On a result of Q. Han, S. Mori and K. Tohge concerning uniqueness of meromorphic functions.

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ABSTRACT

In the paper we prove a result on the uniqueness of meromorphic functions that is related to a result of Q. Han, S. Mori and K. Tohge and is originated from a result of H. Ueda and two subsequent results of G. Brosch.

RESUMEN

En este artículo probamos un resultado de unicidad de funciones meromórficas que se relaciona a un resultado de Q. Han, S. Mori y K. Tohge, y se origina de un resultado de H. Ueda y dos resultados derivados de G. Brosch.

Keywords and Phrases: Meromorphic function, Uniqueness, Weighted sharing.

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1 Introduction, Definitions and Results

Let $f$ and $g$ be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that $f$ and $g$ share the value $a$ CM (counting multiplicities) if $f$, $g$ have the same $a$-points with the same multiplicities. If we do not take the multiplicities into account then $f$, $g$ are said to share the value $a$ IM (ignoring multiplicities). For the standard notations and definitions of the value distribution theory we refer to [5] and [15]. However we require following notations.

**Definition 1.** Let $k$ be a positive integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ and $\overline{E}_k(a; f)$ the collection of those $a$-points of $f$ whose multiplicities does not exceed $k$, with counting multiplicities and with ignoring multiplicities respectively.

**Definition 2.** Let $k$ be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. Then by $N(r, a; f) \leq k$ we denote the counting function of those $a$-points of $f$ (counted with proper multiplicities) whose multiplicities are not greater than $k$. By $\overline{N}(r, a; f) \leq k$ we denote the corresponding reduced counting function.

In an analogous manner we define $N(r, a; f) \geq k$ and $\overline{N}(r, a; f) \geq k$.

Also by $N(r, a; f) = k$ and $\overline{N}(r, a; f) = k$ we denote respectively the counting function and reduced counting function of those $a$-points of $f$ whose multiplicities are exactly $k$.

In 1980 H. Ueda ([14] see also p. 327 [15]) prove the following result.

**Theorem A.** [14] Let $f$ and $g$ be nonconstant entire functions sharing $0, 1$ CM, and $a(\neq 0, 1, \infty)$ be a complex number. If $E_\infty(a; f) \subset E_\infty(a; g)$, then $f$ is a bilinear transformation of $g$.

Improving Theorem A in 1989 G. Brosch [2] proved the following result.

**Theorem B.** [2] Let $f$ and $g$ be two nonconstant meromorphic functions sharing $0, 1, \infty$ CM, and $a(\neq 0, 1, \infty)$ be a complex number. If $E_\infty(a; f) \subset E_\infty(a; g)$, then $f$ is a bilinear transformation of $g$.

Following example shows that in Theorem B the condition $E_\infty(a; f) \subset E_\infty(a; g)$ cannot be replaced by $E_\infty(a; f) \subset E_\infty(b; g)$ for $b \neq a, 0, 1, \infty$.

**Example 1.** Let $f = e^{2z} + e^z + 1$, $g = e^{-2z} + e^{-z} + 1$, $a = \frac{3}{4}$ and $b = 3$. Then $f, g$ share $0, 1, \infty$ CM and $f - a = \frac{1}{4}(2e^z + 1)^2$, $g - b = e^{-2z}(1 + 2e^z)(1 - e^z)$. So $E_\infty(a; f) \subset E_\infty(b; g)$ but $f$ is not a bilinear transformation of $g$.

Considering the possibility $a \neq b$, G. Brosch [2] proved the following theorem.

**Theorem C.** [2] Let $f$ and $g$ be two nonconstant meromorphic functions sharing $0, 1, \infty$ CM, and $a, b$ be two complex numbers such that $a, b \notin (0, 1, \infty)$. If $E_\infty(a; f) = E_\infty(b; g)$, then $f$ is a bilinear transformation of $g$. 
In 2001 the idea of weighted sharing of values was introduced \{cf.\cite{6, 7}\} which provides a scaling between IM sharing and CM sharing of values. We now explain this notion in the following definition.

**Definition 3.** \cite{11} Let \( k \) be a nonnegative integer or infinity. For \( a \in \mathbb{C} \cup \{\infty\} \) we denote by \( E_k(a; f) \) the set of all \( a \)-points of \( f \), where an \( a \)-point with multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k + 1 \) times if \( m > k \). If \( E_k(a; f) = E_k(a; g) \), we say that \( f, g \) share the value \( a \) with weight \( k \).

The definition means that \( z_0 \) is a zero of \( f - a \) with multiplicity \( m(\leq k) \) if and only if \( z_0 \) is a zero of \( g \) with multiplicity \( m(\leq k) \) and \( z_0 \) is a zero of \( f - a \) with multiplicity \( m(> k) \) if and only if \( z_0 \) is a zero of \( g \) with multiplicity \( n(> k) \), where \( m \) is not necessarily equal to \( n \).

We write \( f, g \) share \( (a, k) \) to mean that \( f, g \) share the value \( a \) with weight \( k \). Clearly if \( f, g \) share \( (a, k) \) then \( f, g \) share \( (a, p) \) for all integers \( p \), \( 0 \leq p < k \). Also we note that \( f, g \) share a value \( a \) IM or CM if and only if \( f, g \) share \( (a, 0) \) or \( (a, \infty) \) respectively.

In 2004 using the idea of weighted value sharing T.C. Alzahari and H.X.Yi \cite{11} improved Theorem C in the following manner.

**Theorem D.** \cite{11} Let \( f, g \) be two nonconstant meromorphic functions sharing \( (a_1, 1), (a_2, \infty), (a_3, \infty) \), where \( \{a_1, a_2, a_3\} = \{0, 1, \infty\} \), and let \( a, b \) be two finite complex numbers such that \( a, b \notin \{0, 1\} \). If \( \overline{E}_\infty(a; f) = \overline{E}_\infty(a; g) \), then \( f \) is a bilinear transformation of \( g \). Moreover \( f \) and \( g \) satisfy exactly one of the following relations:

(i) \( f \equiv g \);
(ii) \( fg \equiv 1 \);
(iii) \( bf \equiv ag \);
(iv) \( f + g \equiv 1 \);
(v) \( f \equiv ag \);
(vi) \( f \equiv (1 - a)g + a \);
(vii) \( (1 - b)f \equiv (1 - a)g + (a - b) \);
(viii) \( (1 - a + g)f \equiv ag \);
(ix) \( f((b - a)g + (a - 1)b) \equiv a(b - 1)g \);
(x) \( f(\lambda) \equiv g; \)

The cases (ii) and (v) may occur if \( ab = 1 \), cases (iv) and (viii) may occur if \( a + b = 1 \), cases (vi) and (x) may occur if \( ab = a + b \).

Improving Theorem D recently I.Lahiri and P.Sahoo [12] proved the following theorem.

**Theorem E.** [12] Let \( f, g \) be two distinct nonconstant meromorphic functions sharing \((a_1, 1), (a_2, 1), (a_3, k)\), where \((a_1, a_2, a_3) = \{0, 1, \infty\} \) and \((m - 1)(mk - 1) > (1 + m)^2\). If for two values \( a, b \not\in \{0, 1, \infty\} \) the functions \( f - a \) and \( g - b \) share \((0, 0)\), then \( f, g \) share \((0, 0), (1, \infty), (\infty, \infty)\) and \( f - a, g - b \) share \((0, \infty)\). Also there exists a non-constant entire function \( \lambda \) such that \( f \) and \( g \) are one of the following forms:

(i) \( f = ae^\lambda \) and \( g = be^{-\lambda}, \) where \( ab = 1; \)

(ii) \( f = 1 + ae^\lambda \) and \( g = 1 + (1 - \frac{1}{\lambda})e^{-\lambda}, \) where \( ab = a + b; \)

(iii) \( f = \frac{a}{a + e} \) and \( g = \frac{e^\lambda}{1 - b + e^\lambda}, \) where \( a + b = 1; \)

(iv) \( f = \frac{e^\lambda - a}{e^\lambda + b} \) and \( g = \frac{b}{1 - e^{-\lambda}, \) where \( ab = 1; \)

(v) \( f = \frac{be^{-\lambda} - a}{be^{-\lambda} - b} \) and \( g = \frac{e^\lambda - a}{ae^\lambda - a}, \) where \( a \neq b; \)

(vi) \( f = \frac{1}{a - e^\lambda} \) and \( g = \frac{e^\lambda}{e^\lambda - b}, \) where \( ab = a + b; \)

(vii) \( f = \frac{b - a}{(b - 1)(1 - e\lambda)} \) and \( g = \frac{(b - a)e^\lambda}{(a - 1)(1 - e\lambda)}, \) where \( a \neq b; \)

(viii) \( f = a + e^\lambda \) and \( g = b(1 + \frac{1 - b}{a}), \) where \( a + b = 1; \)

(ix) \( f = e^\lambda - \frac{a(b - 1)}{a - b} \) and \( g = \frac{b(a - 1)}{a - b}(1 - \frac{a(b - 1)}{(b - a)}e\lambda), \) where \( a \neq b; \)

Q.Han, S.Mori and K.Tohge [4] further improved Theorem C, Theorem D, Theorem E and proved the following.

**Theorem F.** [4] Let \( f \) and \( g \) be two distinct nonconstant meromorphic functions sharing \((a_1, k_1), (a_2, k_2) \) and \((a_3, k_3)\) for three distinct values \( a_1, a_2, a_3 \in \mathbb{C} \cup \{\infty\}\), where \( k_1k_2k_3 > k_1 + k_2 + k_3 + 2\). Furthermore if \( \mathbb{E}_{k_1}(a_4; f) = \mathbb{E}_{k_1}(a_5; g) \) for values \( a_4, a_5 \) in \( \mathbb{C} \cup \{\infty\} \setminus \{a_1, a_2, a_3\} \) and for some positive integer \( k(\geq 2) \), then \( f \) is a bilinear transformation of \( g. \)
Example [1] with \( a = b = \frac{3}{4} \) shows that the conclusion of Theorem F does not hold for \( k = 1 \). This suggests that some further investigation is necessary for the case \( k = 1 \). In the paper we take up this problem and prove the following result.

**Theorem 1.1.** Let \( f, g \) be two distinct nonconstant meromorphic functions sharing \( (a_1, k_1), (a_2, k_2), (a_3, k_3) \) where \( a_1, a_2, a_3 \in \mathbb{C} \cup \{\infty\} \) are distinct and \( k_1k_2k_3 > k_1 + k_2 + k_3 + 2 \). Further let \( E_{11}(a; f) \subset E_{\infty}(b; g) \) for two complex numbers \( a, b \notin \{a_1, a_2, a_3\} \) and \( E_{11}(0; f') \subset E_{\infty}(0; g') \). Then \( f \) is a bilinear transformation of \( g \).

If, in particular, \( \{a_1, a_2, a_3\} = \{0, 1, \infty\} \), then there exists a non-constant entire function \( \lambda \) such that \( f \) and \( g \) assume exactly one of the following forms:

(i) \( f = ae^\lambda \) and \( g = be^{-\lambda} \) where \( ab = 1 \);

(ii) \( f = 1 + ae^\lambda \) and \( g = 1 + (1 - \frac{1}{b})e^{-\lambda} \) where \( ab = a + b \);

(iii) \( f = \frac{a}{e^\lambda - a} \) and \( g = \frac{e^\lambda - a}{1 - b + e^\lambda} \) where \( a + b = 1 \):

(iv) \( f = \frac{e^\lambda - a}{a e^\lambda - a} \) where \( E_{\infty}(a; f) = \phi \);

(v) \( f = \frac{b e^\lambda - a}{b e^\lambda} \) and \( g = \frac{b e^\lambda - a}{a e^\lambda - a} \) where \( a \neq b \);

(vi) \( f = \frac{a}{1 - e^\lambda} \) and \( g = \frac{ae^\lambda}{(a - 1)(1 - e^\lambda)} \) where \( E_{\infty}(a; f) = \phi \);

(vii) \( f = \frac{b - a}{(b - 1)(1 - e^\lambda)} \) and \( g = \frac{(b - a)e^\lambda}{(a - 1)(1 - e^\lambda)} \) where \( a \neq b \);

(viii) \( f = a + e^\lambda \) and \( g = (1 - a)/(1 + \frac{a}{e^\lambda}) \) where \( E(a; f) = \phi \);

(ix) \( f = e^\lambda - \frac{a(b - 1)}{a - b} \) and \( g = \frac{b(a - 1)}{a - b}(1 - \frac{a(b - 1)}{b - a}e^\lambda) \) where \( a \neq b \);

Considering Example [1] we see that the condition \( E_{11}(0; f') \subset E_{\infty}(0, g') \) is essential for Theorem [1.1]

## 2 Lemmas

In the section we present some necessary lemmas.

**Lemma 2.1.** [3] Let \( f \) and \( g \) share \( \{0, 0\}, \{1, 0\}, \{\infty, 0\} \). Then \( T(r, f) \leq 3T(r, g) + S(r, f) \) and \( T(r, g) \leq 3T(r, f) + S(r, f) \).

From this we conclude that \( S(r, f) = S(r, g) \). Henceforth we denote either of them by \( S(r) \).
Lemma 2.2. [16] Let $f$ and $g$ share $(0,k_1), (1,k_2), (\infty, k_3)$ and $f \neq g$, where $k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2$. Then

$$\overline{N}(r,0; f \geq 2) + \overline{N}(r,1; f \geq 2) + \overline{N}(r,\infty; f \geq 2) = S(r).$$

Following can be proved in the line of Theorem 3.2 of [11].

Lemma 2.3. Let $f$ and $g$ be two distinct nonconstant meromorphic functions sharing $(0,k_1), (1,k_2), (\infty, k_3)$, where $k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2$. If $N_0(r) + N_1(r) \geq \lambda T(r,f) + S(r)$ for some $\lambda > \frac{1}{2}$, then $f$ is a bilinear transformation of $g$ and

$$N_0(r) + N_1(r) = T(r,f) + S(r) = T(r,g) + S(r),$$

where $N_0(r)(N_1(r))$ denotes the counting function of those simple(multiple) zeros of $f-g$ which are not the zeros of $f(f-1)$ and $\frac{1}{f}$.

Lemma 2.4. [13] Let $f$ and $g$ be two distinct nonconstant meromorphic functions sharing $(0,0), (1,0), (\infty, 0)$. Further suppose that $f$ is a bilinear transformation of $g$ and $\mathcal{E}_1(a,f) \subset \mathcal{T}_\infty(b,g)$, where $a, b \notin \{0, 1, \infty\}$. Then there exists a nonconstant entire function $\lambda$ such that $f$ and $g$ assume exactly one of the forms given in Theorem 1.1.

Following can be proved in the line of Lemma 2.3 [13].

Lemma 2.5. Let $f$ and $g$ share $(0,k_1), (1,k_2), (\infty, k_3)$ and $f \neq g$, where $k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2$.

If $f$ is not a bilinear transformation of $g$, then for a complex number $a \notin \{0, 1, \infty\}$ each of the following holds:

(i) $N(r, a; f \geq 3) + N(r, a; g \geq 3) = S(r)$;

(ii) $T(r,f) = N(r, a; f \leq 2) + S(r)$;

(iii) $T(r,g) = N(r, a; g \leq 2) + S(r)$.

In the line of Lemma 5 [9] we can prove the following.

Lemma 2.6. Let $f, \ g \ share \ (0,k_1), (1,k_2), (\infty, k_3)$ and $f \neq g$, where $k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2$.

If $\alpha = \frac{f-1}{g-1}$ and $\beta = \frac{a}{f}$, then $\overline{N}(r, a; \alpha) = S(r)$ and $\overline{N}(r, a; \beta) = S(r)$ for $a = 0, \infty$.

Following is an analogue of Lemma 2.6 [13].

Lemma 2.7. Let $f$ and $g$ be two distinct meromorphic functions sharing $(0,k_1), (1,k_2), (\infty, k_3)$, where $k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2$. Then $T(r, \frac{a^{(r)}}{\alpha}) + T(r, \frac{b^{(r)}}{\beta}) = S(r)$, where $p$ is a positive integer and $\alpha, \beta$ are defined as in Lemma 2.6.
Using the techniques of [8] and [10] we can prove the following.

**Lemma 2.8.** Let \( f, g \) share \( (0, k_1), (1, k_2), (\infty, k_3) \) and \( f \neq g \), where \( k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2 \). If \( f \) is not a bilinear transformation of \( g \), then each of the following holds:

(i) \( T(r, f) + T(r, g) = N(r, f; f \leq 1) + N(r, 1; f \leq 1) + N(r, \infty; f \leq 1) + N_0(r) + S(r) \),

(ii) \( T(r, f) = N(r, 0; g' \leq 1) + N_0(r) + S(r) \),

(iii) \( T(r, g) = N(r, 0; f' \leq 1) + N_0(r) + S(r) \),

(iv) \( N_1(r) = S(r) \),

(v) \( N_0(r, 0; g' \geq 2) = S(r) \),

(vi) \( N_0(r, 0; f' \geq 2) = S(r) \),

(vii) \( N(r, 0; g' \geq 2) = S(r) \),

(viii) \( N(r, 0; f' \geq 2) = S(r) \),

(ix) \( N(r, 0; f - g \geq 2) = S(r) \),

(x) \( N(r, 0; f - g \mid f = \infty) = S(r) \),

where \( N_0(r, 0; g' \geq 2) \) is the counting function of those multiple zeros of \( g'(f') \) which are not the zeros of \( f(f - 1) \) and \( N(r, 0; f - g \mid f = \infty) \) is the counting function of those zeros of \( f - g \) which are poles of \( f \).

### 3 Proof of Theorem [1.1]

**Proof.** If necessary considering a bilinear transformation we may choose \( \{a_1, a_2, a_3\} = \{0, 1, \infty\} \). We now consider the following cases

**CASE 1.** Let \( a = b \). If possible, we suppose that \( f \) is not a bilinear transformation of \( g \). We put

\[
\Phi = \frac{f'(f - a)}{f(f - 1)} - \frac{g'(g - a)}{g(g - 1)}.
\]

Let \( \Phi \neq 0 \). Since \( \Phi = a \frac{f'}{f} + (1 - a) \frac{g'}{g} \), by Lemma 2.7 we get \( T(r, \Phi) = S(r) \). Since \( E_1(a; f) \subseteq E_{\infty}(a; g) \) and \( E_1(0; f') \subseteq E_{\infty}(0; g') \), it follows that

\[
N(r, a; f \leq 2) \leq 2N(r, 0; \Phi) = S(r),
\]

which contradicts (ii) of Lemma 2.5. Therefore \( \Phi \equiv 0 \) and so

\[
\frac{f'(f - a)}{f(f - 1)} = \frac{g'(g - a)}{g(g - 1)} \quad (3.1)
\]
If $z_0$ is a double zero of $g - \alpha$, then from (3.1) we see that $z_0$ is a common zero of $f'$ and $g$. Hence $z_0$ is a zero of $\frac{\alpha'}{\alpha} = \frac{f'}{f} - \frac{g'}{g}$. So by (i) of Lemma 2.5 and Lemma 2.7 we get

$$N(r, \alpha; g \mid z \geq 2) = 2N(r, \alpha; \frac{\alpha'}{\alpha}) + S(r)$$

Again if $z_1$ is a zero of $g'$ which is not a zero of $g(g - 1)(g - \alpha)$, then from (3.1) and the hypotheses of the theorem it follows that $z_1$ is a zero of $f'$ and so of $\frac{\alpha'}{\alpha}$. Hence from Lemma 2.2 Lemma 2.7 and (3.2) we get

$$N(r, \alpha; g' \mid z \leq 1) \leq N(r, \alpha; g \mid z \geq 2) + \overline{N}(r, \alpha; f \mid z \geq 2) + N(r, \alpha; \frac{\alpha'}{\alpha}) = S(r).$$

Now from (ii) and (iv) of Lemma 2.8 and (3.3) we obtain

$$N_0(r) + N_1(r) = T(r, f) + S(r),$$

which is impossible by Lemma 2.2. Therefore $f$ is a bilinear transformation of $g$ and so by Lemma 2.4 $f$ and $g$ take one of the forms (i)-(iv), (vi) and (viii).

**CASE 2.** Let $\alpha \neq \beta$. If $f$ is a bilinear transformation of $g$, then by Lemma 2.4 $f$ and $g$ assume one of the forms (i)-(ix). So we suppose that $f$ is not a bilinear transformation of $g$. Following two subcases come up for consideration.

**Subcase (i)** Let $N(r, \alpha; f \mid z \geq 2) \neq S(r)$.

We put $\Psi = \frac{f'(f - b)}{f(f - 1)} - \frac{g'(g - b)}{g(g - 1)}$. Since a double zero of $f - \alpha$ is a zero of $f'$ and so a zero of $g'$, if $\Psi \neq 0$, then we get by Lemma 2.5(i) and Lemma 2.7

$$N(r, \alpha; f \mid z \geq 2) \leq 2N(r, \alpha; \Psi) + S(r) = S(r)$$

which is a contradiction. Hence $\Psi \equiv 0$ and so

$$\frac{f'(f - b)}{f(f - 1)} = \frac{g'(g - b)}{g(g - 1)}.$$

This shows that $f - \alpha$ has no simple zero because $E_{11}(\alpha; f) \subseteq \mathfrak{T}_{\infty}(\alpha; g)$.

Since $\frac{\alpha'}{\alpha} = \frac{f'}{f} - \frac{g'}{g}$, and $E_{11}(\alpha; f') \subseteq \mathfrak{T}_{\infty}(\alpha; g')$, it follows that a double zero of $f - \alpha$ is a zero of $\frac{\alpha'}{\alpha}$. So by Lemma 2.7 we get $N(r, \alpha; f \mid z \geq 2) \leq 2N(r, \alpha; \frac{\alpha'}{\alpha}) = S(r)$, which contradicts (ii) of Lemma 2.5.

**Subcase (ii)** Let $N(r, \alpha; f \mid z \geq 2) = S(r)$. Since $f$ is not a bilinear transformation of $g$, we see that $\alpha$, $\beta$ and $\alpha\beta$ are non-constant. Also we note that $f = \frac{1 - \alpha}{1 - \alpha\beta}$ and $g = \frac{(1 - \alpha)\beta}{1 - \alpha\beta}$.
We put \( F = (f - a)(1 - \alpha \beta) = \alpha \beta - \alpha + 1 - a \) and \( w = \frac{F'}{F} \). Also we note that \( F = (f - a)\frac{g - f}{f(g - 1)} \).

Since by Lemma 2.6 \( N(r, \infty; F) = S(r) \) and \( w \) has only simple poles (if there is any), we get

\[
T(r, w) = m(r, w) + N(r, w) = N(r, 0; F) + S(r). \tag{3.4}
\]

Now by Lemma 2.2 and (ix), (x) of Lemma 2.8 we obtain

\[
N(r, 0; F | \geq 2) \leq N(r, a; f | \geq 2) + N(r, 0; f - g | \geq 2) + N(r, \infty; f | \geq 2) + N(r, 0; f - g | f = \infty) = S(r). \tag{3.5}
\]

Hence from (3.4) and (3.5) we get

\[
T(r, w) = N(r, 0; F | \leq 1) + S(r)
\]

\[
= N(r, a; f | \leq 1) + N_0(r) + N_2(r) + S(r), \tag{3.6}
\]

where \( N_2(r) \) is the counting function of those simple poles of \( f \) which are non-zero regular points of \( f - g \).

From the definitions of \( \alpha \) and \( \beta \) we get

\[
\left\{ g - \frac{\alpha' \beta}{(\alpha \beta)'} \right\} \left( \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} \right) = \frac{f'(g - f)}{f(f - 1)}. \tag{3.7}
\]

From (3.7) we see that a simple pole of \( f \) which is a non-zero regular point of \( f - g \) is a regular point of \( \left\{ g - \frac{\alpha' \beta}{(\alpha \beta)'} \right\} \left( \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} \right) \). Hence it is either a pole of \( \frac{\alpha' \beta}{(\alpha \beta)'} \) or a zero of \( \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} \).

Therefore by Lemma 2.7 and the first fundamental theorem we get

\[
N_2(r) \leq T \left( r, \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} \right) + T \left( r, \frac{\alpha' \beta}{(\alpha \beta)'} \right)
\]

\[
\leq T \left( r, \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} \right) + T \left( r, \frac{1}{1 + \frac{\alpha \beta r}{(\alpha \beta)'^2}} \right)
\]

\[
\leq 2T \left( r, \frac{\alpha'}{\alpha} \right) + 2T \left( r, \frac{\beta'}{\beta} \right) + O(1)
\eqn = S(r).
\]

So from (3.6) we get

\[
T(r, w) = N(r, a; f | \leq 1) + N_0(r) + S(r). \tag{3.8}
\]

By (ii) of Lemma 2.5 we get from (3.8)

\[
T(r, w) = T(r, f) + N_0(r) + S(r). \tag{3.9}
\]
Let
\[\tau_1 = \frac{a - 1}{b - 1}(\xi - b\delta),\]
\[\tau_2 = \frac{1}{2}\cdot\frac{a - 1}{b - 1}(\xi' + \xi^2 - b(\delta' + \delta^2))\]
and \[\tau_3 = \frac{1}{6}\cdot\frac{a - 1}{b - 1}(\xi'' + 3\xi\xi' + \xi^3 - b(\delta'' + 3\delta\delta' + \delta^3)),\]
where \(\xi = \frac{\alpha'}{\alpha}\) and \(\delta = \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\). By Lemma 2.4 we see that \(T(r, \xi) = S(r)\) and \(T(r, \delta) = S(r)\).

If \(\tau_1 = 0\), from (3.7) we get
\[(g - b)\delta = \frac{f'(g - f)}{f(f - 1)}.\]  
(3.10)

Since \(E_1(a; f) \subseteq E(b; g)\), it follows from (3.10) that a simple zero of \(f - a\), which is neither a zero nor a pole of \(\delta\), is a zero of \(g - b\) and so is a zero of \(f'\). Hence \(N(r, a; |f| \leq 1) = S(r)\), which contradicts (ii) of Lemma 2.5. Therefore \(\tau_1 \neq 0\).

Let \(z_0\) be a simple zero of \(f - a\) and \(\tau_1(z_0) \neq 0\). Then \(g(z_0) = b\) and so \(\alpha(z_0) = \frac{a - 1}{b - 1}\) and \(\beta(z_0) = \frac{b}{a}\) Expanding \(F\) around \(z_0\) in Taylor’s series we get
\[-F(z) = \tau_1(z_0)(z - z_0) + \tau_2(z_0)(z - z_0)^2 + \tau_3(z_0)(z - z_0)^3 + O((z - z_0)^4).\]
Hence in some neighbourhood of \(z_0\) we obtain
\[w(z) = \frac{1}{z - z_0} + \frac{B(z_0)}{2} + C(z_0)(z - z_0) + O((z - z_0)^2),\]
where \(B = \frac{2\tau_2}{\tau_1}\) and \(C = \frac{2\tau_3}{\tau_1} - \left(\frac{\tau_2}{\tau_1}\right)^2\).

We put
\[H = w' + w^2 - Bw - A,\]  
(3.11)
where \(A = 3C - \frac{B^2}{4} - B'\).

Clearly \(T(r, A) + T(r, B) + T(r, C) = S(r)\) and since \(w = \frac{F'}{F}\) and \(F = (f - a)\frac{g - f}{f(g - 1)}\), we get by Lemma 2.1 and (3.9) that \(S(r, w) = S(r)\).

Let \(H \neq 0\). Then it is easy to see that \(z_0\) is a zero of \(H\). So
\[
N(r, a; f |\leq 1) \leq N(r, 0; H) + S(r) \leq T(r, H) + S(r) = N(r, H) + S(r).
\]  
(3.12)
From (ii) of Lemma 2.8 and (3.12) we get

$$T(r, f) \leq N(r, H) + S(r).$$

(3.13)

Let $z_1$ be a pole of $F$. Then $z_1$ is a simple pole of $w$. So if $z_1$ is not a pole of $A$ and $B$, then $z_1$ is at most a double pole of $H$. Hence by Lemma 2.8 we get

$$N(r, \infty; H | F = \infty) \leq 2N(r, \infty; F) + S(r) = S(r),$$

(3.14)

where $N(r, \infty; H | F = \infty)$ denotes the counting function of those poles of $H$ which are also poles of $F$.

Let $z_2$ be a multiple zero of $F$. Then $z_2$ is a simple pole of $w$. So if $z_2$ is not a pole of $A$ and $B$, then $z_2$ is a pole of $H$ of multiplicity at most two. Hence by (3.13) we get

$$N(r, \infty; H | F = 0, \geq 2) \leq 2N(r, 0; F \geq 2) + S(r) = S(r),$$

(3.15)

where $N(r, \infty; H | F = 0, \geq 2)$ denotes the counting function of those poles of $H$ which are multiple zeros of $F$.

Let $z_3$ be a simple zero of $F$ which is not a pole of $A$ and $B$. Then in some neighbourhood of $z_3$ we get $F(z) = (z - z_3)h(z)$, where $h$ is analytic at $z_3$ and $h(z_3) \neq 0$. Hence in some neighbourhood of $z_3$ we obtain

$$H(z) = \left(\frac{2h'}{h} - B\right) \frac{1}{z - z_3} + h_1,$$

where $h_1 = \left(\frac{h'}{h}\right)' + \left(\frac{h'}{h}\right)^2 - \frac{Bh'}{h} - A$.

This shows that $z_3$ is at most a simple pole of $H$. Since a simple zero of $f - a$ is a zero of $H$ and $N(r, 0; F | f = t) \leq N(r, 0; f - g | \geq 2)$ for $t = 0, 1$ and $F = (f - a) \frac{g - f}{f(g - 1)}$, we get from (3.14) and (3.15) in view of (ix) of Lemma 2.8

$$N(r, H) = N(r, \infty; H | F = \infty) + N(r, \infty; H | F = 0) + S(r)$$
$$\leq N(r, 0; F \geq 1) - N(r, a; f \leq 1) + S(r)$$
$$= N_0(r) + N_2(r) + S(r)$$
$$= N_0(r) + S(r),$$

(3.16)

where $N(r, 0; F | f = t)$ denotes the counting function of those zeros of $F$ which are zeros of $f - t$ and $N(r, \infty; H | F = 0)$ denotes the counting function of those poles of $H$ which are zeros of $F$.

From (3.13) and (3.16) we obtain $T(r, f) \leq N_0(r) + S(r)$, which by (iv) of Lemma 2.8 and
Lemma 2.3 implies a contradiction. Therefore $H \equiv 0$ and so
\[ w' + w^2 - Bw - A \equiv 0 \]
i.e., \[ \frac{w'}{w} \equiv \frac{A}{w} - w + B \]
i.e., \[ F'' \equiv AF + BF'. \]
Since $F' = a(\alpha\beta)' - \alpha'$ and $F'' = a(\alpha\beta)'' - \alpha''$, we get from above
\[ K\alpha\beta + L\alpha \equiv A(f - a)(1 - \alpha\beta), \tag{3.17} \]
where $K = a\left(\frac{(\alpha\beta)''}{\alpha\beta} - b\frac{(\alpha\beta)'}{\alpha\beta}\right)$ and $L = b\frac{\alpha'}{\alpha} - \frac{\alpha''}{\alpha}$.

By Lemma 2.7, we see that $T(r, K) = S(r)$ and $T(r, L) = S(r)$. Since $\alpha\beta = \frac{g(f - 1)}{f(g - 1)}$ and $\alpha = \frac{f - 1}{g - 1}$, we get from (3.17)
\[ Kg + Lf \equiv \frac{A(f - a)(g - f)}{(f - 1)} \tag{3.18} \]

Let $z_0$ be a simple zero of $f - a$ which is not a pole of $A$. Since $E_1(a; f) \subset E_\infty(b; g)$, it follows from (3.18) that $z_0$ is a zero of $bK + aL$. Hence
\[ N(r, a; f \mid \leq 1) \leq N(r, 0; bK + aL) + N(r, \infty; A) \equiv S(r), \]
which contradicts (ii) of Lemma 2.6. This proves the theorem. \hfill \Box

References


