Reproducing inversion formulas for the Dunkl-Wigner transforms

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ABSTRACT

We define and study the Fourier-Wigner transform associated with the Dunkl operators, and we prove for this transform a reproducing inversion formulas and a Plancherel formula. Next, we introduce and study the extremal functions associated to the Dunkl-Wigner transform.

RESUMEN

Definimos y estudiamos la transformada de Fourier-Wigner asociada a los operadores de Dunkl, y probamos una fórmula de inversion y una formula de Plancherel para esta transformada. Luego introducimos y estudiamos las funciones extremales asociadas a la transformada de Dunkl-Wigner.

Keywords and Phrases: Dunkl transform; Dunkl-Wigner transform; inversion formulas; extremal functions.

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1 Introduction

In this paper, we consider \( \mathbb{R}^d \) with the Euclidean inner product \( \langle ., . \rangle \) and norm \( |y| := \sqrt{\langle y, y \rangle} \). For \( \alpha \in \mathbb{R}^d \backslash \{0\} \), let \( \sigma_\alpha \) be the reflection in the hyperplane \( H_\alpha \subset \mathbb{R}^d \) orthogonal to \( \alpha \):

\[
\sigma_\alpha y := y - \frac{2(\alpha, y)}{|\alpha|^2} \alpha.
\]

A finite set \( \mathbb{R}e \subset \mathbb{R}^d \backslash \{0\} \) is called a root system, if \( \mathbb{R}e \cap \mathbb{R}e_\alpha = \{-\alpha, \alpha\} \) and \( \sigma_\alpha \mathbb{R}e = \mathbb{R}e \) for all \( \alpha \in \mathbb{R}e \). We assume that it is normalized by \( |\alpha|^2 = 2 \) for all \( \alpha \in \mathbb{R}e \). For a root system \( \mathbb{R}e \), the reflections \( \sigma_\alpha, \alpha \in \mathbb{R}e \), generate a finite group \( G \). The Coxeter group \( G \) is a subgroup of the orthogonal group \( O(d) \). All reflections in \( G \), correspond to suitable pairs of roots. For a given \( \beta \in \mathbb{R}^d \backslash \bigcup_{\alpha \in \mathbb{R}e} H_\alpha \), we fix the positive subsystem \( \mathbb{R}e_+ := \{\alpha \in \mathbb{R}e : \langle \alpha, \beta \rangle > 0\} \). Then for each \( \alpha \in \mathbb{R}e \) either \( \alpha \in \mathbb{R}e_+ \) or \( -\alpha \in \mathbb{R}e_+ \).

Let \( k : \mathbb{R}e \to \mathbb{C} \) be a multiplicity function on \( \mathbb{R}e \) (a function which is constant on the orbits under the action of \( G \)). As an abbreviation, we introduce the index \( \gamma = \gamma_k := \sum_{\alpha \in \mathbb{R}e_+} k(\alpha) \).

Throughout this paper, we will assume that \( k(\alpha) \geq 0 \) for all \( \alpha \in \mathbb{R}e \). Moreover, let \( w_k \) denote the weight function \( w_k(y) := \prod_{\alpha \in \mathbb{R}e_+} |(\alpha, y)|^{2k(\alpha)} \), for all \( y \in \mathbb{R}^d \), which is \( G \)-invariant and homogeneous of degree \( 2\gamma \).

Let \( c_k \) be the Mehta-type constant given by \( c_k := \left( \int_{\mathbb{R}^d} e^{-|y|^2/2} w_k(y) dy \right)^{-1} \). We denote by \( \mu_k \) the measure on \( \mathbb{R}^d \) given by \( d\mu_k(y) := c_k w_k(y) dy \); and by \( L^p(\mu_k) \), \( 1 \leq p \leq \infty \), the space of measurable functions \( f \) on \( \mathbb{R}^d \), such that

\[
\|f\|_{L^p(\mu_k)} := \left( \int_{\mathbb{R}^d} |f(y)|^p d\mu_k(y) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,
\]

and by \( L^\infty(\mu_k) \) the subspace of \( L^p(\mu_k) \) consisting of radial functions.

For \( f \in L^1(\mu_k) \) the Dunkl transform of \( f \) is defined (see \[3\]) by

\[
\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y)f(y) d\mu_k(y), \quad x \in \mathbb{R}^d,
\]

where \( E_k(-ix, y) \) denotes the Dunkl kernel. (For more details see the next section.)

The Dunkl translation operators \( \tau_x, x \in \mathbb{R}^d \), \[13\] are defined on \( L^2(\mu_k) \) by

\[
\mathcal{F}_k(\tau_x f)(y) = E_k(ix, y)\mathcal{F}_k(f)(y), \quad y \in \mathbb{R}^d.
\]

Let \( g \in L^2_{rad}(\mu_k) \). The Dunkl-Wigner transform \( V_g \) is the mapping defined for \( f \in L^2(\mu_k) \) by

\[
V_g(f)(x, y) := \int_{\mathbb{R}^d} f(t)\overline{\tau_x g_{k,y}(-t)} d\mu_k(t),
\]
where
\[ g_{k,y}(z) := \mathcal{F}_k\left( \sqrt{\tau_y |\mathcal{F}_k(g)|^2} \right)(z). \]

We study some of its properties, and we prove reproducing inversion formulas for this transform. Next, building on the ideas of Matsuura et al. [6], Saitoh [11, 13] and Yamada et al. [20], and using the theory of reproducing kernels [10], we give best approximation of the mapping \( V_g \) on the Sobolev-Dunkl spaces \( H^s(\mu_k) \). More precisely, for all \( \lambda > 0 \), \( h \in L^2(\mu_k \otimes \mu_k) \), the infimum
\[
\inf_{f \in H^s(\mu_k)} \left\{ \lambda \|f\|^2_{H^s(\mu_k)} + \|h - V_g(f)\|^2_{L^2(\mu_k \otimes \mu_k)} \right\},
\]
is attained at one function \( f^{*}_{\lambda,h} \), called the extremal function, and given by
\[
f^{*}_{\lambda,h}(y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E_k(zy) \sqrt{\tau_1 |\mathcal{F}_k(g)|^2(z)} \mathcal{F}_k(h(\cdot, t))(z) \frac{d\mu_k(t) d\mu_k(z)}{\lambda (1 + |z|^2)^{s/2} + \|g\|^2_{L^2(\mu_k \otimes \mu_k)}}.
\]

In the Dunkl setting, the extremal functions are studied in several directions [14, 15, 16, 17].

In the classical case, the Fourier-Wigner transforms are studied by Weyl [21] and Wong [22].

In the Bessel-Kingman hypergroups, these operators are studied by Dachraoui [1].

This paper is organized as follows. In Section 2, we recall some properties of harmonic analysis for the Dunkl operators. Next, we define the Fourier-Wigner transform \( V_g \) in the Dunkl setting, and we have established for it a reproducing inversion formulas. In Section 3, we introduce and study the extremal functions associated to the Dunkl-Wigner transform \( V_g \).

## 2 The Dunkl-Wigner transform

The Dunkl operators \( D_j \): \( j = 1, ..., d \), on \( \mathbb{R}^d \) associated with the finite reflection group \( G \) and multiplicity function \( k \) are given, for a function \( f \) of class \( C^1 \) on \( \mathbb{R}^d \), by
\[
D_j f(y) := \frac{\partial}{\partial y_j} f(y) + \sum_{\alpha \in \Re_+} k(\alpha) \alpha_j \frac{f(y) - f(\sigma_{\alpha} y)}{\langle \alpha, y \rangle}.
\]

For \( y \in \mathbb{R}^d \), the initial problem \( D_j u(\cdot, y)(x) = y_j u(x, y), j = 1, ..., d, \) with \( u(0, y) = 1 \) admits a unique analytic solution on \( \mathbb{R}^d \), which will be denoted by \( E_k(x, y) \) and called Dunkl kernel [2, 4]. This kernel has a unique analytic extension to \( \mathbb{C}^d \times \mathbb{C}^d \) (see [7]). The Dunkl kernel has the Laplace-type representation [8]
\[
E_k(x, y) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} d\Gamma_x(z), \quad x \in \mathbb{R}^d, y \in \mathbb{C}^d,
\]
where \( \langle y, z \rangle := \sum_{i=1}^d y_i z_i \) and \( \Gamma_x \) is a probability measure on \( \mathbb{R}^d \), such that \( \text{supp}(\Gamma_x) \subset \{ z \in \mathbb{R}^d : |z| \leq |x| \} \). In our case,
\[
|E_k(ix, y)| \leq 1, \quad x, y \in \mathbb{R}^d.
\]
The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and was introduced by Dunkl in [3], where already many basic properties were established. Dunkl’s results were completed and extended later by De Jeu [4]. The Dunkl transform of a function $f$ in $L^1(\mu_k)$, is defined by

$$F_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y)f(y)d\mu_k(y), \quad x \in \mathbb{R}^d.$$ 

We notice that $F_0$ agrees with the Fourier transform $F$ that is given by

$$F(f)(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle x, y \rangle}f(y)dy, \quad x \in \mathbb{R}^d.$$ 

Some of the properties of Dunkl transform $F_k$ are collected below (see [3, 4]).

**Theorem 2.1.** (i) $L^1 - L^\infty$-boundedness. For all $f \in L^1(\mu_k)$, $F_k(f) \in L^\infty(\mu_k)$, and

$$\|F_k(f)\|_{L^\infty(\mu_k)} \leq \|f\|_{L^1(\mu_k)}.$$ 

(ii) Inversion theorem. Let $f \in L^1(\mu_k)$, such that $F_k(f) \in L^1(\mu_k)$. Then

$$f(x) = F(F_k(f))(-x), \quad \text{a.e. } x \in \mathbb{R}^d.$$ 

(iii) Plancherel theorem. The Dunkl transform $F_k$ extends uniquely to an isometric isomorphism of $L^2(\mu_k)$ onto itself. In particular, we have

$$\|f\|_{L^2(\mu_k)} = \|F_k(f)\|_{L^2(\mu_k)}.$$ 

(iv) Parseval theorem. For $f, g \in L^2(\mu_k)$, we have

$$(f, g)_{L^2(\mu_k)} = \langle F_k(f), F_k(g)\rangle_{L^2(\mu_k)}.$$ 

The Dunkl transform $F_k$ allows us to define a generalized translation operators on $L^2(\mu_k)$ by setting

$$F_k(\tau_x f)(y) = E_k(ix, y)F_k(f)(y), \quad y \in \mathbb{R}^d.$$ 

It is the definition of Thangavelu and Xu given in [18]. It plays the role of the ordinary translation $\tau_x f = f(x+.)$ in $\mathbb{R}^d$, since the Euclidean Fourier transform satisfies $F(\tau_x f)(y) = e^{ixy}F(f)(y)$. Note that from (2.2) and Theorem 2.1 (iii), the definition (2.3) makes sense, and

$$\|\tau_x f\|_{L^2(\mu_k)} \leq \|f\|_{L^2(\mu_k)}, \quad f \in L^2(\mu_k).$$ 

$$\text{(2.4)}$$
Rösler [9] introduced the Dunkl translation operators for radial functions. If \( f \) are radial functions, \( f(x) = F(|x|) \), then

\[
\tau_x f(y) = \int_{\mathbb{R}^d} F\left( \sqrt{|x|^2 + |y|^2 + 2(y, z)} \right) d\Gamma_x(z); \quad x, y \in \mathbb{R}^d,
\]

where \( \Gamma_x \) is the representing measure given by (2.1).

This formula allows us to establish the following results [18, 19].

**Proposition 2.2.** (i) For all \( p \in [1, 2] \) and for all \( x \in \mathbb{R}^d \), the Dunkl translation \( \tau_x : L^p_{\text{rad}}(\mu_k) \to L^p(\mu_k) \) is a bounded operator, and for \( f \in L^p_{\text{rad}}(\mu_k) \), we have

\[
\|\tau_x f\|_{L^p(\mu_k)} \leq \|f\|_{L^p_{\text{rad}}(\mu_k)}.
\]

(ii) Let \( f \in L^1_{\text{rad}}(\mu_k) \). Then, for all \( x \in \mathbb{R}^d \), we have

\[
\int_{\mathbb{R}^d} \tau_x f(y) d\mu_k(y) = \int_{\mathbb{R}^d} f(y) d\mu_k(y).
\]

The Dunkl convolution product \(*_k\) of two functions \( f \) and \( g \) in \( L^2(\mu_k) \) is defined by

\[
f \ast_k g(x) := \int_{\mathbb{R}^d} \tau_x f(-y) g(y) d\mu_k(y), \quad x \in \mathbb{R}^d.
\]  

(2.5)

We notice that \(*_k\) generalizes the convolution \(*\) that is given by

\[
f \ast g(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x - y) g(y) dy, \quad x \in \mathbb{R}^d.
\]

The Proposition 2.2 allows us to establish the following properties for the Dunkl convolution on \( \mathbb{R}^d \) (see [19]).

**Proposition 2.3.** (i) Assume that \( p \in [1, 2] \) and \( q, r \in [1, \infty] \) such that \( 1/p + 1/q = 1 + 1/r \). Then the map \( (f, g) \to f \ast_k g \) extends to a continuous map from \( L^p_{\text{rad}}(\mu_k) \times L^q(\mu_k) \) to \( L^r(\mu_k) \), and

\[
\|f \ast_k g\|_{L^r(\mu_k)} \leq \|f\|_{L^p_{\text{rad}}(\mu_k)} \|g\|_{L^q(\mu_k)}.
\]

(ii) For all \( f \in L^1_{\text{rad}}(\mu_k) \) and \( g \in L^2(\mu_k) \), we have

\[
\mathcal{F}_k(f \ast_k g) = \mathcal{F}_k(f) \mathcal{F}_k(g).
\]

(iii) Let \( f \in L^2_{\text{rad}}(\mu_k) \) and \( g \in L^2(\mu_k) \). Then \( f \ast_k g \) belongs to \( L^2(\mu_k) \) if and only if \( \mathcal{F}_k(f) \mathcal{F}_k(g) \) belongs to \( L^2(\mu_k) \), and

\[
\mathcal{F}_k(f \ast_k g) = \mathcal{F}_k(f) \mathcal{F}_k(g), \quad \text{in the } L^2(\mu_k) \text{ -- case.}
\]
(iv) Let \( f \in L^2_{rad}(\mu_k) \) and \( g \in L^2(\mu_k) \). Then
\[
\int_{\mathbb{R}^d} |f \ast g(x)|^2 d\mu_k(x) = \int_{\mathbb{R}^d} |F_k(f)(z)|^2 |F_k(g)(z)|^2 d\mu_k(z),
\]
where both sides are finite or infinite.

Let \( g \in L^2_{rad}(\mu_k) \) and \( y \in \mathbb{R}^d \). The modulation of \( g \) by \( y \) is the function \( g_{k,y} \) defined by
\[
g_{k,y}(z) := F_k \left( \sqrt{\tau_y |F_k(g)|^2} \right)(z), \quad z \in \mathbb{R}^d.
\]
Thus,
\[
\|g_{k,y}\|_{L^2(\mu_k)} = \|g\|_{L^2_{rad}(\mu_k)}.
\]

Let \( g \in L^2_{rad}(\mu_k) \). The Fourier-Wigner transform associated to the Dunkl operators, is the mapping \( V_g \) defined for \( f \in L^2(\mu_k) \) by
\[
V_g(f)(x,y) := \int_{\mathbb{R}^d} f(t) \tau_x g_{k,y}(-t) d\mu_k(t), \quad x,y \in \mathbb{R}^d.
\]

**Proposition 2.4.** Let \((f,g) \in L^2(\mu_k) \times L^2_{rad}(\mu_k)\).

(i) \( V_g(f)(x,y) = \overline{g_{k,y}} \ast_k f(x) \).

(ii) \( V_g(f)(x,y) = \int_{\mathbb{R}^d} E_k(ix,z) F_k(f)(z) \sqrt{\tau_y |F_k(g)|^2(z)} d\mu_k(z) \).

(iii) The function \( V_g(f) \) belongs to \( L^\infty(\mu_k \otimes \mu_k) \), and
\[
\|V_g(f)\|_{L^\infty(\mu_k \otimes \mu_k)} \leq \|f\|_{L^2(\mu_k)} \|g\|_{L^2_{rad}(\mu_k)}.
\]

**Proof.** (i) follows from (2.5), (2.7) and the fact that \( \tau_x g_{k,y}(-t) = \tau_x \overline{g_{k,y}}(-t) \).

(ii) By Theorem 2.1 (iv) and (2.3) we have
\[
V_g(f)(x,y) = \int_{\mathbb{R}^d} E_k(ix,z) F_k(f)(z) \overline{F_k(g_{k,y})(-z)} d\mu_k(z).
\]
We obtain the result from the fact that
\[
\overline{F_k(g_{k,y})(-z)} = F_k(\overline{g_{k,y}})(z) = \sqrt{\tau_y |F_k(g)|^2(z)}.
\]

(iii) follows from (2.7), by using Hölder’s inequality, (2.4) and (2.6). \( \square \)

**Theorem 2.5.** Let \( g \in L^2_{rad}(\mu_k) \).

(i) Plancherel formula: For every \( f \in L^2(\mu_k) \), we have
\[
\|V_g(f)\|_{L^2(\mu_k \otimes \mu_k)} = \|g\|_{L^2_{rad}(\mu_k)} \|f\|_{L^2(\mu_k)}.
\]
(ii) Parseval formula: For every \( f, h \in L^2(\mu_k) \), we have
\[
\langle V_g(f), V_g(h) \rangle_{L^2(\mu_k \otimes \mu_k)} = \|g\|_{L^2_{\tau g}(\mu_k)}^2 \langle f, h \rangle_{L^2(\mu_k)}.
\]

(iii) Inversion formula: For all \( f \in L^1 \cap L^2(\mu_k) \) such that \( \mathcal{F}_k(f) \in L^1(\mu_k) \), we have
\[
f(z) = \frac{1}{\|g\|_{L^2_{\tau g}(\mu_k)}^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V_g(f)(x, y) \tau_z g_{k, y}(-x) d\mu_k(x) d\mu_k(y).
\]

Proof. (i) From Theorem 2.1 (iii), Proposition 2.2 (ii), Proposition 2.3 (iv) and Proposition 2.4 (i), we obtain
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_g(f)(x, y)|^2 d\mu_k(x) d\mu_k(y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}_k(g_k, y)(z)|^2 |\mathcal{F}_k(f)(z)|^2 d\mu_k(z) d\mu_k(y)
\]
\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tau_y |\mathcal{F}_k(g)|^2(z) |\mathcal{F}_k(f)(z)|^2 d\mu_k(z) d\mu_k(y)
\]
\[
= \|g\|_{L^2_{\tau g}(\mu_k)}^2 \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(z)|^2 d\mu_k(z).
\]

(ii) follows from (i) by polarization.

(iii) From Theorem 2.1 (iv), Proposition 2.3 (ii) and (iii), we have
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V_g(f)(x, y) \tau_z g_{k, y}(-x) d\mu_k(x) d\mu_k(y)
\]
\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tau_y |\mathcal{F}_k(g)|^2(t) \mathcal{F}_k(f)(t) E_k(iz, t) d\mu_k(t) d\mu_k(y).
\]

Then, by Fubini’s theorem, Theorem 2.1 (ii) and Proposition 2.2 (ii) we deduce that
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V_g(f)(x, y) \tau_z g_{k, y}(-x) d\mu_k(x) d\mu_k(y) = \|g\|_{L^2_{\tau g}(\mu_k)}^2 \int_{\mathbb{R}^d} \mathcal{F}_k(f)(t) E_k(iz, t) d\mu_k(t)
\]
\[
= \|g\|_{L^2_{\tau g}(\mu_k)}^2 f(z).
\]

In the following we establish reproducing inversion formula of Calderón’s type for the Dunkl-Wigner transform on \( \mathbb{R}^d \).

**Theorem 2.6.** Let \( \Delta = \prod_{j=1}^d [a_j, b_j], -\infty < a_j < b_j < \infty \); and let \( g \in L^2_{\tau g}(\mu_k) \) such that \( \mathcal{F}_k(g) \in L^\infty(\mu_k) \). Then, for \( f \in L^2(\mu_k) \), the function \( f_\Delta \) given by
\[
f_\Delta(z) = \frac{1}{\|g\|_{L^2_{\tau g}(\mu_k)}^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V_g(f)(x, y) \tau_z g_{k, y}(-x) d\mu_k(x) d\mu_k(y),
\]
belongs to $L^2(\mu_k)$ and satisfies
\[
\lim_{a_j \to -\infty} \lim_{b_j \to +\infty} \|f - f\|_{L^2(\mu_k)} = 0.
\] (2.8)

**Proof.** From Theorem 2.1 (iii), Proposition 2.3 (iv) and Proposition 2.4 (i), we have
\[
f_{\Delta}(z) = \frac{1}{\|g\|_{L^2(\mu_k)}} \int_{\mathbb{R}^d} \tau_y |\mathcal{F}_k(g)|^2(t)\mathcal{F}_k(f)(t)E_k(iz, t)d\mu_k(t)d\mu_k(y).
\]

By Fubini’s theorem we get
\[
f_{\Delta}(z) = \int_{\mathbb{R}^d} K_{\Delta}(t)\mathcal{F}_k(f)(t)E_k(iz, t)d\mu_k(t).
\] (2.9)

where
\[
K_{\Delta}(t) = \frac{1}{\|g\|_{L^2(\mu_k)}} \int_{\Delta} \tau_y |\mathcal{F}_k(g)|^2(t)d\mu_k(y).
\]

It is easily to see that $||K_{\Delta}||_{L^\infty(\mu_k)} \leq 1$. On the other hand, by Hölder’s inequality, we deduce that
\[
|K_{\Delta}(t)|^2 \leq \frac{\mu_k(\Delta)}{\|g\|_{L^2(\mu_k)}} \int \tau_y |\mathcal{F}_k(g)|^2(t)^2d\mu_k(y).
\]

Hence, by (2.4) we find
\[
\|K_{\Delta}\|_{L^2(\mu_k)}^2 \leq \frac{\mu_k(\Delta)}{\|g\|_{L^2(\mu_k)}} \int |\mathcal{F}_k(g)(t)|^4d\mu_k(t) \leq \frac{(\mu_k(\Delta))^2 \|\mathcal{F}_k(g)\|_{L^\infty(\mu_k)}}{\|g\|_{L^2(\mu_k)}}.
\]

Thus $K_{\Delta} \in L^\infty \cap L^2(\mu_k)$. Therefore and by (2.9) we obtain
\[
\mathcal{F}_k(f_{\Delta})(t) = K_{\Delta}(t)\mathcal{F}_k(f)(t).
\]

From this relation and Theorem 2.1 (iii), it follows that $f_{\Delta} \in L^2(\mu_k)$ and
\[
\|f_{\Delta} - f\|_{L^2(\mu_k)}^2 = \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(t)|^2(1 - K_{\Delta}(t))^2d\mu_k(t).
\]

But by Proposition 2.2 (ii) we have
\[
\lim_{a_j \to -\infty} \lim_{b_j \to +\infty} K_{\Delta}(t) = 1, \quad \text{for all } t \in \mathbb{R}^d,
\]
and
\[
|\mathcal{F}_k(f)(t)|^2(1 - K_{\Delta}(t))^2 \leq |\mathcal{F}_k(f)(t)|^2, \quad \text{for all } t \in \mathbb{R}^d.
\]

So, the relation (2.8) follows from the dominated convergence theorem. \qed
3 Extremal functions for the mapping $V_g$

Let $s \geq 0$. We define the Sobolev-Dunkl space of order $s$, that will be denoted $H^s(\mu_k)$, as the set of all $f \in L^2(\mu_k)$ such that $(1 + |z|^2)^{s/2} \mathcal{F}_k(f) \in L^2(\mu_k)$. The space $H^s(\mu_k)$ provided with the inner product

$$
\langle f, g \rangle_{H^s(\mu_k)} = \int_{\mathbb{R}^d} (1 + |z|^2)^{s/2} \mathcal{F}_k(f)(z) \overline{\mathcal{F}_k(g)(z)} \, d\mu_k(z),
$$

and the norm

$$
\|f\|_{H^s(\mu_k)} = \left[ \int_{\mathbb{R}^d} (1 + |z|^2)^{s} |\mathcal{F}_k(f)(z)|^2 \, d\mu_k(z) \right]^{1/2}.
$$

The space $H^s(\mu_k)$ satisfies the following properties.

(a) $H^0(\mu_k) = L^2(\mu_k)$.

(b) For all $s > 0$, the space $H^s(\mu_k)$ is continuously contained in $L^2(\mu_k)$ and $\|f\|_{L^2(\mu_k)} \leq \|f\|_{H^s(\mu_k)}$.

(c) For all $s$, $t > 0$, such that $t > s$, the space $H^t(\mu_k)$ is continuously contained in $H^s(\mu_k)$ and $\|f\|_{H^s(\mu_k)} \leq \|f\|_{H^t(\mu_k)}$.

(d) The space $H^s(\mu_k)$, $s \geq 0$ provided with the inner product $\langle \cdot, \cdot \rangle_{H^s(\mu_k)}$ is a Hilbert space.

**Remark 3.1.** For $s > \gamma + d/2$, the function $y \to (1 + |z|^2)^{-s/2}$ belongs to $L^2(\mu_k)$. Hence for all $f \in H^s(\mu_k)$, we have $\|\mathcal{F}_k(f)\|_{L^2(\mu_k)} \leq \|f\|_{H^s(\mu_k)}$, and by Hölder’s inequality

$$
\|\mathcal{F}_k(f)\|_{L^1(\mu_k)} \leq \left[ \int_{\mathbb{R}^d} \frac{d\mu_k(z)}{(1 + |z|^2)^s} \right]^{1/2} \|f\|_{H^s(\mu_k)}.
$$

Then the function $\mathcal{F}_k(f)$ belongs to $L^1 \cap L^2(\mu_k)$, and therefore

$$
f(x) = \int_{\mathbb{R}^d} E_k(ix, z) \mathcal{F}_k(f)(z) \, d\mu_k(z), \quad \text{a.e.} \ x \in \mathbb{R}^d.
$$

Let $\lambda > 0$. We denote by $\langle \cdot, \cdot \rangle_{\lambda, H^s(\mu_k)}$ the inner product defined on the space $H^s(\mu_k)$ by

$$
\langle f, h \rangle_{\lambda, H^s(\mu_k)} := \lambda \langle f, h \rangle_{H^s(\mu_k)} + \langle V_g(f), V_g(h) \rangle_{L^2(\mu_k \otimes \mu_k)},
$$

and the norm $\|f\|_{\lambda, H^s(\mu_k)} := \sqrt{\langle f, f \rangle_{\lambda, H^s(\mu_k)}}$.

In the next we suppose that $g \in L^2_{rad}(\mu_k)$. By Theorem 2.5 (ii), the inner product $\langle \cdot, \cdot \rangle_{\lambda, H^s(\mu_k)}$ can be written

$$
\langle f, h \rangle_{\lambda, H^s(\mu_k)} = \lambda \langle f, h \rangle_{H^s(\mu_k)} + \|g\|_{L^2_{rad}(\mu_k)}^2 \langle f, h \rangle_{L^2(\mu_k)}.
$$

**Theorem 3.2.** Let $\lambda > 0$ and $s > \gamma + d/2$ and let $g \in L^2_{rad}(\mu_k)$. The space $(H^s(\mu_k), \langle \cdot, \cdot \rangle_{\lambda, H^s(\mu_k)})$ has the reproducing kernel

$$
K_s(x, y) = \int_{\mathbb{R}^d} \frac{E_k(ix, z)E_k(-iy, z) \, d\mu_k(z)}{\lambda(1 + |z|^2)^s + \|g\|_{L^2_{rad}(\mu_k)}^2},
$$

where $E_k$ is the Dunkl basis function.
that is

(i) For all \( y \in \mathbb{R}^d \), the function \( x \rightarrow K_s(x, y) \) belongs to \( H^s(\mu_k) \).

(ii) The reproducing property: for all \( f \in H^s(\mu_k) \) and \( y \in \mathbb{R}^d \),

\[
(f, K_s(., y))_{\lambda, H^s(\mu_k)} = f(y).
\]

**Proof.** (i) Let \( y \in \mathbb{R}^d \). From (2.2), the function \( \Phi_y : \mathbb{R}^d \rightarrow \mathbb{R}^{d+1} \) belongs to \( L^1 \cap L^2(\mu_k) \). Then, the function \( K_s \) is well defined and by Theorem 2.1 (ii), we have

\[
K_s(x, y) = \mathcal{F}_k^{-1}(\Phi_y)(x), \quad x \in \mathbb{R}^d.
\]

From Theorem 2.1 (iii), it follows that \( K_s(., y) \) belongs to \( L^2(\mu_k) \), and we have

\[
\mathcal{F}_k(K_s(., y))(z) = \frac{\mathcal{E}_k(-iy, z)}{\lambda(1+|z|^2)^s + \|g\|_{L^2_{\mu_k}}^2}, \quad z \in \mathbb{R}^d.
\]

Then by (2.2), we obtain

\[
|\mathcal{F}_k(K_s(., y))(z)| \leq \frac{1}{\lambda(1+|z|^2)^s},
\]

and

\[
\|K_s(., y)\|_{H^s(\mu_k)}^2 \leq \frac{1}{\lambda^2} \int_{\mathbb{R}^d} \frac{d\mu_k(z)}{(1+|z|^2)^s} < \infty.
\]

This proves that for all \( y \in \mathbb{R}^d \) the function \( K_s(., y) \) belongs to \( H^s(\mu_k) \).

(ii) Let \( f \in H^s(\mu_k) \) and \( y \in \mathbb{R}^d \). From (3.1) and (3.3), we have

\[
(f, K_s(., y))_{\lambda, H^s(\mu_k)} = \int_{\mathbb{R}^d} \mathcal{E}_k(iy, z)\mathcal{F}_k(f)(z)d\mu_k(z),
\]

and from Remark 3.1, we obtain the reproducing property:

\[
(f, K_s(., y))_{\lambda, H^s(\mu_k)} = f(y).
\]

This completes the proof of the theorem. \( \square \)

The main result of this subsection can then be stated as follows.

**Theorem 3.3.** Let \( s > \gamma + d/2 \) and \( g \in L^2_{\mu_k} \). For any \( h \in L^2(\mu_k) \) and for any \( \lambda > 0 \), there exists a unique function \( f^*_{\lambda, g} \) where the infimum

\[
\inf_{f \in H^s(\mu_k)} \left\{ \lambda\|f\|_{H^s(\mu_k)}^2 + \|h - V_g(f)\|_{L^2(\mu_k)}^2 \right\}
\]

is attained. Moreover, the extremal function \( f^*_{\lambda, h} \) is given by

\[
f^*_{\lambda, h}(y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x, t)Q_s(x, y, t)d\mu_k(t)d\mu_k(x),
\]
where
\[ Q_s(x, y, t) = \int_{\mathbb{R}^d} \frac{E_k(ix, z)E_k(iy, z)}{\lambda(1 + |z|^2)^s} \|g\|_{L^2_{rad}(\mu_k)}^2 \, d\mu_k(z). \]

**Proof.** The existence and unicity of the extremal function \( f^*_{\lambda, h} \) satisfying (3.4) is given by Kimeldorf and Wahba [5], Matsuura et al. [6] and Saitoh [12]. Especially, \( f^*_{\lambda, h} \) is given by the reproducing kernel of \( H^s(\mu_k) \) with \( \|\cdot\|_{H^s(\mu_k)} \) norm as
\[ f^*_{\lambda, h}(y) = \langle h, V_g(K_s(\cdot, y)) \rangle_{L^2(\mu_k \otimes \mu_k)}, \tag{3.5} \]
where \( K_s \) is the kernel given by (3.2).

But by Proposition 2.4 (ii) and (3.3), we have
\[ V_g(K_s(\cdot, y))(x, t) = \int_{\mathbb{R}^d} E_k(ix, z)F_k(K_s(\cdot, y))(z) \sqrt{\tau_1[F_k(g)]^2(z)} \, d\mu_k(z) \]
\[ = \int_{\mathbb{R}^d} E_k(ix, z)E_k(-iy, z) \sqrt{\tau_1[F_k(g)]^2(z)} \, d\mu_k(z). \]
This clearly yields the result. \( \square \)

**Theorem 3.4.** Let \( s > \gamma + d/2 \) and \( g \in L^2_{rad}(\mu_k) \). For any \( h \in L^2(\mu_k \otimes \mu_k) \) and for any \( \lambda > 0 \), we have
(i) \( |f^*_{\lambda, h}(y)| \leq \frac{\|h\|_{L^2(\mu_k \otimes \mu_k)}}{2\sqrt{\lambda}} \left( \int_{\mathbb{R}^d} \frac{d\mu_k(z)}{(1 + |z|^2)^{s/2}} \right)^{1/2} \)
(ii) \( \|f^*_{\lambda, h}\|_{L^2_{rad}(\mu_k)}^2 \leq \frac{1}{4\lambda} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |h(x, t)|^2 e^{(|x|^2 + |t|^2)/2} d\mu_k(t) d\mu_k(x). \)

**Proof.** (i) From (3.5) and Theorem 2.5 (i), we have
\[ |f^*_{\lambda, h}(y)| \leq \|h\|_{L^2(\mu_k \otimes \mu_k)} \|V_g(K_s(\cdot, y))\|_{L^2(\mu_k \otimes \mu_k)} \]
\[ \leq \|h\|_{L^2(\mu_k \otimes \mu_k)} \|g\|_{L^2_{rad}(\mu_k)} \|K_s(\cdot, y)\|_{L^2(\mu_k)}. \]
Then, by Theorem 2.1 (iii) and (3.3), we deduce that
\[ |f^*_{\lambda, h}(y)| \leq \|h\|_{L^2(\mu_k \otimes \mu_k)} \|g\|_{L^2_{rad}(\mu_k)} \|F_k(K_s(\cdot, y))\|_{L^2(\mu_k)} \]
\[ \leq \|h\|_{L^2(\mu_k \otimes \mu_k)} \|g\|_{L^2_{rad}(\mu_k)} \left( \int_{\mathbb{R}^d} \frac{d\mu_k(z)}{\lambda(1 + |z|^2)^s + \|g\|_{L^2_{rad}(\mu_k)}^2} \right)^{1/2}. \]
Using the fact that \( \lambda(1 + |z|^2)^s + \|g\|_{L^2_{rad}(\mu_k)}^2 \geq 4\lambda(1 + |z|^2)^s + \|g\|_{L^2_{rad}(\mu_k)}^2 \), we obtain the result.

(ii) We write
\[ f^*_{\lambda, h}(y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-(|x|^2 + |t|^2)/4} e^{(|x|^2 + |t|^2)/4} h(x, t) Q_s(x, y, t) d\mu_k(t) d\mu_k(x). \]
Applying Hölder’s inequality, we obtain
\[ |f^*_{\lambda, h}(y)|^2 \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |h(x, t)|^2 e^{(|x|^2 + |t|^2)/2} |Q_s(x, y, t)|^2 d\mu_k(t) d\mu_k(x). \]
Thus and from Fubini-Tonnelli’s theorem, we get

\[ \|f^*_\lambda,h\|_{L^2(\mu_\lambda)}^2 \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |h(x, t)|^2 e^{\left(\|x\|^2 + |t|^2\right)/2} \|Q_s(x, t)\|^2_{L^2(\mu_\lambda)} d\mu_\lambda(t) d\mu_\lambda(x). \]

The function \( z \to \frac{E_k(-ix, z) \sqrt{\tau_1(\mathcal{F}_k(g)^2(z))}}{\lambda(1 + |z|^2)^s + \|g\|_{L^2_{\text{rad}}(\mu_\lambda)}^2} \) belongs to \( L^1 \cap L^2(\mu_\lambda) \), then by Theorem 2.1 (ii), we get

\[ Q_s(x, y, t) = \mathcal{F}_k^{-1}\left( \frac{E_k(-ix, z) \sqrt{\tau_1(\mathcal{F}_k(g)^2(z))}}{\lambda(1 + |z|^2)^s + \|g\|_{L^2_{\text{rad}}(\mu_\lambda)}^2} \right)(y). \]

Thus, by Theorem 2.1 (iii) we deduce that

\[ \|Q_s(x, t)\|_{L^2(\mu_\lambda)} = \int_{\mathbb{R}^d} |\mathcal{F}_k(Q_s(x, t))(z)|^2 d\mu_\lambda(z) \leq \int_{\mathbb{R}^d} \frac{\tau_1(\mathcal{F}_k(g)^2(z)) d\mu_\lambda(z)}{\lambda(1 + |z|^2)^s + \|g\|_{L^2_{\text{rad}}(\mu_\lambda)}^2}. \]

Then

\[ \|Q(x, t)\|_{L^2(\mu_\lambda)} \leq \frac{1}{4\lambda \|g\|_{L^2_{\text{rad}}(\mu_\lambda)}^2} \int_{\mathbb{R}^d} \tau_1(\mathcal{F}_k(g)^2(z)) d\mu_\lambda(z) \leq \frac{1}{4\lambda}. \]

From this inequality we deduce the result. \( \square \)

**Theorem 3.5.** Let \( s > \gamma + d/2 \) and \( g \in L^2_{\text{rad}}(\mu_\lambda) \). For any \( h \in L^2(\mu_\lambda \otimes \mu_\lambda) \) and for any \( \lambda > 0 \), we have

(i) \( f^*_\lambda,h(y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{E_k(iy, z) \sqrt{\tau_1(\mathcal{F}_k(g)^2(z))} \mathcal{F}_k(h(x, t))(z)}{\lambda(1 + |z|^2)^s + \|g\|_{L^2_{\text{rad}}(\mu_\lambda)}^2} d\mu_\lambda(t) d\mu_\lambda(z). \)

(ii) \( \mathcal{F}_k(f^*_\lambda,h)(z) = \int_{\mathbb{R}^d} \frac{\sqrt{\tau_1(\mathcal{F}_k(g)^2(z))} \mathcal{F}_k(h(x, t))(z)}{\lambda(1 + |z|^2)^s + \|g\|_{L^2_{\text{rad}}(\mu_\lambda)}^2} d\mu_\lambda(t). \)

(iii) \( \|f^*_\lambda,h\|_{H^s(\mu_\lambda)} \leq \frac{1}{2\sqrt{\lambda}} \|h\|_{L^2(\mu_\lambda \otimes \mu_\lambda)}. \)

**Proof.** (i) From Theorem 3.3 and Fubini’s theorem, we have

\[ f^*_\lambda,h(y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{E_k(iy, z) \sqrt{\tau_1(\mathcal{F}_k(g)^2(z))}}{\lambda(1 + |z|^2)^s + \|g\|_{L^2_{\text{rad}}(\mu_\lambda)}^2} \left( \int_{\mathbb{R}^d} h(x, t) E_k(-ix, z) d\mu_\lambda(x) \right) d\mu_\lambda(t) d\mu_\lambda(z). \]

(ii) The function \( z \to \int_{\mathbb{R}^d} \frac{\sqrt{\tau_1(\mathcal{F}_k(g)^2(z))} \mathcal{F}_k(h(x, t))(z)}{\lambda(1 + |z|^2)^s + \|g\|_{L^2_{\text{rad}}(\mu_\lambda)}^2} d\mu_\lambda(t) \) belongs to \( L^1 \cap L^2(\mu_\lambda) \). Then

\[ \|f^*_\lambda,h\|_{H^s(\mu_\lambda)} \leq \frac{1}{2\sqrt{\lambda}} \|h\|_{L^2(\mu_\lambda \otimes \mu_\lambda)}. \]
by Theorem 2.1 (ii) and (iii), it follows that $f_{\lambda,h}^*$ belongs to $L^2(\mu_k)$, and

$$\mathcal{F}_{k}(f_{\lambda,h}^*)(z) = \frac{\int_{\mathbb{R}^d} \frac{1}{\mu_k} \frac{\tau_t[f_{\lambda,h}^*(g)]^2(z)\mathcal{F}_{k}(h(.,t))(z)\mathcal{F}_{k}(h(.,t))(z)}{\lambda(1+|z|^2)^s+\|g\|_{L^2_{\text{rad}}(\mu_k)}^2} \, \mu_k(t)}{\lambda(1+|z|^2)^s+\|g\|_{L^2_{\text{rad}}(\mu_k)}^2}.$$

(iii) From (ii), Hölder’s inequality and (2.6) we have

$$|\mathcal{F}_{k}(f_{\lambda,h}^*)(z)|^2 \leq \frac{\|g\|_{L^2_{\text{rad}}(\mu_k)}^2}{\lambda(1+|z|^2)^s+\|g\|_{L^2_{\text{rad}}(\mu_k)}^2} \int_{\mathbb{R}^d} |\mathcal{F}_{k}(h(.,t))(z)|^2 \, \mu_k(t).$$

Thus,

$$\|f_{\lambda,h}^*\|^2_{H^s(\mu_k)} \leq \int_{\mathbb{R}^d} \frac{(1+|z|^2)^s\|g\|_{L^2_{\text{rad}}(\mu_k)}^2}{\lambda(1+|z|^2)^s+\|g\|_{L^2_{\text{rad}}(\mu_k)}^2} \left[\int_{\mathbb{R}^d} |\mathcal{F}_{k}(h(.,t))(z)|^2 \, \mu_k(t)\right] \, \mu_k(z) \leq \frac{1}{4\lambda} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \left|\mathcal{F}_{k}(h(.,t))(z)\right|^2 \, \mu_k(t)\right] \, \mu_k(z) = \frac{1}{4\lambda} \|h\|_{L^2(\mu_k \otimes \mu_k)}^2,$$

which ends the proof. □

**Theorem 3.6.** Let $s > \gamma + d/2$ and $g \in L^2_{\text{rad}}(\mu_k)$. For any $h \in L^2(\mu_k \otimes \mu_k)$ and for any $\lambda > 0$, we have

$$V_g(f_{\lambda,h}^*)(x,y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{E_k(ix,z)\sqrt{\tau_t[f_{\lambda,h}^*(g)]^2(z)\tau_y[f_{\lambda,h}^*(g)]^2(z)}\mathcal{F}(h(.,t))(z)}{\lambda(1+|z|^2)^s+\|g\|_{L^2_{\text{rad}}(\mu_k)}^2} \, \mu_k(t) \, \mu_k(z).$$

**Proof.** From Proposition 2.4 (ii), we have

$$V_g(f_{\lambda,h}^*)(x,y) = \int_{\mathbb{R}^d} \frac{E_k(ix,z)\mathcal{F}_{k}(f_{\lambda,h}^*)(z)}{\lambda(1+|z|^2)^s+\|g\|_{L^2_{\text{rad}}(\mu_k)}^2} \, \mu_k(z).$$

Then by Theorem 3.5 (ii), we obtain the result. □

**References**


