Weighted pseudo almost automorphic solutions of fractional functional differential equations

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ABSTRACT
In this paper we discuss the existence of weighted pseudo almost automorphic solution of fractional order functional differential equations. Using the fixed point theorem we establish existence and uniqueness of solution to the problem under consideration. The results obtained extend the theory of almost automorphic solutions to a more general class of weighted pseudo almost automorphic solutions. These extensions allow to treat infinite dimensional dynamics such as fractional wave and heat equation which are presented in the paper. At the end we give several example to illustrate the analytical findings.

RESUMEN
En este artículo discutimos la existencia de una solución seudo casi automórfica con peso de ecuaciones diferenciales funcionales de orden fraccional. Usando el teorema del punto fijo, establecemos la existencia y unicidad de la solución del problema en estudio. Los resultados obtenidos extienden la teoría de soluciones casi automórficas a clases más generales de soluciones seudo casi automórficas con peso. Estas extensiones permiten estudiar dinámicas infinito-dimensional como la onda fraccionaria y la ecuación del calor, las cuales se presentan en este artículo. Al final, mostramos varios ejemplos para ilustrar los resultados analíticos obtenidos.

Keywords and Phrases: Fractional differential equation, Fixed point theorem, Almost automorphic functions, Abstract differential equations.

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1 Introduction

In this work we consider the following functional differential equations of fractional order $\alpha \in (1, 2)$,

$$
D^\alpha_t u(t) = Au(t) + D^{\alpha-1}f(t, u(t), u_t), \quad t \in \mathbb{R},
$$

$$
u(t) = \phi(t), \quad t \in (-\infty, 0],
$$

$$
u_t(\theta) = u(t + \theta), \quad \theta \in (-\infty, 0],$$

where $f : \mathbb{R} \times X \times X \to X$, $\phi \in C^0((-\infty, 0], \mathbb{R})$ and $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space $X$. With motivation coming from a wide range of engineering and physical applications, fractional differential equations have recently attracted great attention of mathematicians and scientists. This kind of equations is a generalization of ordinary differential equations to arbitrary non integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy and science. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation. Fractional differential equations find numerous applications in the field of visco-elasticity, feedback amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modelling encompassing different branches of physics, chemistry and biological sciences. Many physical processes appear to exhibit fractional order behavior that may vary with time or space. The fractional calculus has allowed the operations of integration and differentiation to any fractional order. The order may take on any real or imaginary value. The existence and uniqueness of solutions to fractional differential equations have been shown by many authors. Agarwal et al. have shown the existence of weighted pseudo almost periodic solutions of semilinear fractional differential equations.

Since Bohr introduced the concept of almost periodic functions, there have been many important generalizations of this functions in the past few decades. The generalization includes pseudo almost periodic functions, where the function can be decomposable in two part. These functions are further generalized to weighted pseudo almost periodic function by Diagana, where the weighted mean of the second component is zero. Another direction of generalization is almost automorphic functions introduced by Bochner. The pseudo almost automorphic functions are natural generalization of almost automorphic functions and introduced by Liang et al. These functions are further generalized by Blot et al. and named weighted pseudo almost automorphic. The authors in have proved very important properties of these functions including composition theorem and completeness property. The study of weighted pseudo almost automorphic solutions of various kind of differential equations are very new and an attractive area of research. For more details on theory and applications of these functions we refer to and references therein. The existence and uniqueness of almost automorphic and pseudo almost automorphic solutions have been established by many authors, for example and references therein.
The problem considered in this work is motivated by the work of Claudio Cuevas and Carlos Lizama [17] work in which they have considered the following fractional differential equations

$$D^\alpha_t u(t) = Au(t) + D^{\alpha - 1}_t f(t, u(t)), \quad t \in \mathbb{R},$$

and proved the existence of almost automorphic solutions under certain assumptions. In this paper we discuss existence and uniqueness of weighted pseudo almost automorphic solutions of problem (1). The concept of Stepanov like pseudo almost periodicity is introduced by Diagana [21, 22], which is a generalization of pseudo almost periodicity. Further Stepanov like almost automorphy has been introduced by N’Guerekata and Pankov [29].

2 Preliminaries

Denote $B(X)$ be the Banach space of all linear and bounded operators on $X$ endowed with the norm $\| \cdot \|_{B(X)}$ and $C = C(\mathbb{R}, X)$ the set of all continuous functions from $\mathbb{R}$ to $X$.

Let $U$ the collection of all positive integrable functions $\rho : \mathbb{R} \to \mathbb{R}$. For each $\rho \in U$ define

$$m(r, \rho) = \int_{-r}^{r} \rho(s) \, ds.$$

Denote

$U_\infty : \text{The set of all } \rho \in U \text{ such that } \lim_{r \to \infty} m(r, \rho) = \infty$

$U_b : \text{The set of all bounded } \rho \in U_\infty \text{ such that } \inf_{t \in \mathbb{R}} \rho(t) > 0.$

Now we state the definitions of weighted almost automorphic functions.

**Definition 2.1.** A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for every real sequence $(s_n)$, there exists a subsequence $(s_{n_k})$ such that

$$g(t) = \lim_{n \to \infty} f(t + s_{n_k})$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n \to \infty} g(t - s_{n_k}) = f(t)$$

for each $t \in \mathbb{R}$. The set of all almost automorphic functions from $\mathbb{R}$ to $X$ are denoted by $AA(X)$.

The set of all almost automorphic functions from $\mathbb{R}$ to $X$ are denoted by $AA(X)$ and it is a Banach space equipped with the sup norm

$$\| f \|_\infty = \sup_{t \in \mathbb{R}} \| f(t) \|.$$

**Definition 2.2.** A continuous function $f : \mathbb{R} \times X \to \mathbb{R}$ is called almost automorphic in $t$ uniformly for $x$ in compact subsets of $X$ if for every compact subset $K$ of $X$ and every real sequence $(s_n)$, there exists a subsequence $(s_{n_k})$ such that

$$g(t, x) = \lim_{n \to \infty} f(t + s_{n_k}, x)$$
is well defined for each \( t \in \mathbb{R}, x \in K \) and
\[
\lim_{n \to \infty} g(t - s_n, x) = f(t, x)
\]
for each \( t \in \mathbb{R}, x \in K \). Denote by \( AA(\mathbb{R} \times X) \) the set of all such functions.

We denote by
\[
AA_0(X) = \{ f \in BC(\mathbb{R}, X) : \lim_{r \to \infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \rho(t) \| f(t) \| \, dt = 0 \},
\]
and by \( AA_0(\mathbb{R} \times X \times X, X) \) the set of all continuous functions \( f : \mathbb{R} \times X \times X \to X \) such that
\[
\lim_{r \to \infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \rho(t) \| f(t, u, \phi) \| \, dt = 0,
\]
uniformly in \((u, \phi) \in X \times X\).

**Definition 2.3.** A mapping \( f \in BC(\mathbb{R}, X) \) is called weighted pseudo almost automorphic if it can be written as
\[
f = f_1 + f_2,
\]
where \( f_1 \in AA(X) \) and \( f_2 \in AA_0(X) \).

The functions \( f_1 \) and \( f_2 \) are called the almost automorphic and the weighted ergodic perturbation components of \( f \) respectively. The set of all such functions will be denoted by \( PAA(X) \).

**Remark 2.4.** A classical example of pseudo almost automorphic function is
\[
f(t) = \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t} + \frac{1}{1 + t^2}, \quad t \in \mathbb{R}.
\]
One can easily see that this function is not almost periodic.

**Example:** Consider the function
\[
f(t) = \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t} + e^{\alpha t}
\]
It is well known that the function \( \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t} \) is almost automorphic. Now consider the weight function \( \rho \) defined by \( \rho(t) = 1 \) \( t < 0 \) and \( \rho(t) = e^{-\beta t} \) \( t \geq 0 \) for some \( \beta > 0 \). It is easy to verify that
\[
m(r, \rho) = \int_{-r}^{r} \rho(t) \, dt = \int_{-r}^{0} \rho(t) \, dt + \int_{0}^{r} \rho(t) \, dt = r + \frac{1 - e^{-\beta r}}{\beta}.
\]
Thus \( \lim_{r \to \infty} m(r, \rho) = \infty \) which implies that \( \rho \in U_\infty \). Further
\[
\int_{-r}^{r} e^{\alpha t} \rho(t) \, dt = \int_{-r}^{0} e^{\alpha t} \, dt + \int_{0}^{r} e^{\alpha t} e^{-\beta t} \, dt = \frac{1 - e^{-\alpha r}}{\alpha} + \int_{0}^{r} e^{(\alpha - \beta)t} \, dt = \frac{1 - e^{-\alpha r}}{\alpha} + \frac{e^{(\alpha - \beta)r} - 1}{\alpha - \beta}.
\]
Thus for $\alpha \leq \beta$, we have
\[
\lim_{r \to \infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} e^{\alpha t} \rho(t) dt = 0.
\]
Hence $e^{\alpha t} \in \text{PAA}(\mathbb{R}, \rho)$ and so $f(t) \in \text{WPAA}(\mathbb{R})$. It is also interesting to note that
\[
\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} e^{\alpha t} dt = \lim_{r \to \infty} \frac{e^{\alpha r} - e^{-\alpha r}}{2\alpha r} = \infty.
\]
This implies that $f(t)$ does not belong to $\text{PAA}(\mathbb{R})$, the space of all pseudo almost automorphic functions.

**Definition 2.5.** A continuous mapping $f : \mathbb{R} \times X \times X \to X$ is called weighted pseudo almost automorphic in $t \in \mathbb{R}$ uniformly in $(x, \phi) \in X \times X$ if it can be written as $f = f_1 + f_2$, where $f_1 \in \text{AA}(\mathbb{R} \times X \times X, X)$ and $f_2 \in \text{AA}_0(\mathbb{R} \times X \times X, X)$.

We denote the set of all weighted pseudo almost automorphic functions $f : \mathbb{R} \times X \times X \to X$ by $\text{WPAA}(\mathbb{R} \times X \times X)$.

The following theorems are from [13].

**Theorem 2.6.** The decomposition of a weighted pseudo almost automorphic function is unique for any $\rho \in U_b$.

**Theorem 2.7.** Let $\text{WPAA}(\mathbb{R}, \rho) \ni f = g + \phi$ where $\rho \in U_{\infty}$ and assume that $f(t, u)$ is uniformly continuous in any bounded subset $K$ of $X$ uniformly in $t \in \mathbb{R}$ and $g(t, u)$ is uniformly continuous in any bounded subset $K$ of $X$ uniformly in $t \in \mathbb{R}$. Then if $u \in \text{WPAA}(\mathbb{R}, \rho)$, implies $f(\cdot, u(\cdot)) \in \text{WPAA}(\mathbb{R}, \rho)$.

The above theorem holds if both functions $f, g$ are Lipschitz continuous in $u$ uniformly in $t \in \mathbb{R}$. The weight one functions that is $\rho = 1$, are called pseudo almost automorphic.

### 3 Weighted pseudo almost automorphic solutions

**Assumptions:** Let us consider the the following assumptions:

(A1) The function $f : \mathbb{R} \times X \times X \to X$ is weighted pseudo almost automorphic with respect to $t$ uniformly in $(u, \phi) \in X \times X$, and there exists $0 < L < 1$, such that
\[
\|f(t, u, \phi) - f(t, v, \psi)\| \leq L(\|u - v\| + \|\phi - \psi\|).
\]

(A2) The function $f$ is bounded.

**Lemma 3.1.** Let $\{S(t)\}_{t>0} \subset B(X)$ be a strongly continuous family of bounded and linear operators such that $\|S(t)\| \leq \phi(t)$ for almost all $t \in \mathbb{R}^+$ with $\phi \in L^1(\mathbb{R}^+)$. If $f : \mathbb{R} \to X$ is a weighted pseudo almost automorphic function then $\int_{-\infty}^{\infty} S(t-s)f(s)ds \in \text{WPAA}(X)$. 

A closed and linear operator $A$ is said to be sectorial of type $\omega$ and angle $\theta$ if there exists $0 < \theta < \frac{\pi}{2}, M > 0$ and $\omega \in \mathbb{R}$ such that its resolvent exists outside the sector
\[ \omega + S_\theta := \{ \omega + \lambda : \lambda \in \mathbb{C}, \ |\arg(-\lambda)| < \theta \}, \]
and
\[ \| (\lambda - A)^{-1} \| \leq \frac{M}{|\lambda - \omega|}, \ \lambda \notin \omega + S_\theta. \]

Sectorial operators are well studied in the literature. For a recent reference including several examples and properties we refer the reader to [30]. Note that an operator $A$ is sectorial of type $\omega$ if and only if $\omega I - A$ is sectorial of type $0$.

The equation (1) can be thought as a limiting case of the following equation
\[ v'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Av(s)ds + f(t, u(t), u_t), \ t \geq 0, \ v_t(\theta) = \phi(t), t \in (-\infty, 0), \]
(4)
in the sense that the solutions are asymptotic to each other as $t \to \infty$. If we consider that the operator $A$ is sectorial of type $\omega$ with $\theta \in [0, \pi(1 - \frac{\alpha}{2})]$, then problem (4) is well posed [19]. Thus we can use variation of parameter formulae to get
\[ v(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s)f(s, u(s), u_s)ds, \ t \geq 0, \]
where
\[ S_\alpha(t) = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} \lambda^{\alpha-1}(\lambda^{\alpha} I - A)^{-1} d\lambda, \ t \geq 0, \]
where the path $\gamma$ lies outside the sector $\omega + S_\theta$. If $S_\alpha(t)$ is integrable then the solution is given by
\[ u(t) = \int_{-\infty}^t S_\alpha(t-s)f(s, u(s), u_s)ds. \]

Now one can easily see that
\[ v(t) - u(t) = S_\alpha(t)u_0 - \int_t^\infty S_\alpha(s)f(t-s, u(t-s), u_{t-s}). \]
Hence for $f \in L^p([\mathbb{R}^+ \times X \times X, X])$, $p \in [1, \infty)$ we have $v(t) - u(t) \to 0$ as $t \to \infty$.

**Definition 3.2.** A function $u : \mathbb{R} \to X$ is said to be a mild solution to (7) if the function $S_\alpha(t-s)f(s, u(s), u_s)$ is integrable on $(\infty, t)$ for each $t \in \mathbb{R}$ and
\[ u(t) = \int_{-\infty}^t S_\alpha(t-s)f(s, u(s), u_s)ds, \]
for each $t \in \mathbb{R}$.
Recently, Cuesta in [19], Theorem-1, has proved that if $A$ is a sectorial operator of type $\omega < 0$ for some $M > 0$ and $\theta \in [0, \pi(1 - \frac{\omega}{\alpha}))$, then there exists $C > 0$ such that
\[
\|S_\alpha(t)\| \leq \frac{CM}{1 + |\omega|t^\alpha}
\]
for $t \geq 0$. Also the following relation [17], Theorem-3.4 holds,
\[
\int_0^\infty \frac{dt}{1 + |\omega|t^\alpha} = \frac{|\omega|^{-\frac{1}{\alpha}}}{\alpha \sin \frac{\pi}{\alpha}}
\]
for $\alpha \in (1, 2)$.

Define the operator
\[
Fu(t) = \int_{-\infty}^t S_\alpha(t - s)f(s, u(s), u_s)ds, \quad t \in \mathbb{R}.
\]
First thing we observe about the operator $F$ is boundedness and continuity. Indeed,
\[
\|Fu\| \leq \int_{-\infty}^t \|S_\alpha(t - s)\| \times \|f(s, u(s), u_s)\| ds
\]
\[
\leq \int_0^\infty \|S_\alpha(s)\| \|f(t - s, u(t - s), u_{t-s})\| ds
\]
\[
\leq CM \int_0^\infty \frac{1}{1 + |\omega|s^\alpha} \|f(t - s, u(t - s), u_{t-s})\| ds
\]
\[
\leq CM\|f\| \int_0^\infty \frac{1}{1 + |\omega|s^\alpha} ds = \frac{CM\|f\| \omega^{-\frac{1}{\alpha}}}{\alpha \sin \frac{\pi}{\alpha}}
\]
Thus $F$ is bounded. Further, we have
\[
\|Fu(t + h) - Fu(t)\|
\]
\[
= \left\| \int_{-\infty}^t S_\alpha(t + h - s)f(s, u(s), u_s)ds - \int_{-\infty}^t S_\alpha(t - s)f(s, u(s), u_s)ds \right\|
\]
\[
\leq \int_{-\infty}^t \|S_\alpha(t - s)\| \times \|f(s + h, u(s + h), u_{s+h}) - f(s, u(s), u_s)\| ds
\]
\[
\leq \int_0^\infty \|S_\alpha(s)\| \times \|f(t - s + h, u(t - s + h), u_{t-s+h}) - f(t - s, u(t - s), u_{t-s})\| ds
\]
\[
\leq CM\sup_{t \in \mathbb{R}} \|f(t - s + h, u(t - s + h), u_{t-s+h}) - f(t - s, u(t - s), u_{t-s})\|
\]
\[
\times \int_0^\infty \frac{1}{1 + |\omega|s^\alpha} ds
\]
\[
= \frac{CM\omega^{-\frac{1}{\alpha}}}{\alpha \sin \frac{\pi}{\alpha}} \sup_{t \in \mathbb{R}} \|f(t - s + h, u(t - s + h), u_{t-s+h}) - f(t - s, u(t - s), u_{t-s})\|,
\]
which goes to zero as $h \to 0$ and hence $F$ is continuous.

It is easy to see that the operator $F$ maps $\text{WPAA}(X)$ to $\text{WPAA}(X)$, which we represent in the form of a lemma as follows.

**Lemma 3.3.** The operator $F$ maps $\text{WPAA}(X)$ to $\text{WPAA}(X)$ if $f \in \text{WPAA}(X)$.

**Proof:** As $f \in \text{WPAA}(X)$, we can decompose it into two parts $f_1 \in AA(X)$ and $f_2 \in AA_0(X)$. Now define the operators

\[
F_1 u(t) = \int_{-\infty}^{t} S_\alpha(t - s)f_1(s, u(s), u_s) ds, \quad t \in \mathbb{R}
\]

and

\[
F_2 u(t) = \int_{-\infty}^{t} S_\alpha(t - s)f_2(s, u(s), u_s) ds, \quad t \in \mathbb{R}.
\]

Also for every sequence $t_n$ there exists a subsequence $t_{n_k}$ such that

\[
f_1(t + t_{n_k}, u, \psi) \to g_1(t, u, \psi),
\]

\[
g_1(t - t_{n_k}, u, \psi) \to f_1(t, u, \psi), \quad u, \psi \in D,
\]

where $D$ is a compact subset of $X \times X$.

\[
F_1 u(t + t_{n_k}) = \int_{-\infty}^{t + t_{n_k}} S_\alpha(t + t_{n_k} - s)f_1(s, u(s), u_s) ds
\]

\[
= \int_{-\infty}^{t} S_\alpha(t - s)f_1(s + t_{n_k}, u(s + t_{n_k}), u_{s + t_{n_k}}) ds
\]

\[
\to \int_{-\infty}^{t} S_\alpha(t - s)g_1(s, u(s), u_s) ds
\]

\[
= (F^* u)(t).
\]

Thus

\[
(F_1 u)(t + t_{n_k}) \to (F^* u)(t).
\]

Similarly one can get

\[
(F^* u)(t - t_{n_k}) \to (F_1 u)(t).
\]

Now we need to show

\[
\lim_{r \to \infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \int_{-\infty}^{t} \rho(s)|S_\alpha(t - s)||f_2(s, u(s), u_s)|| ds dt = 0.
\]

Consider

\[
\frac{1}{m(r, \rho)} \int_{-r}^{r} \int_{-\infty}^{t} \rho(s)|S_\alpha(t - s)||f_2(s, u(s), u_s)|| ds dt \leq I_1(r) + I_2(r),
\]

where

\[
I_1(r) = \frac{1}{m(r, \rho)} \int_{-r}^{r} dt \int_{-r}^{t} \rho(s)|S_\alpha(t - s)||f_2(s, u(s), u_s)|| ds
\]
and
\[
I_2(r) = \frac{1}{m(r, \rho)} \int_{-r}^{r} dt \int_{-\infty}^{r} \rho(s) \|S_\alpha(t-s)\| \|f_2(s, u(s), u_s)\| ds.
\]
Thus we have
\[
I_1(r) \leq \frac{1}{m(r, \rho)} \int_{-r}^{r} \rho(\xi) \|f_2(\xi, u(\xi), u_\xi)\| d\xi \int_{s}^{r} \|S_\alpha(t-\xi)\| dt
\]
\[
\leq \frac{1}{m(r, \rho)} \int_{-r}^{r} \rho(\xi) \|f_2(\xi, u(\xi), u_\xi)\| d\xi \int_{0}^{r-s} \|S_\alpha(t)\| dt
\]
\[
\leq \frac{CM}{m(r, \rho)} \int_{-r}^{r} \rho(\xi) \|f_2(\xi, u(\xi), u_\xi)\| d\xi \int_{0}^{\infty} \|S_\alpha(t)\| dt
\]
\[
\leq \frac{M_1}{m(r, \rho)} \int_{-r}^{r} \rho(\xi) \|f_2(\xi, u(\xi), u_\xi)\| d\xi,
\]
for some positive constant $M_1$. The above calculations imply that
\[
\lim_{r \to \infty} I_1(r) = 0
\]
as $f_2 \in \mathbb{AA}_0(\mathbb{R} \times X \times X)$. Now consider
\[
I_2(r) \leq \frac{1}{m(r, \rho)} \int_{-r}^{r} dt \int_{t+r}^{\infty} \rho(t-s) \|S_\alpha(s)\| \|f_2(t-s, u(t-s), u_{t-s})\| ds
\]
\[
\leq \frac{1}{m(r, \rho)} \int_{-r}^{r} dt \int_{2r}^{\infty} \rho(t-s) \|S_\alpha(s)\| \|f_2(t-s, u(t-s), u_{t-s})\| ds
\]
\[
\leq \|f_2\| \int_{2r}^{\infty} \|S_\alpha(s)\| ds.
\]
From the above analysis we get
\[
\lim_{r \to \infty} I_2(r) = 0.
\]
Thus we have
\[
\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \|F_2(u)(t)\| dt = 0.
\]
Hence the result is proved.

**Theorem 3.4.** Problem (7) has a unique solution in $\mathbb{WPAA}(X)$ under the assumption $(A_1)$ provided that
\[
\frac{2L|\omega|^{\frac{1}{\alpha}} \pi}{\alpha \sin \frac{\alpha}{\alpha}} < 1.
\]
Proof: In order to prove that the operator $F$ has a fixed point, consider
\[
\|F u_1(t) - F u_2(t)\| \\
\leq \int_{-\infty}^{t} \|S_\alpha(t-s)\| \left(\|f(s,u_1(s),u_{1s}) - f(s,u_2(s),u_{2s})\| ds \\
\leq L \int_{-\infty}^{t} \|S_\alpha(t-s)\| \left(\|u_1(s) - u_2(s)\| + \|u_{1s} - u_{2s}\|_{B(X)}\right) ds \\
\leq 2L\|u_1 - u_2\| \int_{0}^{\infty} \|S_\alpha(t)\| dt.
\]
Thus for $2L \int_{0}^{\infty} |S_\alpha(t)|dt < 1$, the problem (1) has an unique solution. We have mentioned that
\[
\int_{0}^{\infty} \frac{1}{\frac{1}{2} + |\omega|t^\alpha} = \frac{|\omega|^{\frac{1}{2} + \frac{\pi}{\alpha}}} {\frac{\sin \frac{\pi}{\alpha}}}{}
\]
for $\alpha \in (1,2)$. Thus the above condition reduces to $\frac{2CML|\omega|^{\frac{1}{2} + \frac{\pi}{\alpha}}}{\alpha \sin \frac{\pi}{\alpha}} < 1$.

Remark 3.5. One can easily show that for $f$ Stepanov almost automorphic, the problem (1) has a unique stepanov almost automorphic solutions under the same condition as in both Theorems.

Remark 3.6. It is to note that for differential equation
\[
\frac{du(t)}{dt} = Au(t) + f(t,u(t), u(t+\tau,x))), t \in \mathbb{R} \\
u(0) = u_0,
\]
where $A$ generates an exponentially stable $C_0$ semigroup $\{T(t)\}_{t \geq 0}$, we can conclude that, if $f$ if Lipschitz continuous, bounded and Weyl almost automorphic or Weyl pseudo almost automorphic, then there exists a unique Weyl almost automorphic or Weyl pseudo almost automorphic solution accordingly of the problem provided that $\frac{L \cdot M_1}{\delta} < 1$, where $\|T(t)\| \leq M_1e^{-\delta t}$ for some $M \geq 1$ and $\delta > 0$.

4 Examples

Example-1: Consider the following fractional order partial differential equation for $\alpha \in (1,2),$
\[
\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial x^2} + \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}(g(t,u(t,x),u(t-\tau,x))), \quad \tau > 0, \\
t \in \mathbb{R}, \quad x \in (0,\pi) \\
u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R}, \\
u(t,x) = \phi(t,x) \quad t \in [-\tau,0],
\]
where $g$ is a weighted pseudo almost automorphic function in $t$. Also assume that $g$ satisfies Lipschitz condition in both variable with Lipschitz constant $L_g$. Using the transformation $u(t)x = \cdot$
satisfies Lipschitz condition in both variable with Lipschitz constant \( L_p > 0 \) where

\[
\frac{2L_g|\omega_1|^\frac{1}{\alpha}\pi}{\alpha \sin \frac{\pi}{\alpha}} < 1.
\]

Thus under all the required assumption on \( g \), the existence of weighted almost automorphic solutions is ensured accordingly.

**Example-2:** Consider the following fractional order delay relaxation oscillation equation for \( \alpha \in (1, 2) \),

\[
\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial x^2} - p u(t,x) + \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}} f(t,u(t,x),u(t-\tau,x)), \quad \tau > 0,
\]

\[
t \in \mathbb{R}, \quad x \in (0, \pi)
\]

\[
\begin{aligned}
&u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R}, \\
&u(t,x) = \phi(t,x) \quad t \in [-\tau,0],
\end{aligned}
\]

where \( p > 0 \) and \( f \) is a weighted pseudo almost automorphic function in \( t \). Also assume that \( f \) satisfies Lipschitz condition in both variable with Lipschitz constant \( L_f \). Using the transformation

\[
u(t,x) = u(t,x)
\]

and define

\[
A = \frac{\partial^2 u}{\partial x^2} - p u,
\]

\( u \in D(A) \), where

\[
D(A) = \left\{ u \in L^2((0,\pi), \mathbb{C}), \ u' \in L^2((0,\pi), \mathbb{C}), \ u'' \in L^2((0,\pi), \mathbb{C}), \ u(0) = u(\pi) = 0 \right\},
\]

the above equation can be transform into

\[
\frac{d^\alpha u(t)}{dt^\alpha} = Au(t) + \frac{d^{\alpha-1}}{dt^{\alpha-1}} g(t,u(t),u_t(-\tau)),
\]

\( t \in \mathbb{R} \) and \( u(t) = \phi(t) \ t \in [-\tau,0] \). It is to note that \( A \) generates an analytic semigroup \( \{T(t) : t \geq 0 \} \) on \( X \), where \( X = L^2((0,\pi), \mathbb{R}) \). Hence \( pI - A \) is sectorial of type \( \omega < p < 0 \). Further \( A \) has discrete spectrum with eigenvalues of the form \( -k^2; k \in \mathbb{N} \), and corresponding normalized eigenfunctions given by \( z_k(x) = (\frac{\pi}{\alpha})^\frac{1}{\alpha} \sin(kx) \). As \( A \) is analytic, let us assume that it is sectorial of type \( \omega_1 \) and let the following relation holds

\[
\frac{2L_f|\omega|^\frac{1}{\alpha}\pi}{\alpha \sin \frac{\pi}{\alpha}} < 1.
\]
Thus under all the required assumption on \( f \), the existence of weighted almost automorphic solutions is ensured accordingly.

**Example-3:** Consider the following abstract differential equations of fractional order over a complex Banach space \((X, \| \cdot \|)\),

\[
\frac{d^\alpha u(t)}{dt^\alpha} = Au(t) + \frac{d^{\alpha-1}}{dt^{\alpha-1}}(g(t,u(t)) + Ku(t)),
\]

\( t \in \mathbb{R} \), where

\[
Ku(t) = \int_{-\infty}^{t} k(t-s)u(s)ds
\]

and \( A : D(A) \subset X \to X \) is a linear densely defined operator of sectorial type on a complex Banach space \( X \). We assume that \( g \) is weighted pseudo almost automorphic in \( t \) uniformly in \( u \) and \( k \) satisfy \( |k(t)| \leq Ce^{-bt} \) for some \( C,b > 0 \). For \( u \) weighted pseudo almost automorphic, it is not difficult to see that \( Ku(t) \) is also weighted pseudo almost automorphic. Let us assume that \( g \) satisfy Lipschitz condition with Lipschitz constant \( L_g \). Now for \( u_1,u_2 \in X \), consider

\[
\|g(t,u_1(t)) - g(t,u_2(t))\| + \|Ku_1(t) - Ku_2(t)\| \leq L_g \|u_1 - u_2\| + \int_{-\infty}^{t} |k(t-s)|u_1(s) - u_2(s)|ds
\]

\[
\leq L_g \|u_1 - u_2\| + \|u_1 - u_2\| \int_{0}^{\infty} |k(s)|ds
\]

\[
\leq L_g \|u_1 - u_2\| + \frac{C}{b} \|u_1 - u_2\|
\]

\[
\leq \left( L_g + \frac{C}{b} \right) \|u_1 - u_2\|. \tag{17}
\]

Thus we have

\[
\|g(t,u_1) - g(t,u_2)\| + \|Ku_1 - Ku_2\| \leq \left( L_g + \frac{C}{b} \right) \|u_1 - u_2\|.
\]

Considering \( t-s = s_1 \) we have

\[
Ku(t) = \int_{0}^{\infty} k(s_1)u(t+s_1) = \int_{0}^{\infty} k(s)u_1(s).
\]

Thus if we take \( g_1(t,u(t),u_1) = g(t,u(t)) + Ku(t) \), the above equation is similar to \([1]\). From the above analysis, one can easily see that \( g_1 \) satisfies Lipschitz condition with Lipschitz constant \( L_g + \frac{C}{b} \). Further assume that \( A \) is sectorial of type \( \omega_2 \) and the following condition hold

\[
\frac{2(L_g + \frac{C}{b})|\omega_2|^\frac{1}{\alpha} \pi}{\alpha \sin \frac{\pi}{\alpha}} < 1.
\]

One can easily see that for \( u \in WPAA(X) \), \( Ku(t) \in WPAA(X) \). Thus we can apply our result to ensure the existence of weighted almost automorphic solution for \( g \) weighted almost automorphic.

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References


