined by $y_1 = y$ and (3.2) for $n \in \mathbb{N}$. Then $\{y_n\}$ converges strongly to w, where w is the unique fixed point of $P_F \circ f_1$.*

We also know the following result; see [1, 5].

Theorem 3.5. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{S_n\}$ a sequence of nonexpansive self-mappings of C with a common fixed point, F the set of common fixed points of $\{S_n\}$. Let $\{\lambda_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ such that*

$$\lambda_n \to 0, \sum_{n=1}^{\infty} \lambda_n = \infty, \text{ and } 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $\{S_n\}$ satisfies the conditions (Z) and (R). Let x and u be points in C and $\{x_n\}$ a sequence defined by $x_1 = x$ and

$$x_{n+1} = \lambda_n u + (1-\lambda_n)\big((1-\beta_n)x_n + \beta_n S_n x_n\big)$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_F(u)$.

Using Theorems 3.1 and 3.5, we also obtain the following:

Theorem 3.6. *Let H, C, $\{S_n\}$, F, $\{\lambda_n\}$, and $\{\beta_n\}$ be the same as in Theorem 3.5. Let θ be a nonnegative real number with $\theta < 1$ and $\{f_n\}$ a sequence of θ-contractions on C which is stable on F. Let y be a point in C and $\{y_n\}$ a sequence defined by $y_1 = y$ and*

$$y_{n+1} = \lambda_n f_n(y_n) + (1-\lambda_n)\big((1-\beta_n)y_n + \beta_n S_n y_n\big)$$

for $n \in \mathbb{N}$. Then $\{y_n\}$ converges strongly to w, where w is the unique fixed point of $P_F \circ f_1$.

Proof. Set $T_n = (1-\beta_n)I + \beta_n S_n$ for $n \in \mathbb{N}$. Then it is clear that each T_n is nonexpansive and $y_{n+1} = \lambda_n f_n(y_n) + (1-\lambda_n)T_n y_n$ for $n \in \mathbb{N}$. Let x and u be points in C and $\{x_n\}$ a sequence defined by $x_1 = x$ and $x_{n+1} = \lambda_n u + (1-\lambda_n)T_n x_n$ for $n \in \mathbb{N}$. Then it follows from Theorem 3.5 that $x_n \to P_F(u)$. Therefore, Theorem 3.1 implies the conclusion. □

4 Convergence theorems by the hybrid steepest descent method

In this section, we deal with the variational inequality problem over the set of common fixed points of a sequence of nonexpansive mappings; see Problem 4.1 below. Then we prove some strong convergence theorems by the hybrid steepest descent method introduced by Yamada [30]. We know many results by using the hybrid steepest descent method; see [2, 14, 17, 29, 31].

Problem 4.1. *Let H be a Hilbert space, $\{T_n\}$ a sequence of nonexpansive self-mappings of H with a common fixed point, F the set of common fixed points of $\{T_n\}$, and $A\colon H \to H$ a κ-strongly monotone and η-lipschitzian mapping, where κ and η are positive real numbers such that $\eta^2 < 2\kappa$. Then find $z \in F$ such that*

$$\langle y - z, Az\rangle \geq 0 \text{ for all } y \in F.$$

Remark 4.2. *The assumption that $\eta^2 < 2\kappa$ in Problem 4.1 is not restrictive. Indeed, suppose that a κ-strongly monotone and η-lipschitzian mapping A is given. Let us choose a positive constant μ such that $\mu < 2\kappa/\eta^2$, and define $\kappa' = \mu\kappa$ and $\eta' = \mu\eta$. Then it is easy to verify that $(\eta')^2 < 2\kappa'$, μA is κ'-strongly monotone and η'-lipschitzian, and moreover, $\langle y - z, Az\rangle \geq 0$ is equivalent to $\langle y - z, \mu Az\rangle \geq 0$ for every $y, z \in H$.*

As a consequence of Theorem 3.1, we can obtain the following theorem, which shows that every sequence generated by the hybrid steepest descent method for Problem 4.1 is convergent whenever so is every sequence generated by the Halpern type iterative method for the sequence of nonexpansive mappings.

Theorem 4.3. *Let H, $\{T_n\}$, F, κ, η, and A be the same as in Problem 4.1. Let $\{\lambda_n\}$ be a sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. Suppose that for any $(x, u) \in H \times H$, the sequence $\{x_n\}$ defined by $x_1 = x$ and*

$$x_{n+1} = \lambda_n u + (1-\lambda_n)T_n x_n \tag{4.1}$$

for $n \in \mathbb{N}$ converges strongly to $P_F(u)$. Let y be a point in H and $\{y_n\}$ a sequence defined by $y_1 = y$ and

$$y_{n+1} = (I - \lambda_n A)T_n y_n \tag{4.2}$$

for $n \in \mathbb{N}$. Then $\{y_n\}$ converges strongly to the unique solution of Problem 4.1.

Proof. Set $f_n = (I - A)T_n$ for $n \in \mathbb{N}$. Since T_n is nonexpansive, f_n is a θ-contraction by Lemma 2.4, where $\theta = \sqrt{1 - 2\kappa + \eta^2}$. By the definition of $\{y_n\}$, it is clear that

$$y_{n+1} = \lambda_n (I - A)T_n x_n + (1 - \lambda_n)T_n y_n = \lambda_n f_n(y_n) + (1 - \lambda_n)T_n y_n$$

for every $n \in \mathbb{N}$. It is also clear that $\{f_n\}$ is stable on F. Thus Theorem 3.1 implies that $\{y_n\}$ converges strongly to $w = P_F\big((I - A)T_1 w\big) = P_F(I - A)w$, which is the unique solution of Problem 4.1 by Lemma 2.4. □

Using Theorem 3.6 and other known results, we also obtain the following:

Theorem 4.4 (Iemoto and Takahashi [17, Theorem 3.1]). *Let H, $\{T_n\}$, F, κ, η, and A be the same as in Problem 4.1. Let $\{\lambda_n\}$ be a sequence in $[0, 1]$ such that*

$$\lambda_n \to 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = \infty$$

and $\{\gamma_n\}$ a sequence in $[a, b]$, where $0 < a \le b < 1$. For each $n \in \mathbb{N}$ and $k \in \{1, 2, \dots, n + 1\}$, let $U_{n,k}$ be a mapping defined by

$$U_{n,k} = \begin{cases} I & \text{if } k = n + 1; \\ U_{n,k} = (1 - \gamma_k)I + \gamma_k T_k U_{n,k+1} & \text{if } k \in \{1, 2, \dots, n\}. \end{cases}$$

Let y be a point in H and $\{y_n\}$ a sequence defined by $y_1 = y$ and

$$y_{n+1} = (I - \lambda_n A)U_{n,1} y_n \tag{4.3}$$

for $n \in \mathbb{N}$. Then $\{y_n\}$ converges strongly to the unique solution of Problem 4.1.

Proof. Set $f_n = (I - A)U_{n,1}$ and $S_n = T_1 U_{n,2}$ for $n \in \mathbb{N}$. Then it is obvious from (4.3) that

$$\begin{aligned} y_{n+1} &= \lambda_n (I - A)U_{n,1} y_n + (1 - \lambda_n)U_{n,1} y_n \\ &= \lambda_n f_n(y_n) + (1 - \lambda_n)\big((1 - \gamma_1)y_n + \gamma_1 S_n y_n\big) \end{aligned}$$

for every $n \in \mathbb{N}$. It is known that

$$\mathrm{Fix}(S_n) = \mathrm{Fix}(U_{n,1}) = \bigcap_{k=1}^{n} \mathrm{Fix}(T_k)$$

by [23, Lemma 3.2]; see also [9, Lemma 4.2]. Hence we have

$$\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(U_{n,1}) = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{n} \mathrm{Fix}(T_k) = F$$

and thus

$$f_n(z) = (I - A)U_{n,1} z = (I - A)z$$

for all $z \in F$. Therefore, $\{f_n\}$ is stable on F. Since $U_{n,1}$ is nonexpansive, f_n is a θ-contraction by Lemma 2.4, where $\theta = \sqrt{1 - 2\kappa + \eta^2}$. It is also known that $\{S_n\}$ satisfies the conditions (Z) and (R); see [3, 5, 9, 11]. Therefore, Theorem 3.6 implies that $\{y_n\}$ converges strongly to $w = P_F\big(f_1(w)\big) = P_F(I - A)w$, which is the unique solution of Problem 4.1 by Lemma 2.4. □

5 Zero point problems and fixed point problems

Motivated by Ceng, Petruşel, and Yao [13], we consider the problem of finding a common solution of the zero point problem for a maximal monotone operator and the fixed point problems for nonexpansive mappings. Then, by using Theorem 3.6, we prove the following strong convergence theorem, which is a generalization of [13, Theorem 3.1]; see Remark 5.2 below.

Theorem 5.1. *Let* H *be a Hilbert space,* C *a nonempty closed convex subset of* H, $\{T_n\}$ *a sequence of nonexpansive self-mappings of* C, $A \subset H \times H$ *a maximal monotone operator with* $\mathrm{dom}(A) \subset C$, θ *a nonnegative real number with* $\theta < 1$, *and* f *a* θ*-contraction on* C. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *be sequences in* $[0, 1)$ *such that* $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \le \sup_n \beta_n < 1$, $\alpha_n + \beta_n \le 1$ *for every* $n \in \mathbb{N}$, $0 < \liminf_{n\to\infty} \gamma_n \le \sup_n \gamma_n < 1$, *and* $\gamma_{n+1} - \gamma_n \to 0$. *Let* $\{\rho_n\}$ *be a sequnece of positive real numbers such that* $\inf_n \rho_n > 0$ *and* $\rho_{n+1} - \rho_n \to 0$. *Suppose that* $F = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap A^{-1}0$ *is nonempty and*

$$\lim_{n\to\infty} \sup_{y\in D} \|T_n y - T_m T_n y\| = 0 \text{ and } \lim_{n\to\infty} \sup_{y\in D} \|T_{n+1} y - T_m T_n y\| = 0 \tag{5.1}$$

for any $m \in \mathbb{N}$ *and nonempty bounded subset* D *of* C. *Let* y *be a point in* C *and* $\{y_n\}$ *a sequence defined by* $y_1 = y$ *and*

$$y_{n+1} = \alpha_n f(V_n x_n) + (1 - \alpha_n - \beta_n) x_n + \beta_n T_n V_n x_n \tag{5.2}$$

for $n \in \mathbb{N}$, *where* $V_n = \gamma_n I + (1 - \gamma_n) T_n J_{\rho_n}$ *and* J_{ρ_n} *is the resolvent of* A. *Then* $\{y_n\}$ *converges strongly to the unique fixed point of* $P_F \circ f$.

Proof. Since $\gamma_n \neq 1$ and $\mathrm{Fix}(T_n) \cap \mathrm{Fix}(J_{\rho_n}) = \mathrm{Fix}(T_n) \cap A^{-1}0$ is nonempty, it follows from [8, Corollary 3.9] and [9, Corollary 3.6] that

$$\mathrm{Fix}(V_n) = \mathrm{Fix}(T_n J_{\rho_n}) = \mathrm{Fix}(T_n) \cap \mathrm{Fix}(J_{\rho_n}) = \mathrm{Fix}(T_n) \cap A^{-1}0$$

and

$$\mathrm{Fix}(T_n V_n) = \mathrm{Fix}(T_n) \cap \mathrm{Fix}(V_n) = \mathrm{Fix}(V_n)$$

for every $n \in \mathbb{N}$. Therefore, we have

$$\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n V_n) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(V_n) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap A^{-1}0 = F \neq \emptyset. \tag{5.3}$$

It is clear that each V_n is nonexpansive and thus $f \circ V_n$ is a θ-contraction for every $n \in \mathbb{N}$. Since $f(V_n z) = f(z)$ for all $z \in F$ by (5.3), we see that $\{f \circ V_n\}$ is stable on F.

We next show that $\{T_n V_n\}$ satisfies the condition (R). Let D be a nonempty bounded subset of C. By (5.1), we have

$$\lim_{n\to\infty} \sup_{y\in D} \|T_{n+1} y - T_n y\| \le \lim_{n\to\infty} \sup_{y\in D} \|T_{n+1} y - T_1 T_n y\| + \lim_{n\to\infty} \sup_{y\in D} \|T_1 T_n y - T_n y\| = 0$$

and hence $\{T_n\}$ satisfies the condition (R). Since $\{J_{\rho_n}\}$ satisfies the condition (R) by [5, Example 4.2], Lemma 2.2 shows that $\{T_n J_{\rho_n}\}$ satisfies the condition (R). Thus Lemma 2.3 implies that $\{V_n\}$ satisfies the condition (R). Therefore, it follows from Lemma 2.2 that $\{T_n V_n\}$ satisfies the condition (R).

We next show that $\{T_n V_n\}$ satisfies the condition (Z). Let $\{x_n\}$ be a bounded sequence in C such that $x_n - T_n V_n x_n \to 0$ and $\{x_{n_i}\}$ a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$. It is enough to show that $z \in F$. It follows from [8, Theorem 3.10] that $x_n - T_n x_n \to 0$ and $x_n - V_n x_n \to 0$. Let D be a nonempty bounded subset of C such that $x_n \in D$ for all $n \in \mathbb{N}$. For fixed $m \in \mathbb{N}$, it follows from (5.1) and $x_n - T_n x_n \to 0$ that

$$\begin{aligned}\|x_n - T_m x_n\| &\le \|x_n - T_n x_n\| + \|T_n x_n - T_m T_n x_n\| + \|T_m T_n x_n - T_m x_n\| \\ &\le 2\|x_n - T_n x_n\| + \sup_{y \in D} \|T_n y - T_m T_n y\| \to 0\end{aligned}$$

as $n \to 0$. Thus, by the demiclosedness [15, p.109] of $I - T_m$, $z \in \mathrm{Fix}(T_m)$ and hence $z \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. On the other hand, $x_n - V_n x_n \to 0$ and [9, Corollary 3.2] imply that $x_n - T_n J_{\rho_n} x_n \to 0$ and hence $x_n - J_{\rho_n} x_n \to 0$ by [8, Theorem 3.10]. Thus $z \in A^{-1}0$ because $\{J_{\rho_n}\}$ satisfies the condition (Z); see [8, Lemma 5.1], [10, Lemma 2.1], and [12, Lemma 2.4]. Consequently, we conclude that $z \in F$.

Finally, by assumption, it is obvious that

$$y_{n+1} = \alpha_n f(V_n x_n) + (1 - \alpha_n)\left(\left(1 - \frac{\beta_n}{1 - \alpha_n}\right)x_n + \frac{\beta_n}{1 - \alpha_n} T_n V_n x_n\right)$$

for every $n \in \mathbb{N}$ and

$$0 < \liminf_{n \to \infty} \frac{\beta_n}{1 - \alpha_n} \le \limsup_{n \to \infty} \frac{\beta_n}{1 - \alpha_n} < 1.$$

Thus Theorem 3.6 implies the conclusion. □

Remark 5.2. *Ceng, Petruşel, and Yao* [13] *considered an equilibrium problem for a real-valued function* ϕ *defined on* $C \times C$ *and they adopted the resolvent of* ϕ *in* [13, Theorem 3.1]. *According to* [6], *such an equilibrium problem can be regarded as a zero point problem for a maximal monotone operator* $A \subset H \times H$ *and we know that the resolvent* ϕ *is equivalent to that of* A. *Thus Theorem 5.1 is a generalization of* [13, Theorem 3.1].

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