Lp local uncertainty inequality for the Sturm-Liouville transform

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ABSTRACT

In this paper, we give analogues of local uncertainty inequality for the Sturm-Liouville transform on \([0, \infty]\). A generalization of Donoho-Stark’s uncertainty principle is obtained for this transform.

RESUMEN

En este artículo entregamos resultados análogos de una desigualdad de incertidumbre local de la transformada Sturm-Liouville en \([0, \infty]\). Una generalización del principio de incertidumbre de Donoho-Stark se obtiene de esta transformación.

Keywords and Phrases: Sturm-Liouville transform; local uncertainty principle; Donoho-Stark’s uncertainty principle.

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1 Introduction

We consider the second-order differential operator defined on $[0, \infty[$ by

$$\Delta u := u'' + \frac{A'}{A}u' + \rho^2 u,$$

where $A$ is a nonnegative function satisfying certain conditions and $\rho$ is a nonnegative real number. This operator plays an important role in analysis. For example, many special functions (orthogonal polynomials) are eigenfunctions of an operator of $\Delta$ type. The radial part of the Beltrami-Laplacian in a symmetric space is also of $\Delta$ type. Many aspects of such operators have been studied [2, 10, 17, 18, 19]. In particular, the two references [2, 17] investigate standard constructions of harmonic analysis, such as translation operators, convolution product, and Fourier transform, in connection with $\Delta$.


Building on the ideas of Faris [7] and Price [12, 13], we show a local uncertainty principle for the Sturm-Liouville transform $F$. More precisely, we will show the following result. If $1 < p \leq 2$, $q = p/(p-1)$ and $0 < a < (2\alpha + 2)/q$, there is a constant $K(a)$ such that for every $f \in L^p(\mu)$ and every measurable subset $E \subset [0, \infty[$ such that $0 < \nu(E) < \infty$,

$$\left( \int_E |F(f)(\lambda)|^q d\nu(\lambda) \right)^{1/q} \leq K(a) \left( \nu(E) \right)^{\frac{-2\alpha q}{2\alpha q - q}} \|x^a f\|_{L^p(\mu)},$$

where $\mu$ is the measure given by $d\mu(x) := A(x)dx$, and $\nu$ is the Plancherel measure associated to $F$. (For more details see the next section.) This inequality generalizes the local uncertainty principle for the Hankel transform given by Ghobber et al. [8] and Omri [14].

We shall use the local uncertainty principle (1.1); and building on the techniques of Donoho and Stark [6], we show a continuous-time principles for the $L^p$ theory, when $1 < p \leq 2$.

This paper is organized as follows. In Section 2 we list some basic properties of the Sturm-Liouville transform $F$ (Plancherel theorem, inversion formula,...). In Section 3 we show a local uncertainty principle for the Sturm-Liouville $F$. The Section 4 is devoted to Donoho-Stark’s uncertainty principle for the Sturm-Liouville transform $F$ in the $L^p$ theory, when $1 < p \leq 2$. 
2 The Sturm-Liouville transform $\mathcal{F}$

We consider the second-order differential operator $\Delta$ defined on $]0, \infty[$ by

$$\Delta u := u'' + \frac{A'}{A} u' + \rho^2 u,$$

where $\rho$ is a nonnegative real number and

$$A(x) := x^{2\alpha+1} B(x), \quad \alpha > -1/2,$$

for $B$ a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0) = 1$. Moreover we assume that $A$ and $B$ satisfy the following conditions:

(i) $A$ is increasing and $\lim_{x \to \infty} A(x) = \infty$.

(ii) $\frac{A'}{A}$ is decreasing and $\lim_{x \to \infty} \frac{A'(x)}{A(x)} = 2\rho$.

(iii) There exists a constant $\delta > 0$ such that

$$\frac{A'(x)}{A(x)} = 2\rho + D(x) \exp(-\delta x) \quad \text{if} \quad \rho > 0,$$

$$\frac{A'(x)}{A(x)} = \frac{2\alpha + 1}{x} + D(x) \exp(-\delta x) \quad \text{if} \quad \rho = 0,$$

where $D$ is an infinitely differentiable function on $]0, \infty[$, bounded and with bounded derivatives on all intervals $[x_0, \infty[$, for $x_0 > 0$. This operator was studied in [2, 10, 17], and the following results have been established:

(I) For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases}
\Delta u = -\lambda^2 u \\
u(0) = 1, \quad u'(0) = 0
\end{cases}$$

admits a unique solution, denoted by $\varphi_\lambda$, with the following properties:

• for $x \geq 0$, the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic on $\mathbb{C}$;

• for $\lambda \in \mathbb{C}$, the function $x \mapsto \varphi_\lambda(x)$ is even and infinitely differentiable on $\mathbb{R}$;

• for all $\lambda, x \in \mathbb{R}$,

$$|\varphi_\lambda(x)| \leq 1. \quad \text{(2.1)}$$

(II) For nonzero $\lambda \in \mathbb{C}$, the equation $\Delta u = -\lambda^2 u$ has a solution $\Phi_\lambda$ satisfying

$$\Phi_\lambda(x) = \frac{1}{\sqrt{A(x)}} \exp(i\lambda x) V(x, \lambda),$$

with $\lim_{x \to \infty} V(x, \lambda) = 1$. Consequently there exists a function (spectral function)

$$\lambda \mapsto c(\lambda),$$
such that
\[
\varphi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda} \quad \text{for nonzero } \lambda \in \mathbb{C}.
\]
Moreover there exist positive constants \(k_1, k_2\) and \(k\) such that
\[
k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1}
\]
for all \(\lambda\) such that \(\text{Im}\lambda \leq 0\) and \(|\lambda| \geq k\).

**Notation 2.1.** We denote by

- \(\mu\) the measure defined on \([0, \infty[\) by \(d\mu(x) := A(x)dx\); and by \(L^p(\mu), 1 \leq p \leq \infty\), the space of measurable functions \(f\) on \([0, \infty[\), such that
  \[
  \|f\|_{L^p(\mu)} := \left(\int_0^\infty |f(x)|^p d\mu(x)\right)^{1/p} < \infty, \quad 1 \leq p < \infty,
  \]
  \[
  \|f\|_{L^\infty(\mu)} := \text{ess sup}_{x \in [0, \infty[} |f(x)| < \infty;
  \]

- \(\nu\) the measure defined on \([0, \infty[\) by \(d\nu(\lambda) := \frac{d\lambda}{2\pi|c(\lambda)|^p}\); and by \(L^p(\nu), 1 \leq p \leq \infty\), the space of measurable functions \(f\) on \([0, \infty[\), such that \(\|f\|_{L^p(\nu)} < \infty\).

The Fourier transform associated with the operator \(\Delta\) is defined on \(L^1(\mu)\) by
\[
\mathcal{F}(f)(\lambda) := \int_0^\infty \varphi_\lambda(x)f(x)d\mu(x) \quad \text{for } \lambda \in \mathbb{R}.
\]

Some of the properties of the Fourier transform \(\mathcal{F}\) are collected below (see [2, 10, 17, 18]).

**Theorem 2.2.** (i) \(L^1 - L^\infty\)-boundedness. For all \(f \in L^1(\mu), \mathcal{F}(f) \in L^\infty(\nu)\) and
\[
\|\mathcal{F}(f)\|_{L^\infty(\nu)} \leq \|f\|_{L^1(\mu)}, \tag{2.2}
\]

(ii) Inversion theorem. Let \(f \in L^1(\mu), \) such that \(\mathcal{F}(f) \in L^1(\nu).\) Then
\[
f(x) = \int_0^\infty \varphi_\lambda(x)\mathcal{F}(f)(\lambda)d\nu(\lambda), \quad \text{a.e. } x \in [0, \infty[. \tag{2.3}
\]

(iii) Plancherel theorem. The Fourier transform \(\mathcal{F}\) extends uniquely to an isometric isomorphism of \(L^2(\mu)\) onto \(L^2(\nu).\) In particular,
\[
\|f\|_{L^2(\mu)} = \|\mathcal{F}(f)\|_{L^2(\nu)}, \tag{2.4}
\]

Using relations (2.2) and (2.4) with Marcinkiewicz's interpolation theorem [15, 16], we deduce that for every \(1 \leq p \leq 2,\) and for every \(f \in L^p(\mu),\) the function \(\mathcal{F}(f)\) belongs to the space \(L^q(\nu), q = p/(p-1),\) and
\[
\|\mathcal{F}(f)\|_{L^q(\nu)} \leq \|f\|_{L^p(\mu)}, \tag{2.5}
\]
3 \( L^p \) local uncertainty inequality

This section is devoted to establish a local uncertainty principle for the Sturm-Liouville transform \( \mathcal{F} \), more precisely, we will show the following theorem.

**Theorem 3.1.** If \( 1 < p \leq 2 \), \( q = p/(p-1) \) and \( 0 < a < (2\alpha + 2)/q \), then for all \( f \in L^p(\mu) \) and all measurable subset \( E \subset [0, \infty) \) such that \( 0 < \nu(E) < \infty \),

\[
\left( \int_E |\mathcal{F}(f)(\lambda)|^q d\nu(\lambda) \right)^{1/q} \leq K(a) \left( \nu(E) \right)^{-\frac{2\alpha}{q-1}} \|f\|_{L^p(\mu)},
\]

where

\[
K(a) = \left( qa \right)^{-\frac{2\alpha}{q-1}} \left( 2\alpha + 2 - qa \right) \frac{1}{q} \left( \nu(E) \right)^{\frac{1}{q}}.
\]

**Proof.** For \( r > 0 \), denote by \( \chi_E \), \( \chi_{[0,r]} \) and \( \chi_{[r,\infty)} \) the characteristic functions. Let \( f \in L^p(\mu), 1 < p \leq 2 \) and let \( q = p/(p-1) \). By Minkowski's inequality, for all \( r > 0 \),

\[
\|\mathcal{F}(f)\chi_E\|_{L^q(\nu)} \leq \|\mathcal{F}(f\chi_{[0,r]}\chi_E)\|_{L^q(\nu)} + \|\mathcal{F}(f\chi_{[r,\infty)}\chi_E)\|_{L^q(\nu)}
\]

\[
\leq \left( \nu(E) \right)^{1/q} \|\mathcal{F}(f\chi_{[0,r]}\chi_E)\|_{L^\infty(\nu)} + \|\mathcal{F}(f\chi_{[r,\infty)}\chi_E)\|_{L^\infty(\nu)};
\]

hence it follows from (2.2) and (2.5) that

\[
\|\mathcal{F}(f)\chi_E\|_{L^q(\nu)} \leq \left( \nu(E) \right)^{1/q} \|f\chi_{[0,r]}\|_{L^1(\mu)} + \|f\chi_{[r,\infty)}\|_{L^p(\mu)}. \tag{3.1}
\]

On the other hand, by Hölder's inequality,

\[
\|f\chi_{[0,r]}\|_{L^1(\mu)} \leq \|x^{-a}\chi_{[0,r]}\|_{L^q(\mu)} \|f\|_{L^p(\mu)}.
\]

By hypothesis \( a < (2\alpha + 2)/q \),

\[
\|x^{-a}\chi_{[0,r]}\|_{L^q(\mu)} \leq \frac{r^{-a+(2\alpha+2)/q}}{(2\alpha + 2 - qa)^{1/q}} \left( \sup_{x \in [0,r]} B(x) \right)^{1/q},
\]

and therefore,

\[
\|f\chi_{[0,r]}\|_{L^1(\mu)} \leq \frac{r^{-a+(2\alpha+2)/q}}{(2\alpha + 2 - qa)^{1/q}} \left( \sup_{x \in [0,r]} B(x) \right)^{1/q} \|f\|_{L^p(\mu)}. \tag{3.2}
\]

Moreover,

\[
\|f\chi_{[r,\infty)}\|_{L^p(\mu)} \leq \|x^{-a}\chi_{[r,\infty)}\|_{L^\infty(\mu)} \|f\|_{L^p(\mu)} \leq r^{-a} \|f\|_{L^p(\mu)}. \tag{3.3}
\]

Combining the relations (3.1), (3.2) and (3.3), we deduce that

\[
\|\mathcal{F}(f)\chi_E\|_{L^q(\nu)} \leq \left[ r^{-a} + \left( \nu(E) \right)^{1/q} \frac{r^{-a+(2\alpha+2)/q}}{(2\alpha + 2 - qa)^{1/q}} \left( \sup_{x \in [0,r]} B(x) \right)^{1/q} \right] \|f\|_{L^p(\mu)}.
\]
We choose \( r = r_0 = (qa)^{\frac{q}{2\alpha + 2}} \left( 2\alpha + 2 - qa \right)^{\frac{1-q}{2\alpha}} \left( \nu(E) \right)^{-\frac{1}{2\alpha + 2}} \), we obtain the desired inequality.

\[ \square \]

**Remark 3.2.** (i) The Local uncertainty principle for the Sturm-Liouville transform \( F \) generalizes the local uncertainty principle for the Hankel transform (see [8, 11]).

(ii) If \( 1 < p \leq 2 \) and \( 0 < a < \frac{2\alpha + 2}{q} \), where \( q = \frac{p}{p - 1} \), then for every \( f \in L^p(\mu) \),

\[
\sup_{E \subset [0, \infty], 0 < \nu(E) < \infty} \left[ \left( \nu(E) \right)^{-\frac{q}{2\alpha + 2}} \| F(f) \chi_E \|_{L^q(E)} \right] \leq K(a) \| x^a f \|_{L^q(\mu)}.
\]

The left hand side is known to be an equivalent norm of \( F(f) \) in the Lorentz-space \( L^{p_a, q}(\nu) \), where

\[
p_a = \frac{q(2\alpha + 2)}{2\alpha + 2 - qa}.
\]

## 4 \( L^p \) Donoho-Stark uncertainty principle

Let \( T \) and \( E \) be measurable subsets of \([0, \infty]\). We introduce the time-limiting operator \( P_T \) by

\[
P_T f := f_{\chi_T},
\]

and, we introduce the partial sum operator \( S_E \) by

\[
F(S_E f) = F(f) \chi_E.
\]

**Lemma 4.1.** If \( \nu(E) < \infty \) and \( f \in L^p(\mu), 1 \leq p \leq 2 \),

\[
S_E f(x) = \int_E \phi_\lambda(x) F(f)(\lambda) d\nu(\lambda).
\]

**Proof.** Let \( f \in L^p(\mu), 1 \leq p \leq 2 \) and let \( q = \frac{p}{p - 1} \). Then by (2.1), Hölder’s inequality and (2.5),

\[
\| F(f) \chi_E \|_{L^1(\nu)} = \int_E |F(f)(\lambda)| d\nu(\lambda)
\leq \left( \nu(E) \right)^{1/p} \| F(f) \|_{L^1(\nu)}
\leq \left( \nu(E) \right)^{1/p} \| f \|_{L^p(\mu)},
\]

and

\[
\| F(f) \chi_E \|_{L^2(\nu)} = \left( \int_E |F(f)(\lambda)|^2 d\nu(\lambda) \right)^{1/2}
\leq \left( \nu(E) \right)^{\frac{q-2}{q}} \| F(f) \|_{L^q(\nu)}
\leq \left( \nu(E) \right)^{\frac{q-2}{q}} \| f \|_{L^p(\mu)}.
\]
Thus $\mathcal{F}(f)\chi_E \in L^1(\mu) \cap L^2(\mu)$ and by (4.2),

$$S_\varepsilon f = \mathcal{F}^{-1}\left(\mathcal{F}(f)\chi_E\right).$$

This combined with (2.3) gives the result. \hfill \Box

Let $T$ and $E$ be measurable subsets of $[0, \infty[$. We say that a function $f \in L^p(\mu)$, $1 \leq p \leq 2$, is $\varepsilon$-concentrated to $T$ in $L^p(\mu)$-norm, if there is a measurable function $g(t)$ vanishing outside $T$ such that $\|f - g\|_{L^p(\mu)} \leq \varepsilon\|f\|_{L^p(\mu)}$. Similarly, we say that $\mathcal{F}(f)$ is $\varepsilon$-concentrated to $E$ in $L^q(\nu)$-norm, $q = p/(p-1)$, if there is a function $h(\nu)$ vanishing outside $E$ with $\|\mathcal{F}(f) - h\|_{L^q(\nu)} \leq \varepsilon\|\mathcal{F}(f)\|_{L^q(\nu)}$.

If $f$ is $\varepsilon_T$-concentrated to $T$ in $L^p(\mu)$-norm ($g$ being the vanishing function) then by (4.1),

$$\|f - P_T f\|_{L^p(\mu)} = \left(\int_{[0, \infty[ \setminus T} |f(t)|^p\,d\mu(t)\right)^{1/p} \leq \|f - g\|_{L^p(\mu)} \leq \varepsilon_T \|f\|_{L^p(\mu)}$$

(4.3)

and therefore $f$ is $\varepsilon_T$-concentrated to $T$ in $L^p(\mu)$-norm if and only if

$$\|f - P_T f\|_{L^p(\mu)} \leq \varepsilon_T \|f\|_{L^p(\mu)}.$$

From (4.2) it follows as for $P_T$ that $\mathcal{F}(f)$ is $\varepsilon_E$-concentrated to $E$ in $L^q(\nu)$-norm, $q = p/(p-1)$, if and only if

$$\|\mathcal{F}(f) - \mathcal{F}(S_E f)\|_{L^q(\nu)} \leq \varepsilon_E \|\mathcal{F}(f)\|_{L^q(\nu)}.$$

(4.4)

Let $B_p(E)$, $1 \leq p \leq 2$, be the set of functions $f \in L^p(\mu)$ that are bandlimited to $E$ (i.e. $f \in B_p(E)$ implies $S_1 f = f$).

The spaces $B_p(E)$ satisfy the following property.

**Lemma 4.2.** Let $T$ and $E$ be measurable subsets of $[0, \infty[$ such that $0 < \nu(E) < \infty$. For $f \in B_p(E)$, $1 < p \leq 2$ and $0 < a < (2a + 2)(1 - \frac{1}{p})$,

$$\|P_T f\|_{L^p(\mu)} \leq K(a)\left(\mu(T)\right)^{1/p} \left(\nu(E)\right)^{\frac{1}{p} + \frac{a}{p + a}} \|\chi f\|_{L^p(\mu)},$$

where $K(a)$ is the constant given by Theorem 3.1.

**Proof.** If $\mu(T) = \infty$, the inequality is clear. Assume that $\mu(T) < \infty$. For $f \in B_p(E)$, $1 < p \leq 2$, from Lemma 4.1,

$$f(t) = \int_E \varphi_\lambda(t)\mathcal{F}(f)(\lambda)d\nu(\lambda),$$

and by (2.1), Hölder’s inequality and Theorem 3.1,

$$|f(t)| \leq \left(\nu(E)\right)^{1/p} \left(\int_E |\mathcal{F}(f)(\lambda)|^q d\nu(\lambda)\right)^{1/q} \leq K(a)\left(\nu(E)\right)^{\frac{1}{p} + \frac{a}{p + a}} \|\chi f\|_{L^p(\mu)}.$$
where \( q = \frac{p}{p-1} \). Hence,
\[
\|P_T f\|_{L^p(\mu)} = \left( \int_T |f(t)|^p d\mu(t) \right)^{1/p} \leq K(a) \left( \mu(T) \right)^{1/p} \left( \nu(E) \right)^{\frac{1}{p} + \frac{a}{p-1}} \|x^a f\|_{L^p(\mu)},
\]
which yields the result. \( \Box \)

It is useful to have uncertainty principle for the \( L^p(\mu) \)-norm.

**Theorem 4.3.** Let \( T \) and \( E \) be measurable subsets of \([0, \infty[\) such that \( 0 < \nu(E) < \infty \); and let \( f \in B_p(E), 1 < p \leq 2 \) and \( 0 < \alpha < (2\alpha + 2)(1 - \frac{1}{p}) \). If \( f \) is \( \varepsilon_T \)-concentrated to \( T \), then
\[
\|f\|_{L^p(\mu)} \leq \frac{K(a)}{1 - \varepsilon_T} \left( \mu(T) \right)^{1/p} \left( \nu(E) \right)^{\frac{1}{p} + \frac{a}{p-1}} \|x^a f\|_{L^p(\mu)}.
\]

**Proof.** Let \( f \in B_p(E), 1 < p \leq 2 \). Since \( f \) is \( \varepsilon_T \)-concentrated to \( T \) in \( L^p(\mu) \)-norm, then by (4.3) and Lemma 4.2,
\[
\|f\|_{L^p(\mu)} \leq \varepsilon_T \|f\|_{L^p(\mu)} + \|P_T f\|_{L^p(\mu)} \\
\leq \varepsilon_T \|f\|_{L^p(\mu)} + K(a) \left( \mu(T) \right)^{1/p} \left( \nu(E) \right)^{\frac{1}{p} + \frac{a}{p-1}} \|x^a f\|_{L^p(\mu)}.
\]
Thus,
\[
(1 - \varepsilon_T) \|f\|_{L^p(\mu)} \leq K(a) \left( \mu(T) \right)^{1/p} \left( \nu(E) \right)^{\frac{1}{p} + \frac{a}{p-1}} \|x^a f\|_{L^p(\mu)},
\]
which gives the result. \( \Box \)

Another uncertainty principle for the \( L^p(\mu) \) theory is obtained.

**Theorem 4.4.** Let \( E \) be measurable subset of \([0, \infty[\) such that \( 0 < \nu(E) < \infty \); and let \( f \in L^p(\mu), 1 < p \leq 2 \) and \( 0 < \alpha < (2\alpha + 2)(1 - \frac{1}{p}) \). If \( F(f) \) is \( \varepsilon_E \)-concentrated to \( E \) in \( L^q(\nu) \)-norm, \( q = \frac{p}{p-1} \), then
\[
\|F(f)\|_{L^q(\nu)} \leq \frac{K(a)}{1 - \varepsilon_E} \left( \nu(E) \right)^{\frac{a}{p-1}} \|x^a f\|_{L^p(\mu)}.
\]

**Proof.** Let \( f \in L^p(\mu), 1 < p \leq 2 \). Since \( F(f) \) is \( \varepsilon_E \)-concentrated to \( E \) in \( L^q(\nu) \)-norm, \( q = \frac{p}{p-1} \), then by (4.4) and Theorem 3.1,
\[
\|F(f)\|_{L^q(\nu)} \leq \varepsilon_E \|F(f)\|_{L^q(\nu)} + \left( \int_E |F(f)(\lambda)|^q d\nu(\lambda) \right)^{1/q} \\
\leq \varepsilon_E \|F(f)\|_{L^q(\nu)} + K(a) \left( \nu(E) \right)^{\frac{a}{p-1}} \|x^a f\|_{L^p(\mu)}.
\]
Thus,
\[
(1 - \varepsilon_E) \|F(f)\|_{L^q(\nu)} \leq K(a) \left( \nu(E) \right)^{\frac{a}{p-1}} \|x^a f\|_{L^p(\mu)},
\]
which proves the result. \( \Box \)

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