Continuity via $\Lambda^*_I$-open sets

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ABSTRACT
Sanabria, Rosas and Carpintero [7] introduced the notions of $\Lambda^*_I$-sets and $\Lambda^*_I$-closed sets using ideals on topological spaces. Given an ideal $I$ on a topological space $(X, \tau)$, a subset $A \subset X$ is said to be $\Lambda^*_I$-closed if $A = U \cap F$ where $U$ is a $\Lambda^*_I$-set and $F$ is a $\tau^*$-closed set. In this work we use sets that are complements of $\Lambda^*_I$-closed sets, which are called $\Lambda^*_I$-open, to characterize new variants of continuity namely $\Lambda^*_I$-continuous, quasi-$\Lambda^*_I$-continuous and $\Lambda^*_I$-irresolute functions.

RESUMEN
Sanabria, Rosas y Carpintero [7] introdujeron las nociones de conjuntos $\Lambda^*_I$ y conjuntos $\Lambda^*_I$-cerrados usando ideales sobre espacios topológicos. Dado un ideal $I$ sobre un espacio topológico $(X, \tau)$, un subconjunto $A \subset X$ se llama $\Lambda^*_I$-cerrado si $A = U \cap F$ donde $U$ es un $\Lambda^*_I$-conjunto y $F$ es un conjunto $\tau^*$-cerrado. En este trabajo usamos conjuntos que son complementos de conjuntos $\Lambda^*_I$-cerrados, los cuales son llamados $\Lambda^*_I$-abiertos, para caracterizar nuevas variantes de continuidad, denominadas, funciones $\Lambda^*_I$-continuas y funciones $\Lambda^*_I$- irresolutas.

Keywords and Phrases: Local function, $\Lambda^*_I$-open sets, $\Lambda^*_I$-irresolute functions.

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1 Introduction

The theory of ideal on topological spaces has been the subject of many studies in recent years. It was the works of Hamlet and Jankovic [5], Abd El-Monsef, Lashien and Nasef [1] and Hatir and Noiri [2] which motivated the research in applying topological ideals to generalize the most basic properties in general topology. In 2002, Hatir and Noiri [2] introduced the notion of semi-I-open sets in topological spaces. Also, Hatir and Noiri [3] investigated semi-I-open and semi-I-continuous functions. Quite recently, Sanabria, Rosas and Carpintero [7] have introduced the notions of \( \Lambda^I_1 \)-sets and \( \Lambda^I_2 \)-closed sets to obtain characterizations of two low separation axioms, namely semi-I-\( T_1 \) and semi-I-\( T_{1/2} \) spaces. In this article we introduce the notion of \( \Lambda^I_{\text{sio}} \)-open sets in order to characterize new variants of continuity in ideal topological spaces.

2 Preliminaries

Throughout this paper, \( P(X), Cl(A) \) and \( Int(A) \) denote the power set of \( X \), the closure of \( A \) and the interior of \( A \), respectively. An ideal \( I \) on a topological space \( (X, \tau) \) is a nonempty collection of subsets of \( X \) which satisfies the following two properties:

1. \( A \in I \) and \( B \subset A \) implies \( B \in I \);
2. \( A \in I \) and \( B \in I \) implies \( A \cup B \in I \).

A topological space \( (X, \tau) \) with an ideal \( I \) on \( X \) is called an ideal topological space and is denoted by \( (X, \tau, I) \). Given an ideal topological space \( (X, \tau, I) \), a set operator \( (\cdot)^* : P(X) \rightarrow P(X) \), called a local function [6] of \( A \) with respect to \( \tau \) and \( I \), is defined as follows: for \( A \subset X \), \( A^*(I, \tau) = \{ x \in X : U \cap A \notin \tau \} \) for every \( U \in \tau(x) \), where \( \tau(x) = \{ U \in \tau : x \in U \} \). When there is no chance for confusion, we will simply write \( A^* \) for \( A^*(I, \tau) \). In general, \( X^* \) is a proper subset of \( X \). A Kuratowski closure operator \( Cl^*(\cdot) \) for a topology \( \tau^*(I, \tau) \), called the *-topology, finer than \( \tau \), is defined by \( Cl^*(A) = A \cup A^*(I, \tau) \) [5]. For any ideal topological space \( (X, \tau, I) \), the collection \( \beta(I, \tau) = \{ V \setminus J : V \in \tau \) and \( J \in I \} \) is a basis for \( \tau^*(I, \tau) \). According to the above, in this article, we denote by \( \tau^* \) to topology \( \tau^*(I, \tau) \) generated by \( Cl^* \), that is, \( \tau^* = \{ U \in P(X) : Cl^*(X - U) = X - U \} \). The elements of \( \tau^* \) are called \( \tau^* \)-open and the complement of a \( \tau^* \)-open is called \( \tau^* \)-closed. It is well known that a subset \( A \) of an ideal topological space \( (X, \tau, I) \) is \( \tau^* \)-closed if and only if \( A^* \subset A \) [5].

**Definition 2.1.** A subset \( A \) of an ideal topological space \( (X, \tau, I) \) is said to be semi-I-open [2] if \( A \subset Cl^*(Int(A)) \). The complement of a semi-I-open set is said to be semi-I-closed. The family of all semi-I-open sets of an ideal topological space \( (X, \tau, I) \) is denoted by \( \text{SIO}(X, \tau) \).

The following three notions has been introduced by Sanabria et al. [7].

**Definition 2.2.** Let \( A \) be a subset of an ideal topological space \( (X, \tau, I) \). A subset \( \Lambda^I_1(A) \) is defined as follows: \( \Lambda^I_1(A) = \cap \{ U : A \subset U, U \in \text{SIO}(X, \tau) \} \).

\( \Lambda^I_1 \)-sets and \( \Lambda^I_2 \)-closed sets to obtain characterizations of two low separation axioms, namely semi-I-\( T_1 \) and semi-I-\( T_{1/2} \) spaces. In this article we introduce the notion of \( \Lambda^I_{\text{sio}} \)-open sets in order to characterize new variants of continuity in ideal topological spaces.
Definition 2.3. Let $(X, \tau, I)$ an ideal topological space. A subset $A$ of $X$ is said to be:

1. $\Lambda^I_1$-set if $A = \Lambda^I_1(A)$.
2. $\Lambda^I_1$-closed if $A = U \cap F$, where $U$ is a $\Lambda^I_1$-set and $F$ is an $\tau^*$-closed set.

Since each open set is semi-I-open, by Propositions 3.1(3) and 4.1 of [7], we have the following implications:

$$\text{Open} \implies \text{semi-I-open} \implies \Lambda^I_1\text{-set} \implies \Lambda^I_1\text{-closed}.$$ 

Lemma 2.1. (Sanabria, Rosas and Carpintero [7]) For an ideal topological space $(X, \tau, I)$, we take $\tau^{\Lambda^I_1} = \{A : A$ is a $\Lambda^I_1$-set of $(X, \tau, I)\}$. Then the pair $(X, \tau^{\Lambda^I_1})$ is an Alexandroff space.

Remark 2.1. According to Lemma 2.1, a subset $A$ of an ideal topological space $(X, \tau, I)$ is open in $(X, \tau^{\Lambda^I_1})$, if $A$ is a $\Lambda^I_1$-set of $(X, \tau, I)$.

Definition 2.4. A subset $A$ of an ideal topological space $(X, \tau, I)$ is called $\Lambda^I_1$-open if $X \setminus A$ is a $\Lambda^I_1$-closed set.

In the sequel, the ideal topological space $(X, \tau, I)$ is simply denoted by $X$. Next we present some results related with $\Lambda^I_1$-open sets.

Lemma 2.2. Every $\tau^*$-open set is $\Lambda^I_1$-open.

Proof. This follows from Proposition 4.1 of [7].

Lemma 2.3. Let $\{B_\alpha : \alpha \in \Delta\}$ be a family of subsets of $X$. If $B_\alpha$ is $\Lambda^I_1$-open for each $\alpha \in \Delta$, then $\bigcup\{B_\alpha : \alpha \in \Delta\}$ is $\Lambda^I_1$-open.

Proof. The proof is an immediate consequence from Proposition 4.2 of [7].

3 New variants of continuity

In this section we use the notions of open, $\Lambda^I_1$-open and $\tau^*$-open sets in order to introduce new forms of continuous functions called $\Lambda^I_1$-continuous, quasi-$\Lambda^I_1$-continuous and $\Lambda^I_1$-irresolute. We study the relationships between these classes of functions and also obtain some properties and characterizations of them.

Definition 3.1. A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be semi-I-irresolute [4], if $f^{-1}(V)$ is a semi-I-open set in $(X, \tau, I)$ for each semi-J-open set of $(Y, \sigma, J)$.

Theorem 3.1. If a function $f : (X, \tau, I) \to (Y, \sigma, J)$ is semi-I-irresolute, then $f : (X, \tau^{\Lambda^I_1}) \to (Y, \sigma^{\Lambda^I_1})$ is continuous.
Proof. Let $V$ be any $\Lambda^2_I$-set of $(Y,\sigma,J)$, that is $V \in \sigma^{\Lambda^2_I}$, then $V = \Lambda^2_I(V) = \bigcap\{W : V \subset W$ and $W$ is semi-J-open in $(Y,\sigma,J)\}$. Since $f$ is semi-I-irresolute, $f^{-1}(W)$ is a semi-I-open set in $(X,\tau,I)$ for each $W$, hence we have

\[
\Lambda^2_I(f^{-1}(V)) = \cap\{U : f^{-1}(V) \subset U \text{ and } U \in SIO(X,\tau)\} \\
\subset \cap\{f^{-1}(W) : f^{-1}(V) \subset f^{-1}(W) \text{ and } W \in SJO(Y,\sigma)\} \\
= f^{-1}(V).
\]

On the other hand, always we have $f^{-1}(V) \subset \Lambda^2_I(f^{-1}(V))$ and so $f^{-1}(V) = \Lambda^2_I(f^{-1}(V))$. Therefore, $f^{-1}(V) \in \tau^{\Lambda^2_I}$ and $f : (X,\tau^{\Lambda^2_I}) \to (Y,\sigma^{\Lambda^2_I})$ is continuous. \hfill $\square$

**Definition 3.2.** A function $f : (X,\tau,I) \to (Y,\sigma,J)$ is called:

1. $\Lambda^2_I$-continuous, if $f^{-1}(V)$ is a $\Lambda^2_I$-open set in $(X,\tau,I)$ for each open set $V$ of $(Y,\sigma,J)$.
2. Quasi-$\Lambda^2_I$-continuous, if $f^{-1}(V)$ is a $\Lambda^2_I$-open set in $(X,\tau,I)$ for each $\sigma^*$-open set $V$ of $(Y,\sigma,J)$.
3. $\Lambda^2_I$-irresolute, if $f^{-1}(V)$ is a $\Lambda^2_I$-open set in $(X,\tau,I)$ for each $\Lambda^2_I$-open set $V$ of $(Y,\sigma,J)$.

**Theorem 3.2.** If $f : (X,\tau,I) \to (Y,\sigma,J)$ is $\Lambda^2_I$-irresolute function, then $f$ is quasi-$\Lambda^2_I$-continuous.

Proof. Let $V$ be a $\sigma^*$-open set of $(Y,\sigma,J)$, then by Lemma 2.2 we have $V$ is a $\Lambda^2_I$-open set of $(Y,\sigma,J)$ and since $f$ is $\Lambda^2_I$-irresolute, $f^{-1}(V)$ is a $\Lambda^2_I$-open set of $(X,\tau,I)$. Therefore, $f$ is quasi-$\Lambda^2_I$-continuous. \hfill $\square$

The following example shows a function quasi-$\Lambda^2_I$-continuous which is not $\Lambda^2_I$-irresolute.

**Example 3.1.** Let $X = \{a,b,c\}$, $\tau = \{\emptyset,\{a,c\},X\}$, $\sigma = \{\emptyset,\{a\},\{a,b\},\{a,c\},X\}$, $I = \{\emptyset,\{c\}\}$ and $J = \{\emptyset,\{b\}\}$. The collection of the $\Lambda^2_I$-open sets of $(X,\tau,I)$ is $\{\emptyset,\{a\},\{a,c\},\{a\},\{b\},X\}$, the collection of the $\sigma^*$-open sets of $(X,\sigma,J)$ is $\{\emptyset,\{a\},\{a,c\},\{a\},\{b\},X\}$ and the collection of the $\Lambda^2_I$-open sets of $(X,\sigma,J)$ is $\{\emptyset,\{a\},\{b\},\{c\},\{a,c\},\{b\},X\}$. The identity function $f : (X,\tau,I) \to (X,\sigma,J)$ is quasi-$\Lambda^2_I$-continuous, but is not $\Lambda^2_I$-irresolute, since $f^{-1}(\{b,c\}) = \{b,c\}$ and $f^{-1}(\{c\}) = \{c\}$ are not $\Lambda^2_I$-open sets.

**Theorem 3.3.** If $f : (X,\tau,I) \to (Y,\sigma,J)$ is quasi-$\Lambda^2_I$-continuous function, then $f$ is $\Lambda^2_I$-continuous.

Proof. Let $V$ be an open set of $(Y,\sigma,J)$, then $V$ is $\sigma^*$-open set of $(Y,\sigma,J)$ and since $f$ is quasi-$\Lambda^2_I$-continuous, $f^{-1}(V)$ is a $\Lambda^2_I$-open set of $(X,\tau,I)$. This shows that $f$ is $\Lambda^2_I$-continuous. \hfill $\square$

The following example shows a function $\Lambda^2_I$-continuous which is not quasi-$\Lambda^2_I$-continuous.

**Example 3.2.** Let $X = \{a,b,c\}$, $\tau = \{\emptyset,\{a,c\},X\}$, $\sigma = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$, $I = \{\emptyset,\{c\}\}$ and $J = \{\emptyset,\{a\}\}$. The collection of the $\Lambda^2_I$-open sets of $(X,\tau,I)$ is $\{\emptyset,\{a\},\{a,c\},\{a\},\{b\},X\}$, the collection of the $\sigma^*$-open sets of $(X,\sigma,J)$ is $\{\emptyset,\{a\},\{b,c\},\{a,b\},\{b\},X\}$. The identity function $f : (X,\tau,I) \to (X,\sigma,J)$ is $\Lambda^2_I$-continuous, but is not quasi-$\Lambda^2_I$-continuous, because $f^{-1}(\{b,c\}) = \{b,c\}$ is not a $\Lambda^2_I$-open set.
Corollary 3.1. If \( f : (X, \tau, I) \to (Y, \sigma, J) \) is a \( \Lambda_1^-\)-irresolute function, then \( f \) is \( \Lambda_1^-\)-continuous.

Proof. This is an immediate consequence of Theorems 3.2 and 3.3.

By the above results, we have the following diagram and none of these implications is reversible:

\[
\Lambda_1^-\text{-irresolute} \implies \text{quasi-}\Lambda_1^-\text{-continuous} \implies \Lambda_1^-\text{-continuous.}
\]

Proposition 3.1. Let \( f : (X, \tau, I) \to (Y, \sigma, J) \) and \( g : (Y, \sigma, J) \to (Z, \theta, K) \) be two functions, where \( I, J, K \) are ideals on \( X,Y,Z \) respectively. Then:

1. \( g \circ f \) is \( \Lambda_1^-\)-irresolute, if \( f \) is \( \Lambda_1^-\)-irresolute and \( g \) is \( \Lambda_1^-\)-irresolute.
2. \( g \circ f \) is \( \Lambda_1^-\)-continuous, if \( f \) is \( \Lambda_1^-\)-irresolute and \( g \) is \( \Lambda_1^-\)-continuous.
3. \( g \circ f \) is \( \Lambda_1^-\)-continuous, if \( f \) is \( \Lambda_1^-\)-continuous and \( g \) is continuous.
4. \( g \circ f \) is quasi-\( \Lambda_1^-\)-continuous, if \( f \) is \( \Lambda_1^-\)-irresolute and \( g \) is quasi-\( \Lambda_1^-\)-continuous.

Proof. (1) Let \( V \) be a \( \Lambda_1^-\)-open set in \( (Z, \theta, K) \). Since \( g \) is \( \Lambda_1^-\)-irresolute, then \( g^{-1}(V) \) is a \( \Lambda_1^-\)-open set in \( (Y, \sigma, J) \), using that \( f \) is \( \Lambda_1^-\)-irresolute, we obtain that \( f^{-1}(g^{-1}(V)) \) is a \( \Lambda_1^-\)-open set in \( (X, \tau, I) \). But \( (g \circ f)^{-1}(V) = (f^{-1} \circ g^{-1})(V) = f^{-1}(g^{-1}(V)) \) and hence, \( (g \circ f)^{-1}(V) \) is a \( \Lambda_1^-\)-open set in \( (X, \tau, I) \). This shows that \( g \circ f \) is \( \Lambda_1^-\)-irresolute.

The proofs of (2), (3) and (4) are similar to the case (1).

In the next three theorems, we characterize \( \Lambda_1^-\)-continuous, quasi-\( \Lambda_1^-\)-continuous and \( \Lambda_1^-\)-irresolute functions, respectively.

Theorem 3.4. For a function \( f : (X, \tau, I) \to (Y, \sigma) \), the following statements are equivalent:

1. \( f \) is \( \Lambda_1^-\)-continuous.
2. \( f^{-1}(B) \) is a \( \Lambda_1^-\)-closed set in \( (X, \tau, I) \) for each closed set \( B \) in \( (Y, \sigma) \).
3. For each \( x \in X \) and each open set \( V \) in \( (Y, \sigma) \) containing \( f(x) \) there exists a \( \Lambda_1^-\)-open set \( U \) in \( (X, \tau, I) \) containing \( x \) such that \( f(U) \subseteq V \).

Proof. (1) \( \Rightarrow \) (2) Let \( B \) be any closed set in \( (Y, \sigma) \), then \( V = Y \setminus B \) is an open set in \( (Y, \sigma) \) and since \( f \) is \( \Lambda_1^-\)-continuous, \( f^{-1}(V) \) is a \( \Lambda_1^-\)-open subset in \( (X, \tau, I) \), but \( f^{-1}(V) = f^{-1}(Y \setminus B) = f^{-1}(Y) \setminus f^{-1}(B) = X \setminus f^{-1}(B) \) and hence, \( f^{-1}(B) \) is a \( \Lambda_1^-\)-closed set in \( (X, \tau, I) \).

(2) \( \Rightarrow \) (1) Let \( V \) be any open set in \( (Y, \sigma) \), then \( B = Y \setminus V \) is a closed set in \( (Y, \sigma) \). By hypothesis, we have \( f^{-1}(B) \) is a \( \Lambda_1^-\)-closed set in \( (X, \tau, I) \), but \( f^{-1}(B) = f^{-1}(Y \setminus V) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V) \) and so, \( f^{-1}(V) \) is a \( \Lambda_1^-\)-open set in \( (X, \tau, I) \). This shows that \( f \) is \( \Lambda_1^-\)-continuous.

(1) \( \Rightarrow \) (3) Let \( x \in X \) and \( V \) any open set in \( (Y, \sigma) \) such that \( f(x) \in V \), then \( x \in f^{-1}(V) \) and since \( f \) is a \( \Lambda_1^-\)-continuous function, \( f^{-1}(V) \) is a \( \Lambda_1^-\)-open set in \( (X, \tau, I) \). If \( U = f^{-1}(V) \), then \( U \) is a \( \Lambda_1^-\)-open
set in \((X, \tau, I)\) containing \(x\) such that \(f(U) = f(f^{-1}(V)) \subset V\).

(3) \(\Rightarrow\) (1) Let \(V\) be any open set in \((Y, \sigma)\) and \(x \in f^{-1}(V)\), then \(f(x) \in V\) and by (3) there exists a \(\Lambda^s_I\)-open set \(U_x\) in \((X, \tau, I)\) such that \(x \in U_x\) and \(f(U_x) \subset V\). Thus, \(x \in U_x \subset f^{-1}(f(U_x)) \subset f^{-1}(V)\) and hence \(f^{-1}(V) = \bigcup\{U_x : x \in f^{-1}(V)\}\). By Lemma 2.3 we have \(f^{-1}(V)\) is a \(\Lambda^s_I\)-open set in \((X, \tau, I)\) and so \(f\) is \(\Lambda^s_I\)-continuous.

**Theorem 3.5.** For a function \(f : (X, \tau, I) \to (Y, \sigma, J)\), the following statements are equivalent:

1. \(f\) is quasi-\(\Lambda^s_I\)-continuous.
2. \(f^{-1}(B)\) is a \(\Lambda^s_I\)-closed set in \((X, \tau, I)\) for each \(\sigma^*\)-closed set \(B\) in \((Y, \sigma, J)\).
3. For each \(x \in X\) and each \(\sigma^*\)-open set \(V\) in \((Y, \sigma, J)\) containing \(f(x)\) there exists a \(\Lambda^s_I\)-open set \(U\) in \((X, \tau, I)\) containing \(x\) such that \(f(U) \subset V\).

**Proof.** The proof is similar to Theorem 3.4.

**Theorem 3.6.** For a function \(f : (X, \tau, I) \to (Y, \sigma, J)\), the following statements are equivalent:

1. \(f\) is \(\Lambda^s_I\)-irresolute.
2. \(f^{-1}(B)\) is a \(\Lambda^s_I\)-closed set in \((X, \tau, I)\) for each \(\Lambda^s_J\)-closed set \(B\) in \((Y, \sigma, J)\).
3. For each \(x \in X\) and each \(\Lambda^s_J\)-open set \(V\) in \((Y, \sigma, J)\) containing \(f(x)\) there exists a \(\Lambda^s_I\)-open set \(U\) in \((X, \tau, I)\) containing \(x\) such that \(f(U) \subset V\).

**Proof.** The proof is similar to Theorem 3.4.

### 4 \(\Lambda^s_I\)-compactness and \(\Lambda^s_I\)-connectedness

In this section, new notions of compactness and connectedness are introduced in terms of \(\Lambda^s_I\)-open sets and semi-I-open sets, in order to study their behavior under the direct images of the new forms of continuity defined in the previous section.

**Definition 4.1.** An ideal topological space \((X, \tau, I)\) is said to be:

1. \(\Lambda^s_I\)-compact if every cover of \(X\) by \(\Lambda^s_I\)-open sets has a finite subcover.
2. \(\tau^*\)-compact if every cover of \(X\) by \(\tau^*\)-open sets has a finite subcover.
3. Semi-I-compact if every cover of \(X\) by semi-I-open sets has a finite subcover.

**Theorem 4.1.** Let \((X, \tau, I)\) be an ideal topological space, the following properties hold:
(1) \((X, \tau, I)\) is \(\Lambda^1_\tau\)-compact if and only if for every collection \(\{A_\alpha : \alpha \in \Delta\}\) of \(\Lambda^1_\tau\)-closed sets in 
\((X, \tau, I)\) satisfying \(\bigcap\{A_\alpha : \alpha \in \Delta\} = \emptyset\), there is a finite subcollection \(A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\) with 
\(\bigcap\{A_{\alpha_k} : k = 1, \ldots, n\} = \emptyset\).

(2) \((X, \tau, I)\) is \(\tau^*\)-compact if and only if for every collection \(\{A_\alpha : \alpha \in \Delta\}\) of \(\tau^*\)-closed sets in 
\((X, \tau, I)\) satisfying \(\bigcap\{A_\alpha : \alpha \in \Delta\} = \emptyset\), there is a finite subcollection \(A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\) with 
\(\bigcap\{A_{\alpha_k} : k = 1, \ldots, n\} = \emptyset\).

(3) \((X, \tau, I)\) is semi-I-compact if and only if for every collection \(\{A_\alpha : \alpha \in \Delta\}\) of semi-I-closed sets in 
\((X, \tau, I)\) satisfying \(\bigcap\{A_\alpha : \alpha \in \Delta\} = \emptyset\), there is a finite subcollection \(A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\) with 
\(\bigcap\{A_{\alpha_k} : k = 1, \ldots, n\} = \emptyset\).

Proof. (1) Let \(\{A_\alpha : \alpha \in \Delta\}\) be a collection of \(\Lambda^1_\tau\)-closed sets such that \(\bigcap\{A_\alpha : \alpha \in \Delta\} = \emptyset\), then 
\(\{X - A_\alpha : \alpha \in \Delta\}\) is a collection of \(\Lambda^1_\tau\)-open sets such that

\[
X = X - \emptyset = X - \bigcap\{A_\alpha : \alpha \in \Delta\} = \bigcup\{X - A_\alpha : \alpha \in \Delta\},
\]

that is, \(\{X - A_\alpha : \alpha \in \Delta\}\) is a cover of \(X\) by \(\Lambda^1_\tau\)-open sets. Since \((X, \tau, I)\) is \(\Lambda^1_\tau\)-compact, there exists a finite subcollection \(X - A_{\alpha_1}, X - A_{\alpha_2}, \ldots, X - A_{\alpha_n}\) such that

\[
X = \bigcup\{X - A_{\alpha_k} : k = 1, \ldots, n\} = X - \bigcap\{A_{\alpha_k} : k = 1, \ldots, n\}.
\]

This shows that \(\bigcap\{A_{\alpha_k} : k = 1, \ldots, n\} = \emptyset\). Conversely, suppose that \(\{U_\alpha : \alpha \in \Delta\}\) is a cover of \(X\) by \(\Lambda^1_\tau\)-open sets, then \(\{X - U_\alpha : \alpha \in \Delta\}\) is a collection of \(\Lambda^1_\tau\)-closed sets such that \(\bigcap\{X - U_\alpha : \alpha \in \Delta\} = X - \bigcup\{U_\alpha : \alpha \in \Delta\} = X - X = \emptyset\). By hypothesis, there exists a finite subcollection \(X - U_{\alpha_1}, X - U_{\alpha_2}, \ldots, X - U_{\alpha_n}\) such that \(\bigcap\{X - U_{\alpha_k} : k = 1, \ldots, n\} = \emptyset\). Follows \(X = X - \emptyset = X - \bigcap\{X - U_{\alpha_k} : k = 1, \ldots, n\} = X - (X - \bigcup\{U_{\alpha_k} : k = 1, \ldots, n\}) = \bigcup\{U_{\alpha_k} : k = 1, \ldots, n\}\). This shows that \((X, \tau, I)\) is \(\Lambda^1_\tau\)-compact.

The proofs of (2) and (3) are similar to the case (1).

\(\square\)

**Theorem 4.2.** Let \((X, \tau, I)\) be an ideal topological space, the following properties hold:

(1) If \((X, \tau^{\Lambda^1_\tau})\) is compact, then \((X, \tau, I)\) is semi-I-compact.

(2) If \((X, \tau, I)\) is \(\Lambda^1_\tau\)-compact, then \((X, \tau, I)\) is \(\tau^*\)-compact.

(3) If \((X, \tau, I)\) is \(\Lambda^1_\tau\)-compact, then \((X, \tau, I)\) is compact.

Proof. (1) Let \(\{U_\alpha : \alpha \in \Delta\}\) any cover of \(X\) by semi-I-open sets, since every \(\alpha \in \Delta\), \(U_\alpha\) is a \(\Lambda^1_\tau\)-set and hence, \(U_\alpha \in \tau^{\Lambda^1_\tau}\) for each \(\alpha \in \Delta\). Since \((X, \tau^{\Lambda^1_\tau})\) is compact, there exists a finite subset \(\Delta_0\) of \(\Delta\) such that \(X = \bigcup\{U_\alpha : \alpha \in \Delta_0\}\). This shows that \((X, \tau)\) is semi-I-compact.

(2) Let \(\{F_\alpha : \alpha \in \Delta\}\) be a collection of \(\tau^*\)-closed sets of \(X\) such that \(\bigcap\{F_\alpha : \alpha \in \Delta\} = \emptyset\). Since every \(\tau^*\)-closed set is \(\Lambda^1_\tau\)-closed, then \(\{F_\alpha : \alpha \in \Delta\}\) is a collection of \(\Lambda^1_\tau\)-closed sets and \((X, \tau, I)\) is \(\Lambda^1_\tau\)-compact. By Theorem 4.1 (1), there exists a finite subset \(\Delta_0\) of \(\Delta\) such that \(\bigcap\{F_\alpha : \alpha \in \Delta_0\} = \emptyset\).
and by Theorem 4.1, we conclude that \((X, \tau, I)\) is \(\tau^\ast\)-compact.

(3) Follows from (2) and the fact that every \(\tau^\ast\)-compact space is compact. \(\square\)

**Theorem 4.3.** If \(f : (X, \tau, I) \rightarrow (Y, \sigma, J)\) is a surjective function, the following properties hold:

(1) If \(f\) is \(\Lambda_I^\ast\)-irresolute and \((X, \tau, I)\) is \(\Lambda_I^\ast\)-compact, then \((Y, \sigma, J)\) is \(\Lambda_J^\ast\)-compact.

(2) If \(f\) is semi-I-irresolute and \((X, \tau, I)\) is semi-I-compact, then \((Y, \sigma, J)\) is semi-J-compact.

(3) If \(f\) is quasi-\(\Lambda_I^\ast\)-continuous and \((X, \tau, I)\) is \(\Lambda_I^\ast\)-compact, then \((Y, \sigma, J)\) is \(\sigma^\ast\)-compact.

(4) If \(f\) is \(\Lambda_I^\ast\)-continuous and \((X, \tau, I)\) is \(\Lambda_I^\ast\)-compact, then \((Y, \sigma, J)\) is compact.

**Proof.**

(1) Let \(\{V_\alpha : \alpha \in \Delta\}\) be a cover of \(Y\) by \(\Lambda_J^\ast\)-open sets. Since \(f\) is \(\Lambda_I^\ast\)-irresolute, \(\{f^{-1}(V_\alpha) : \alpha \in \Delta\}\) is a cover of \(X\) by \(\Lambda_I^\ast\)-open sets and by the \(\Lambda_I^\ast\)-compactness of \((X, \tau, I)\), there exists a finite subset \(\Delta_0\) of \(\Delta\) such that \(X = \bigcup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}\). Since \(f\) is surjective, then \(Y = f(X) = f\left(\bigcup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}\right) = \{f_\alpha : \alpha \in \Delta_0\}\) and this shows that \((Y, \theta, J)\) is \(\Lambda_J^\ast\)-compact.

The proofs of (2), (3) and (4) are similar to case (1). \(\square\)

**Definition 4.2.** An ideal topological space \((X, \tau, I)\) is said to be:

(1) \(\Lambda_I^\ast\)-connected if \(X\) cannot be written as a disjoint union of two nonempty \(\Lambda_I^\ast\)-open sets.

(2) \(\tau^\ast\)-connected if \(X\) cannot be written as a disjoint union of two nonempty \(\tau^\ast\)-open sets.

(3) semi-I-connected if \(X\) cannot be written as a disjoint union of two nonempty semi-I-open sets.

**Theorem 4.4.** Let \((X, \tau, I)\) be an ideal topological space, the following properties hold:

(1) If \((X, \tau^\Lambda_I)\) is connected, then \((X, \tau, I)\) is semi-I-connected.

(2) If \((X, \tau, I)\) is \(\Lambda_I^\ast\)-connected, then \((X, \tau, I)\) is \(\tau^\ast\)-connected.

(3) If \((X, \tau, I)\) is \(\Lambda_I^\ast\)-connected, then \((X, \tau, I)\) is connected.

**Proof.**

(1) Suppose that \((X, \tau, I)\) is not semi-I-connected, then there exist non-empty semi-I-open sets \(A\) and \(B\) such that \(A \cap B = \emptyset\) and \(A \cup B = X\). By Proposition 3.1(3) of \(\cite{7}\), \(A\) and \(B\) are \(\Lambda_I^\ast\)-sets and hence, \((X, \tau^\Lambda_I)\) is not connected.

(2) Suppose that \((X, \tau, I)\) is not \(\tau^\ast\)-connected, then there exist non-empty \(\tau^\ast\)-open sets \(A\) and \(B\) such that \(A \cap B = \emptyset\) and \(A \cup B = X\). By Lemma 2.2 we have \(A\) and \(B\) are \(\Lambda_I^\ast\)-open sets and so, \((X, \tau, I)\) is not \(\Lambda_I^\ast\)-connected.

(3) Follows from (2) and the fact that every \(\tau^\ast\)-connected space is connected. \(\square\)

**Theorem 4.5.** For an ideal topological space \((X, \tau, I)\), the following statements are equivalent:

(1) \((X, \tau, I)\) is \(\Lambda_I^\ast\)-connected.
(2) \( \emptyset \) and \( X \) are the only subsets of \( X \) which are both \( \Lambda^s_1 \)-open and \( \Lambda^s_1 \)-closed.

(3) Every \( \Lambda^s_1 \)-continuous function of \( X \) into a discrete space \( Y \) with at least two points, is a constant function.

**Proof.** (1)\( \Rightarrow \) (2) Let \( V \) be a subset of \( X \) which is both \( \Lambda^s_1 \)-open and \( \Lambda^s_1 \)-closed, then \( X - V \) is both \( \Lambda^s_1 \)-open and \( \Lambda^s_1 \)-closed, so \( X = V \cup (X - V) \). Since \( (X, \tau, I) \) is \( \Lambda^s_1 \)-connected, then one of those sets is \( \emptyset \). Therefore, \( V = \emptyset \) or \( V = X \).

(2)\( \Rightarrow \) (1) Suppose that \( (X, \tau, I) \) is not \( \Lambda^s_1 \)-connected and let \( X = U \cup V \), where \( U \) and \( V \) are disjoint nonempty \( \Lambda^s_1 \)-open sets in \( (X, \tau, I) \), then \( U = X - V \) is both \( \Lambda^s_1 \)-open and \( \Lambda^s_1 \)-closed. By hypothesis, \( U = \emptyset \) or \( U = X \), which is a contradiction. Therefore, \( (X, \tau, I) \) is \( \Lambda^s_1 \)-connected.

(2)\( \Rightarrow \) (3) Let \( f : (X, \tau, I) \rightarrow Y \) be a \( \Lambda^s_1 \)-continuous function, where \( Y \) is a topological space with the discrete topology and contains at least two points, then \( X \) can be cover by a collection of sets which are both \( \Lambda^s_1 \)-open and \( \Lambda^s_1 \)-closed of the form \( \{ f^{-1}(y) : y \in Y \} \), from these, we conclude that there exists a \( y_0 \in Y \) such that \( f^{-1}([y_0]) = X \) and so, \( f \) is a constant function.

(3)\( \Rightarrow \) (2) Let \( W \) be a subset of \( (X, \tau, I) \) which is both \( \Lambda^s_1 \)-open and \( \Lambda^s_1 \)-closed. Suppose that \( W \neq \emptyset \) and let \( f : (X, \tau, I) \rightarrow Y \) be the function defined by \( f(W) = \{ y_1 \} \) and \( f(X - W) = \{ y_2 \} \) for \( y_1, y_2 \in Y \) and \( y_1 \neq y_2 \). Then \( f \) is \( \Lambda^s_1 \)-continuous, since the inverse image de each open set in \( Y \) is \( \Lambda^s_1 \)-open in \( X \). Hence, by (3), \( f \) must be a constant function. It follows that \( X = W \).  

**Theorem 4.6.** If \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \) is a surjective function, the following properties hold:

(1) If \( f \) is a \( \Lambda^s_1 \)-irresolute and \( (X, \tau, I) \) is \( \Lambda^s_1 \)-connected, then \( (Y, \sigma, J) \) is \( \Lambda^s_1 \)-connected.

(2) If \( f \) is a semi-I-irresolute function and \( (X, \tau, I) \) is semi-I-connected, then \( (Y, \sigma, J) \) is semi-I-connected.

(3) If \( f \) is a quasi-\( \Lambda^s_1 \)-continuous function and \( (X, \tau, I) \) is \( \Lambda^s_1 \)-connected, then \( (Y, \sigma, J) \) is \( \sigma^* \)-connected.

(4) If \( f \) is a \( \Lambda^s_1 \)-continuous function and \( (X, \tau, I) \) is \( \Lambda^s_1 \)-connected, then \( (Y, \sigma) \) is connected.

**Proof.** (1) Suppose that \( (Y, \sigma, J) \) is not \( \Lambda^s_1 \)-connected, then there exist nonempty \( \Lambda^s_1 \)-open sets \( H, G \) in \( (Y, \sigma, J) \) such that \( G \cap H = \emptyset \) and \( G \cup H = Y \). Hence, we have \( f^{-1}(G) \cap f^{-1}(H) = \emptyset \), \( f^{-1}(G) \cup f^{-1}(H) = X \) and moreover, \( f^{-1}(G) \) and \( f^{-1}(H) \) are nonempty \( \Lambda^s_1 \)-open sets in \( (X, \tau, I) \). This shows that \( (X, \tau, I) \) is not \( \Lambda^s_1 \)-connected.

The proofs of (2), (3) and (4) are similar to case (1).

**Open problems.** The Theorems 4.2 and 4.4 have been proved using the fact that every semi-I-open set is \( \Lambda^s_1 \)-open and that every \( \tau^* \)-open set is \( \Lambda^s_1 \)-open. But until today, we don’t have any contra example in order to shows that the converse of such Theorems are not true. In that sense we write the following questions.

(1) Does there exists an ideal topological space \( (X, \tau, I) \) which is semi-I-compact (resp. semi-I-connected) but \( (X, \tau^{\Lambda^s_1}) \) is not a compact (resp. connected) space?
(2) Does there exists an ideal topological space \((X, \tau, I)\) which is \(\tau^*\)-compact (resp. \(\tau^*\)-connected) but \((X, \tau)\) is not \(\Lambda^I_\ast\)-compact space (resp. \(\Lambda^I_\ast\)-connected space.)?

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