Calderón’s reproducing Formula For q-Bessel operator

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ABSTRACT

In this paper a Calderón-type reproducing formula for q-Bessel convolution is established using the theory of q-Bessel Fourier transform \[13, 17\], obtained in Quantum calculus.

RESUMEN

En este trabajo se prueba una fórmula de tipo Calderón para convolución q-Bessel, usando la teoría de q-Bessel transformada de Fourier \[13, 17\], obtenida en cálculo cuántico.

Keywords and Phrases: q-Calderon, q-Calculus, q-Bessel Convolution, q-Fourier Bessel transform, q-Measure.

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1 Introduction

Calderón’s formula [1] involving convolutions related to the Fourier transform is useful in obtaining reconstruction formula for wavelet transform, in decomposition of certain spaces and in characterization of Besov spaces [6, 8, 10]. Calderón’s reproducing formula was also established for Bessel operator [4, 5]. This work is a continuation of a last work [9], and we establish formula for $q$-Bessel convolution for both functions and measures which generalize the above one.

In the classical case this formula is expressed for a suitable function $f$ as follows:

$$f(x) = \int_0^\infty (g_t * h_t * f)(x) \frac{dt}{t},$$

(1)

where $g, h \in L^2(\mathbb{R})$ and $g_t(x) = \frac{1}{t} g(\frac{x}{t})$, $h_t(x) = \frac{1}{t} h(\frac{x}{t})$, $t > 0$ satisfying

$$\int_0^\infty \hat{g}(xt) \hat{h}(xt) \frac{dx}{x} = 1, \text{ for all } t \in \mathbb{R} \setminus \{0\},$$

where $\hat{g}$ and $\hat{h}$ is the usual Fourier transform of $g$ and $h$ on $\mathbb{R}$.

If $\mu$ is a finite Borel measure on the real line $\mathbb{R}$, identity (1) has natural generalization as follow

$$f(x) = \int_0^\infty (f * \mu_t)(x) \frac{dt}{t},$$

(2)

where $\mu_t$ is the dilated measure of $\mu$ under some restriction on $\mu$, the $L^p$-norm of (2) has proved in [2]. A general form of (2) has been investigated in [3].

In this paper we study similar questions when in (1) and (2) the classical convolution $*$ is replaced by the $q$-Bessel convolution $*_{\alpha,q}$ on the half line generated by the $q$-Bessel operator defined by

$$\Delta_{q,\alpha} f(x) = \frac{1}{x^{2\alpha+1}} D_q \left[ x^{2\alpha+1} D_q f \right] (q^{-1} x).$$

(3)

In this paper we prove that, for $\varphi$ and $\psi \in L^1_{\alpha,q}(\mathbb{R}_q,+; d_q\sigma(x))$ satisfying

$$\int_0^\infty \mathcal{F}_{\alpha,q}(\varphi)(\xi) \mathcal{F}_{\alpha,q}(\psi)(\xi) \frac{d_q \xi}{\xi} = 1$$

(4)

we have

$$f(x) = \int_0^\infty (f *_{\alpha,q} \varphi_t *_{\alpha,q} \psi_t)(x) \frac{d_q t}{t}, \quad f \in L^1_{\alpha,q}(\mathbb{R}_q,+; d_q\sigma(x)).$$

(5)

where $d_q\sigma(x) = \frac{(1 + q)^{-\alpha}}{t^{\alpha + 1}} x^{2\alpha+1} d_q x = b_{\alpha,q} x^{2\alpha+1} d_q x$, $\varphi_t(x) = \frac{1}{t^{2\alpha+2}} \varphi\left(\frac{x}{t}\right)$.

In particular for $\varphi \in L^1_{\alpha,q}(\mathbb{R}_q,+; d_q\sigma(x))$ such that

$$\int_0^\infty |\mathcal{F}_{\alpha,q}(\varphi)(\xi)|^2 \frac{d_q \xi}{\xi} = 1,$$

(6)
and for a suitable function $f$, put
\[ f^{\varepsilon,\delta}(x) = \int_{\varepsilon}^{\delta} (f \ast_{\alpha,q} \varphi_1 \ast_{\alpha,q} \varphi_1)(x) \frac{d_q t}{t} \] (7)
then
\[ \| f^{\varepsilon,\delta} - f \|_{2,\alpha,q} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \text{ and } \delta \rightarrow \infty. \] (8)
In the case $f \in L^1_{\alpha,q}(\mathbb{R}_q,+\,d_q \sigma(x))$ such that $F_{\alpha,q}f \in L^1_{\alpha,q}(\mathbb{R}_q,+\,d_q \sigma(x))$ one has
\[ \lim_{\varepsilon \rightarrow 0 \atop \delta \rightarrow \infty} f^{\varepsilon,\delta}(x) = f(x), \quad x \in \mathbb{R}. \] (9)

Then we prove that for $\mu \in \mathcal{M}'(\mathbb{R}_q,+)$, such that the $q$-integral
\[ c_{\mu,\alpha,q} = \int_0^{\infty} F_{\alpha,q}(\mu)(\lambda) \frac{d_q \lambda}{\lambda} \] (10)
is finite. Then for all $f \in L^2_{\alpha,q}(\mathbb{R}_q,+\,d_q \sigma(x))$, we have
\[ \lim_{\varepsilon \rightarrow 0 \atop \delta \rightarrow \infty} f^{\varepsilon,\delta} = c_{\mu,\alpha,q} f. \] (11)
where the limit is in $L^2_{\alpha,q}(\mathbb{R}_q,+\,d_q \sigma(x))$. And if $\mu \in \mathcal{M}'(\mathbb{R}_q,+)$ is such that the $q$-integral
\[ \int_0^{\infty} |\mu([0,y])| \frac{d_q y}{y} \] (12)
is finite, for all $f \in L^2_{\alpha,q}(\mathbb{R}_q,+\,d_q \sigma(x))$
\[ \lim_{\varepsilon \rightarrow 0 \atop \delta \rightarrow \infty} f^{\varepsilon,\delta} = c_{\mu,\alpha,q} f, \quad \text{in } L^2_{\alpha,q}(\mathbb{R}_q,+\,d_q \sigma(x)). \] (13)

The outline of this paper is as follows: In Section 2, basic properties of $q$-Bessel transform on $\mathbb{R}_q$ of functions and bounded measure and its underlying $q$-convolution structure are called and introduced here. In Section 3, we give the first main result of the paper, the $q$-Calderon’s reproducing formula for functions. Section 4 is consecrate to establish the same result as in section 3 for finite measures.

2 Preliminaries

In this section we recall some basic result in harmonic analysis related to the $q$-Bessel Fourier transform. Standard reference here is Gasper & Rahman [7].
For \(a, q \in \mathbb{C}\) the \(q\)-shifted factorial \((a; q)_k\) is defined as a product of \(k\) factors:

\[
(a; q)_k = (1 - a)(1 - aq)...(1 - aq^{k-1}), \quad k \in \mathbb{N}^*; \quad (a; q)_0 = 1.
\]

If \(|q| < 1\) this definition remains meaningful for \(k = +\infty\) as a convergent infinite product:

\[
(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).
\]

We also write \((a_1, \ldots, a_r; q)_k\) for the product of \(r\) \(q\)-shifted factorials:

\[
(a_1, \cdots, a_r; q)_k = (a_1; q)_k \cdots (a_r; q)_k \quad (k \in \mathbb{N} \text{ or } k = \infty).
\]

A \(q\)-hypergeometric series is a power series (for the moment still formal) in one complex variable \(z\) with power series coefficients which depend, apart from \(q\), on \(r\) complex upper parameters \(a_1, \ldots, a_r\) and \(s\) complex lower parameters \(b_1, \ldots, b_s\) as follows:

\[
r \varphi_s(a_1, \cdots, a_r; b_1, \cdots, b_s; q, x) = \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_k}{(b_1, \cdots, b_s; q)_k} \left(\frac{-1}{q}\right)^{k+1} \frac{-1}{q} q^{k(k-1)/2} \frac{z^k}{k+1-s} \quad \text{for } r, s \in \mathbb{N}.
\]

2.1 \(q\)-Exponential series

\[
e_q(z) = \varphi_0(0; -; q, z) = \sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k} = \frac{1}{(z; q)_\infty} \quad (|z| < 1)
\]

\[
E_q(z) = \varphi_0(-; -; q, -z) = \sum_{k=0}^{\infty} \frac{q^{1/2} q^{k(k-1)/2} z^k}{(q; q)_k} = (-z; q)_\infty \quad (z \in \mathbb{C}).
\]

2.2 \(q\)-Derivative and \(q\)-Integral

The \(q\)-derivative of a function \(f\) given on a subset of \(\mathbb{R}\) or \(\mathbb{C}\) is defined by:

\[
D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x} \quad (x \neq 0, q \neq 0),
\]

where \(x\) and \(qx\) should be in the domain of \(f\). By continuity we set \((D_q f)(0) = f'(0)\) provided \(f'(0)\) exists.

The \(q\)-shift operators are:

\[
(\Lambda_q f)(x) = f(qx), \quad (\Lambda_q^{-1} f)(x) = f(q^{-1}x).
\]
For $a \in \mathbb{R} \setminus \{0\}$ and a function $f$ given on $(0, a]$ or $[a, 0)$, we define the $q$-integral by

$$\int_0^a f(x) d_q x = (1 - q)a \sum_{n=0}^{\infty} f(a q^n) q^n,$$  \hspace{1cm} (21)

provided the infinite sum converges absolutely (for instance if $f$ is bounded). If $F(a)$ is given by the left-hand side of (21) then $D_q F = f$. The right-hand side of (21) is an infinite Riemann sum.

For a $q$-integral over $(0, \infty)$ we define

$$\int_0^\infty f(x) d_q x = (1 - q) \sum_{n=\infty}^{\infty} f(q^n) q^n.$$  \hspace{1cm} (22)

Note that for $n \in \mathbb{Z}$ and $a \in \mathbb{R}_q$, we have

$$\int_0^\infty f(q^n x) d_q x = \frac{1}{q^n} \int_0^\infty f(x) d_q x,$$

$$\int_0^a f(q^n x) d_q x = \frac{1}{q^n} \int_0^{a q^n} f(x) d_q x.$$  \hspace{1cm} (23)

The $q$-integration by parts is given for suitable functions $f$ and $g$ by:

$$\int_a^b f(x) D_q g(x) d_q x = \left[ f(x) g(x) \right]_a^b - \int_a^b D_q f(x) g(x) d_q x.$$  \hspace{1cm} (24)

The $q$-Logarithm $\log_q$ is given by \[19\]

$$\log_q x = \int_0^{d_q x} x = \frac{1 - q}{\log q} \log x.$$  \hspace{1cm} (25)

For all $a, b \in \mathbb{q^2}$, $a < b$

$$\log_q (b/a) = (1 - q) \sum_{k:a \leq q^k \leq b} 1.$$  \hspace{1cm} (26)

The improper integral is defined in the following way

$$\int_0^{\infty/A} f(x) d_q x = (1 - q) \sum_{n=\infty}^{+\infty} f \left( \frac{q^n}{A} \right) q^n.$$  \hspace{1cm} (27)

We remark that for $n \in \mathbb{Z}$, we have

$$\int_0^{\infty/q^n} f(x) d_q x = \int_0^{\infty} f(x) d_q x.$$  \hspace{1cm} (28)

The following property holds for suitable function $f$

$$\int_0^\infty \int_0^\infty f(x, y) d_q y d_q x = \int_0^\infty \int_0^\infty f(x, y) d_q x d_q y.$$  \hspace{1cm} (29)
2.3 The $q$-gamma function

The $q$-gamma function is defined by [7, 16]
\[
\Gamma_q(z) = \frac{[q; q]_\infty}{(q^z; q)_\infty} (1 - q)^{1-z}, \quad 0 < q < 1; z \neq 0, -1, -2, \ldots
\]
(30)

\[
= \int_0^{(1-q)^{-1}} t^{z-1} E_q(-(1-q)qt) dq t, \quad (\Re z > 0)
\]
(31)

moreover the $q$-duplication formula holds
\[
\Gamma_q(2z)\Gamma_q^2\left(\frac{1}{2}\right) = (1 + q)^{2z-1}\Gamma_q^2(z)\Gamma_q^2\left(z + \frac{1}{2}\right).
\]
(32)

2.4 Some $q$-functional spaces

We begin by putting
\[
\mathbb{R}_{q,+} = \{ +q^k, k \in \mathbb{Z}\}, \quad \tilde{\mathbb{R}}_{q,+} = \{ +q^k, k \in \mathbb{Z}\} \cup \{0\}
\]
(33)

and we denote by

- $L_{p,q}^p(\mathbb{R}_{q,+})$, $p \in [1, +\infty[$, (resp. $L_{\infty,q}^\infty(\mathbb{R}_{q,+})$) the space of functions $f$ such that,
\[
\|f\|_{p,q} = \left(\int_0^\infty |f(x)|^p \, dq \, \sigma(x)\right)^{\frac{1}{p}} < +\infty.
\]
(34)

(resp. $\|f\|_{\infty,q} = \text{ess sup}_{x \in \mathbb{R}_{q,+}} |f(x)| < +\infty$).
(35)

- $S_{q,*}(\mathbb{R}_q)$ the $q$-analogue of Schwartz space of even functions defined on $\mathbb{R}_q$ such that $D_{q,x}^k f(x)$ is continuous in 0 for all $k \in \mathbb{N}$ and
\[
N_{q,n,k}(f) = \sup_{x \in \mathbb{R}_q} |(1 + x^2)^n D_{q,x}^k f(x)| < +\infty.
\]
(36)

- The $q$-analogue of the tempered distributions is introduced in [12] as follow:

(i) A $q$-distribution $T$ in $\mathbb{R}_q$ is said to be tempered if there exists $C_q > 0$ and $k \in \mathbb{N}$ such that:
\[
|\langle T, f \rangle| \leq C_q N_{q,n,k}(f); \quad f \in S_{q,*}(\mathbb{R}_q).
\]
(37)

(ii) A linear form $T$: $S_{q,*}(\mathbb{R}_q) \longrightarrow \mathbb{C}$ is said continuous if there exist $C_q > 0$ and $k \in \mathbb{N}$ such that:
\[
|\langle T, f \rangle| \leq C_q N_{q,n,k}(f); \quad f \in S_{q,*}(\mathbb{R}_q).
\]
(38)
• $S_{q,*}^\prime(\mathbb{R}_q)$ the space of even $q$-tempered distributions in $\mathbb{R}_q$. That is the topological dual of $S_{q,*}(\mathbb{R}_q)$.

• $D_{q,*}(\mathbb{R}_q)$ the space of even functions infinitely $q$-differentiable on $\mathbb{R}_q$ with compact support in $\mathbb{R}_q$. We equip this space with the topology of the uniform convergence of the functions and their $q$-derivatives.

• $C_{q,*}^{0,0}(\mathbb{R}_q)$ the space of even functions $f$ defined on $\mathbb{R}_q$ continuous on $0$, infinitely $q$-differentiable and such that $\|f\|_{C_{q,*}^{0,0}} = \sup_{x \in \mathbb{R}_q} |f(x)| < +\infty.$

• $H_{q,*}(\mathbb{R}_q)$ the space of even functions $f$ defined on $\mathbb{R}_q$ continuous on $0$ with compact support such that $\|f\|_{H_{q,*}} = \sup_{x \in \mathbb{R}_q} |f(x)| < +\infty.$

2.5 $q$-Bessel function

The following properties of the normalized $q$-Bessel function is given (see [13]) by

$$j_\alpha(x; q^2) = \Gamma_{q^2}(\alpha + 1) \sum_{k=0}^{\infty} \frac{(-1)^k q^k(k-1)}{q^2(k+1)\Gamma_{q^2}(\alpha + k + 1)} \left(\frac{x}{1 + q}\right)^{2k}.$$  (41)

This function is bounded and for every $x \in \mathbb{R}_q$ and $\alpha > -\frac{1}{2}$ we have

$$|j_\alpha(x; q^2)| \leq \frac{1}{(|q|; q^2)_\infty},$$  (42)

$$\left(\frac{1}{x}D_q\right)j_\alpha(\cdot; q^2)(x) = -\frac{(1-q)}{1 - q^{2\alpha+2}} j_{\alpha+1}(qx; q^2),$$  (43)

$$\left(\frac{1}{x}D_q\right)(x^{2\alpha}j_\alpha(x; q^2)) = \frac{1-q^{2\alpha}}{1-q} x^{(\alpha-1)} j_{\alpha-1}(x; q^2),$$  (44)

$$|D_qj_\alpha(x; q^2)| \leq \frac{(1-q)}{(1-q^{2\alpha+2}) (q; q^2)_\infty^2} \frac{x}{x^{2\alpha+2}}.$$  (45)

We remark that for $\lambda \in \mathbb{C}$, the function $j_\alpha(\lambda x; q^2)$ is the unique solution of the $q$-differential system

$$\begin{cases}
\Delta_{q,\alpha} U(x, q) = -\lambda^2 U(x, q), \\
U(0, q) = 1; \ D_{q,x} U(x, q)|_{x=0} = 0,
\end{cases}$$  (46)
where $\Delta_{q,\alpha}$ is the q-Bessel operator defined by

\[
\Delta_{q,\alpha} f(x) = \frac{1}{x^{2\alpha+1}} D_q \left[ x^{2\alpha+1} D_q f \right] (q^{-1}x)
\]

\[
= q^{2\alpha+1} \Delta_q f(x) + \frac{1-q^{2\alpha+1}}{(1-q)q^{-1}} D_q f(q^{-1}x),
\]

where

\[
\Delta_q f(x) = \Lambda^{-1} D_q^2 f(x) = (D_q^2 f)(q^{-1}x),
\]

and for $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}_{q,+}$,

\[
\Delta_{q,\alpha}^k j_\alpha(\lambda x; q^2) = (-1)^k \lambda^{2k} j_\alpha(\lambda x; q^2).
\]

### 2.6 q-Bessel Translation operator

$T_{\alpha,q}^x, x \in \mathbb{R}_{q,+}$ is the q-generalized translation operator associated with the q-Bessel transform is introduced in [13] and rectified in [17], where it is defined by the use of Jackson’s q-integral and the q-shifted factorial as

\[
T_{\alpha,q}^x f(y) = \int_0^{+\infty} f(t) D_{\alpha,q}(x, y, t)t^{2\alpha+1}d_q t, \quad \alpha > -1
\]

with

\[
D_{\alpha,q}(x, y, z) = c_{\alpha,q}^2 \int_0^{+\infty} j_\alpha(x t; q^2) j_\alpha(y t; q^2) j_\alpha(z t; q^2)t^{2\alpha+1}d_q t
\]

where

\[
c_{\alpha,q} = \frac{1}{1-q} \left( \frac{q^{2\alpha+2}; q^2}{q^2; q^2} \right)_\infty.
\]

In particular the following product formula holds

\[
T_{\alpha,q}^x j_\alpha(y, q^2) = j_\alpha(x, q^2) j_\alpha(y, q^2).
\]

It is shown in [18] that for $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, $T_{\alpha,q}^x f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ and

\[
\|T_{\alpha,q}^x f\|_{1,\alpha,q} = \|f\|_{1,\alpha,q}.
\]

### 2.7 The q-convolution and the q-Bessel Fourier transform

The q-Bessel Fourier transform $F_{\alpha,q}$ and the q-Bessel convolution product are defined for suitable functions $f, g$ as follows

\[
F_{\alpha,q}(f) (\lambda) = \int_0^{+\infty} f(x) j_\alpha(\lambda x; q^2) d_q \sigma(x),
\]
The \( q \)-Bessel Fourier transform \( \mathcal{F}_{\alpha,q} \) is a modified version of the \( q \)-analogue of the Hankel transform defined in [13].

It is shown in [13, 17, 14], that the \( q \)-Bessel Fourier transform \( \mathcal{F}_{\alpha,q} \) satisfies the following properties:

**Proposition 2.1.** If \( f \in L^1_{\alpha,q}(\mathbb{R}_+,+) \), then \( \mathcal{F}_{\alpha,q}(f) \in C_{q,*}\,0(\mathbb{R}_+,+) \) and

\[ \| \mathcal{F}_{\alpha,q}(f) \|_{C_{q,*}\,0} \leq B_{\alpha,q} \| f \|_{1,\alpha,q}, \]

where

\[ B_{\alpha,q} = \frac{1}{1-q} \frac{(-q^2; q^2)_\infty(-q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty}. \]

**Proposition 2.2.** Given two functions \( f, g \in L^1_{\alpha,q}(\mathbb{R}_+,+) \), then

\[ f \ast_{\alpha,q} g \in L^1_{\alpha,q}(\mathbb{R}_+,+) \]

and

\[ \mathcal{F}_{\alpha,q}(f \ast_{\alpha,q} g) = \mathcal{F}_{\alpha,q}(f) \mathcal{F}_{\alpha,q}(g). \]

**Theorem 2.3.** \((Inversion formula)\)

1. If \( f \in L^1_{\alpha,q}(\mathbb{R}_+,+) \) such that \( \mathcal{F}_{\alpha,q}(f) \in L^1_{\alpha,q}(\mathbb{R}_+,+) \), then for all \( x \in \mathbb{R}_+,+ \), we have

\[ f(x) = \int_0^\infty \mathcal{F}_{\alpha,q}(f)(y)j_\alpha(xy; q^2)d_q\sigma(y). \]

2. \( \mathcal{F}_{\alpha,q}(f) \) is an isomorphism of \( S_{*,q}(\mathbb{R}_q) \) and \( \mathcal{F}_{\alpha,q}^{-1}(f) = Id. \)

• Note that the inversion formula is valid for \( f \in L^1_{\alpha,q}(\mathbb{R}_+,+) \) without the additional condition \( \mathcal{F}_{\alpha,q}(f) \in L^1_{\alpha,q}(\mathbb{R}_+,+) \).

\( \mathcal{F}_{\alpha,q}(f) \) can be extended to \( L^2_{\alpha,q}(\mathbb{R}_+,+) \) and we have the following theorem:

**Theorem 2.4.** \((q\text{-}Plancherel theorem)\)

\( \mathcal{F}_{\alpha,q}(f) \) is an isomorphism of \( L^2_{\alpha,q}(\mathbb{R}_+,+) \), we have \( \| \mathcal{F}_{\alpha,q}(f) \|_{2,\alpha,q} = \| f \|_{2,\alpha,q} \) for \( f \in L^2_{\alpha,q}(\mathbb{R}_+,+) \) and \( \mathcal{F}_{\alpha,q}^{-1}(f) = \mathcal{F}_{\alpha,q}(f) \).

**Proposition 2.5.**

(i) For \( f \in L^p_{\alpha,q}(\mathbb{R}_+,+) \), \( p \in [1,\infty] \), \( g \in L^1_{\alpha,q}(\mathbb{R}_+,+) \) we have \( f \ast_{\alpha,q} g \in L^p_{\alpha,q}(\mathbb{R}_+,+) \) and \( \| f \ast_{\alpha,q} g \|_{p,\alpha,q} \leq \| f \|_{p,\alpha,q} \| g \|_{1,\alpha,q}. \)

(ii) \( \int_0^\infty \mathcal{F}_{\alpha,q}(f)(\xi)g(\xi)d_q\sigma(\xi) = \int_0^\infty f(\xi)\mathcal{F}_{\alpha,q}(g)(\xi)d_q\sigma(\xi) \); \( f, g \in L^1_{\alpha,q}(\mathbb{R}_+,+) \).
Proposition 2.8. \( F_{\alpha,q}(T_{q,x}^\alpha f)(\xi) = j_\alpha(\xi x; q^2)F_{\alpha,q}(f)(\xi); \ f \in L^1_{\alpha,q}(\mathbb{R}_q^+). \)

Specially, we choose \( q \in [0, q_0] \) where \( q_0 \) is the first zero of the function \[ q \mapsto \frac{\log(1-q)}{\log q} \in \mathbb{Z}. \]

Definition 2.6. \[ [11, 9] \] A bounded complex even measure \( \mu \) on \( \mathbb{R}_q \) is a bounded linear functional \( \mu \) on \( \mathcal{H}_{q^*}(\mathbb{R}_q) \), i.e., for all \( f \in \mathcal{H}_{q^*}(\mathbb{R}_q) \), we have
\[
|\mu(f)| \leq C\|f\|_{\mathcal{H}_{q^*}},
\]
where \( C > 0 \) is a positive constant.

Denote the space of all such measure by \( \mathcal{M}^\prime(\mathbb{R}_q^+) \).

Note that \( \mu \in \mathcal{M}^\prime(\mathbb{R}_q^+) \) can be identified with a function \( \tilde{\mu} \) on \( \mathbb{R}_q^+ \) such that \( \tilde{\mu} \) restricted to \( \mathbb{R}_q^+ \) is \( L^1_{\alpha,q}(\mathbb{R}_q^+) \):

\[
\mu(f) = \mu([0])f(0) + \int_0^\infty \tilde{\mu}(x)f(x)d\mu(x), \ (f \in \mathcal{H}_{q^*}(\mathbb{R}_q)).
\]
For \( \mu \in \mathcal{M}^\prime(\mathbb{R}_q^+) \) denote \( ||\mu|| = |\mu|_{\mathbb{R}_q^+} \) where \( |\mu| \) is the absolute value of \( \mu \).

Definition 2.7. The q-Bessel Fourier transform of a measure \( \mu \) in \( \mathcal{M}^\prime(\mathbb{R}_q^+) \) is defined for all \( \varphi \in \mathcal{S}_{q^*}(\mathbb{R}_q) \) by
\[
F_{\alpha,q}\mu(\lambda) = b_{\alpha,q} \int_0^{+\infty} \tilde{\mu}(x)j_\alpha(\lambda x; q^2)d\mu(x).
\]
The q-Bessel convolution product of a measure \( \mu \in \mathcal{M}^\prime(\mathbb{R}_q^+) \) and a suitable function \( f \) on \( \mathbb{R}_q^+ \) is defined by
\[
\mu *_{\alpha,q} f(x) = \int_0^\infty T_{q,x}^\alpha f(y)d\mu(y).
\]

Proposition 2.8. (1) The q-Bessel Fourier transform \( F_{\alpha,q} \) of a measure \( \mu \) in \( \mathcal{M}^\prime(\mathbb{R}_q^+) \) is the q-tempered distribution \( F_{\alpha,q}\mu \) given by:
\[
\langle F_{\alpha,q}\mu, \varphi \rangle = \langle \mu, F_{\alpha,q}\varphi \rangle = \int_0^{+\infty} F_{\alpha,q}\varphi(\lambda)d\mu(\lambda).
\]
(2) For all \( x, \lambda \in \mathbb{R}_q^+ \) we have
\[
T_{q,x}^\alpha F_{\alpha,q}\mu(\lambda) = b_{\alpha,q} \int_0^{+\infty} j_\alpha(\lambda x; q^2)j_\alpha(\lambda t; q^2)d\mu(t).
\]
(3) For all \( \mu \in \mathcal{M}^\prime(\mathbb{R}_q^+) \), \( F_{\alpha,q}\mu \) is continuous on \( \mathbb{R}_q^+ \), and
\[
\lim_{\lambda \to \infty} F_{\alpha,q}\mu(\lambda) = \mu([0]).
\]
\( F_{\alpha,q} \) maps one to one \( \mathcal{M}^\prime(\mathbb{R}_q^+) \) into \( \mathcal{C}_b(\mathbb{R}_q^+) \), (the space of continuous and bounded functions on \( \mathbb{R}_q^+ \)).
(4) If $\mu \in \mathcal{M}'(\mathbb{R}_q, +)$ and $f \in L^p_{\alpha, q}(\mathbb{R}_q, +)$, $p = 1, 2$ then $\mu *_{\alpha, q} f \in L^p_{\alpha, q}(\mathbb{R}_q, +)$ and
\[
\|\mu *_{\alpha, q} f\|_{p, \alpha, q} \leq \|\mu\| \|f\|_{p, \alpha, q}.
\] (58)

(5) For all $\mu \in \mathcal{M}'(\mathbb{R}_q, +)$ and $f \in L^p_{\alpha, q}(\mathbb{R}_q, +)$, $p = 1, 2$ we have

\[
\mathcal{F}_{\alpha, q}(\mu *_{\alpha, q} f) = \mathcal{F}_{\alpha, q}(\mu) \mathcal{F}_{\alpha, q}(f).
\] (59)

**Definition 2.9.** Let $\mu \in \mathcal{M}'(\mathbb{R}_q, +)$ and $a > 0$. We define the $q$-dilated measure $\mu_a$ of $\mu$ by
\[
\int_0^\infty \varphi(x) d_q \mu_a(x) = \int_0^\infty \varphi(ax) d_q \mu(x), \quad \varphi \in \mathcal{H}_{q,*}(\mathbb{R}_q).
\] (60)

**Proposition 2.10.** (i) When $\mu = f(x) x^{2\alpha + 1} d_q x$, with $f \in L^1_{\alpha, q}(\mathbb{R}_q, +)$, the measure $\mu_a$, $a > 0$, is given by the function
\[
f_a(x) = \frac{1}{a^{2\alpha + 2}} f\left(\frac{x}{a}\right), \quad x \geq 0.
\] (61)

(ii) Let $\mu \in \mathcal{M}'(\mathbb{R}_q, +)$, then
\[
\mathcal{F}_{\alpha, q}(\mu_a)(\lambda) = \mathcal{F}_{\alpha, q}(\mu)(a\lambda), \quad \text{for all } \lambda \geq 0.
\] (62)

(iii) For $\mu \in \mathcal{M}'(\mathbb{R}_q, +)$ and $f \in L^p_{\alpha, q}(\mathbb{R}_q, +), p = 1, 2$ we have
\[
\lim_{a \to 0} \mu_a *_{\alpha, q} f = \mu(\overline{\mathbb{R}}_q, +) f.
\] (63)

where the limit is in $L^p_{\alpha, q}(\mathbb{R}_q, +)$.

(iv) Let $g \in L^1_{\alpha, q}(\mathbb{R}_q, +)$ and $f \in L^p_{\alpha, q}(\mathbb{R}_q, +), 1 < p < \infty$. Then
\[
\lim_{a \to \infty} f *_{\alpha, q} g_a = 0
\] (64)

where the limit is in $L^p_{\alpha, q}(\mathbb{R}_q, +)$.

**Proof.** Statement of (i) and (ii) are obvious. A standard argument gives (iii). Let us verify (iv). If $f, g \in D_{q,*}(\mathbb{R}_q)$ then by (58) and (61) we have
\[
\|f *_{\alpha, q} g_a\|_{p, \alpha, q} \leq \|f\|_{1, \alpha, q} \|g_a\|_{p, \alpha, q} = a^{\frac{2(1 + \alpha)}{p} - 1} \|f\|_{1, \alpha, q} \|g\|_{p, \alpha, q} \to 0, \quad \text{as } a \to \infty.
\]

For arbitrary $g \in L^1_{\alpha, q}(\mathbb{R}_q, +)$ and $f \in L^p_{\alpha, q}(\mathbb{R}_q, +)$ the result follows by density. \qed

Given a measure $\mu \in \mathcal{M}'(\mathbb{R}_q, +)$. Denote
\[
c_{\mu, \alpha, q} = \int_0^\infty \mathcal{F}_{\alpha, q}(\mu)(\lambda) d_q \lambda / \lambda.
\] (65)
3 q-Calderón’s formula for functions

In this section, we establish the q-Calderón’s reproducing identity for functions using the properties of q-Fourier Bessel transform $F_{\alpha,q}$ and q-Bessel convolution $*_{\alpha,q}$.

**Theorem 3.1.** Let $\varphi$ and $\psi \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ be such that following admissibility condition holds
\[
\int_0^{\infty} F_{\alpha,q}(\varphi)(\xi) F_{\alpha,q}(\psi)(\xi) \frac{dq\xi}{\xi} = 1
\] (66)
then for all $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, the following Calderón’s reproducing identity holds:
\[
f(x) = \int_0^{\infty} (f *_{\alpha,q} \varphi_t *_{\alpha,q} \psi_t)(x) \frac{dq_t}{t}.
\] (67)

**Proof.** Taking q-Bessel Fourier transform of the right-hand side of (67), we get
\[
F_{\alpha,q}\left[ \int_0^{\infty} (f *_{\alpha,q} \varphi_t *_{\alpha,q} \psi_t)(x) \frac{dq_t}{t} \right] (\xi) = \int_0^{\infty} F_{\alpha,q}(f)(\xi) F_{\alpha,q}(\varphi_t)(\xi) F_{\alpha,q}(\psi_t)(\xi) \frac{dq_t}{t} = F_{\alpha,q}(f)(\xi).
\]
Now, by putting $t\xi = s$, we get
\[
\int_0^{\infty} F_{\alpha,q}(\varphi)(t\xi) F_{\alpha,q}(\psi)(t\xi) \frac{dq_t}{t} = \int_0^{\infty} F_{\alpha,q}(\varphi)(s) F_{\alpha,q}(\psi)(s) \frac{dq_s}{s} = 1.
\]
Hence, the result follows. \qed

The equality (67) can be interpreted in the following $L^2$-sense.

**Theorem 3.2.** Suppose $\varphi \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ and satisfies
\[
\int_0^{\infty} \left| F_{\alpha,q}(\varphi)(\xi) \right|^2 \frac{dq\xi}{\xi} = 1.
\] (68)

For $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+}) \cap L^2_{\alpha,q}(\mathbb{R}_{q,+})$, suppose that
\[
f^{\varepsilon,\delta}(x) = \int_{\varepsilon}^{\delta} (f *_{\alpha,q} \varphi_t *_{\alpha,q} \varphi_t)(x) \frac{dq_t}{t}
\] (69)
then
\[
\| f^{\varepsilon,\delta} - f \|_{2,\alpha,q} \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{and} \quad \delta \to \infty.
\] (70)
Proof. Taking q-Bessel Fourier transform of both sides of (69) and using Fubini’s theorem, we get

$$\mathcal{F}_{\alpha,q}(f^{\epsilon,\delta})(\xi) = \mathcal{F}_{\alpha,q}(f)(\xi) \int_{\epsilon}^{\delta} [\mathcal{F}_{\alpha,q}(\varphi)(t\xi)]^2 \frac{dq}{t}$$

by Proposition 2.5 we have

$$\|\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f\|_{2,\alpha,q} \leq \|\varphi_t *_{\alpha,q} \varphi_t \|_{1,\alpha,q} \|f\|_{2,\alpha,q} \leq \|\varphi_t\|_{1,\alpha,q} \|f\|_{2,\alpha,q}.$$ 

Now using above inequality, Minkowski’s inequality and relation (29), we get

$$\|f^{\epsilon,\delta}\|_{2,\alpha,q}^2 = \int_{\epsilon}^{\delta} \int_{0}^{\infty} [(\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f)(x)]^2 d_q t^2 d_q \sigma(x) \leq \int_{\epsilon}^{\delta} \|\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f\|_{2,\alpha,q} \frac{dq}{t} \leq \|\varphi_t\|_{1,\alpha,q} \|f\|_{2,\alpha,q} \int_{\epsilon}^{\delta} \frac{dq}{t} \leq \|\varphi_t\|_{1,\alpha,q} \|f\|_{2,\alpha,q} \log \left(\frac{\delta}{\epsilon}\right).$$

Hence, by Theorem 2.4, we get

$$\lim_{\epsilon \to 0} \lim_{\delta \to \infty} \|f^{\epsilon,\delta} - f\|_{2,\alpha,q}^2 = \lim_{\epsilon \to 0} \lim_{\delta \to \infty} \|\mathcal{F}_{\alpha,q}(f^{\epsilon,\delta}) - \mathcal{F}_{\alpha,q}(f)\|_{2,\alpha,q}^2 = \lim_{\epsilon \to 0} \int_{0}^{\infty} [\mathcal{F}_{\alpha,q}(f)](\xi) \left(1 - \int_{\epsilon}^{\delta} [\mathcal{F}_{\alpha,q}(\varphi)(t\xi)]^2 \frac{dq}{t} \right)^2 d_q \sigma(x) = 0.$$

Since $|\mathcal{F}_{\alpha,q}(f)(\xi) \left(1 - \int_{\epsilon}^{\delta} [\mathcal{F}_{\alpha,q}(\varphi)(t\xi)]^2 \frac{dq}{t} \right)| \leq |\mathcal{F}_{\alpha,q}(f)(\xi)|$, therefore, by the dominated convergence theorem, the result follows.

The reproducing identity (67) holds in the pointwise sense under different sets of nice conditions.

**Theorem 3.3.** Suppose $f, \mathcal{F}_{\alpha,q} f \in L_{1,\alpha,q}^1(\mathbb{R}_{q,+})$. Let $\varphi \in L_{1,\alpha,q}^1(\mathbb{R}_{q,+})$ and satisfies

$$\int_{0}^{\infty} [\mathcal{F}_{\alpha,q}\varphi(t\xi)]^2 \frac{dq}{t} = 1 \quad (71)$$
then
\[
\lim_{\varepsilon \to 0 \atop \delta \to \infty} f^{\varepsilon,\delta}(x) = f(x),
\]
(72)
where \( f^{\varepsilon,\delta} \) is given by (69).

Proof. by Proposition 2.5, we have
\[
\| \varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f \|_{1,\alpha,q} \leq \| \varphi_t \|_{1,\alpha,q}^2 \| f \|_{1,\alpha,q}.
\]
Now
\[
\| f^{\varepsilon,\delta} \|_{1,\alpha,q} = \int_0^\infty \int_\varepsilon^\delta (\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f)(x) \frac{d_q t}{t} d_q \sigma(x)
\]
\[
\leq \int_\varepsilon^\delta \| (\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f)(x) \|_{1,\alpha,q} \frac{d_q t}{t}
\]
\[
\leq \| \varphi_t \|_{1,\alpha,q}^2 \| f \|_{1,\alpha,q} \log \frac{\delta}{\varepsilon}.
\]
Therefore, \( f^{\varepsilon,\delta} \in L_{1,\alpha,q}(\mathbb{R}_q,+) \). Also using Fubini’s theorem and taking \( q \)- Bessel Fourier transform of \( f^{\varepsilon,\delta} \), we get
\[
\mathcal{F}_{\alpha,q} f^{\varepsilon,\delta}(\xi) = \int_0^\infty \int_\varepsilon^\delta \left( \int_\varepsilon^\delta \left( (\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f)(x) \frac{d_q t}{t} \right) d_q \sigma(x) \right) d_q \sigma(\xi)
\]
\[
= \int_\varepsilon^\delta \int_0^\infty j_\alpha(x \xi; q^2) \left( (\varphi_t *_{\alpha,q} \varphi_t *_{\alpha,q} f)(x) \frac{d_q t}{t} \right) d_q \sigma(\xi)
\]
\[
= \int_\varepsilon^\delta \mathcal{F}_{\alpha,q} \varphi_t(\xi) \mathcal{F}_{\alpha,q} \varphi_t(\xi) \mathcal{F}_{\alpha,q} f(\xi) \frac{d_q t}{t}
\]
\[
= \mathcal{F}_{\alpha,q} f(\xi) \int_\varepsilon^\delta \mathcal{F}_{\alpha,q} \varphi(t \xi) \frac{d_q t}{t}.
\]
Therefore by (71), \(| \mathcal{F}_{\alpha,q} f^{\varepsilon,\delta}(\xi) | \leq | \mathcal{F}_{\alpha,q} f(\xi) | \). It follows that \( \mathcal{F}_{\alpha,q} f^{\varepsilon,\delta} \in L_{1,\alpha,q}(\mathbb{R}_q,+) \). By inversion, we have
\[
f(x) - f^{\varepsilon,\delta}(x) = \int_0^\infty j_\alpha(x \xi; q^2) \left[ \mathcal{F}_{\alpha,q} f(\xi) - \mathcal{F}_{\alpha,q} f^{\varepsilon,\delta}(\xi) \right] d_q \sigma(\xi).
\]
(73)
Putting
\[
g^{\varepsilon,\delta}(x, \xi) = j_\alpha(x \xi; q^2) \left[ \mathcal{F}_{\alpha,q} f(\xi) - \mathcal{F}_{\alpha,q} f^{\varepsilon,\delta}(\xi) \right]
\]
(74)
\[
= j_\alpha(x \xi; q^2) \mathcal{F}_{\alpha,q} f(\xi) \left[ 1 - \int_\varepsilon^\delta \left( \mathcal{F}_{\alpha,q} \varphi(t \xi) \right)^2 \frac{d_q t}{t} \right],
\]
we get

\[ f(x) - f^{\epsilon, \delta}(x) = \int_0^\infty j_{\alpha}(x\xi; q^2) \left[ \mathcal{F}_{\alpha,q}(f)(\xi) - \mathcal{F}_{\alpha,q}(f^{\epsilon, \delta})(\xi) \right] d_q \sigma(\xi) \]

\[ = \int_0^\infty g^{\epsilon, \delta}(x, \xi) d_q \sigma(\xi). \]

Now using (71) and (74), we get

\[ \lim_{\delta \to \infty} g^{\epsilon, \delta}(x, \xi) = 0. \] (75)

Since \(|g^{\epsilon, \delta}(x, \xi)| \leq \frac{1}{(q; q^2)^{2\alpha}} |\mathcal{F}_{\alpha,q}(f)(\xi)|\), the dominated convergence theorem yields the result. 

\[ \square \]

4 \quad q\text{-Calderón’s formula for finite measures}

It is now possible to define analogues to (2) for the q-Bessel convolution \(*_{\alpha,q}\) and investigate its convergence in the \(L^2_{\alpha,q}(\mathbb{R}_{q,+})\) q-norm. To this end we need some technical lemmas

**Lemma 4.1.** Let \(\mu \in \mathcal{M}'(\mathbb{R}_{q,+})\), for \(0 < \epsilon < \delta < \infty\) define

\[ G_{\epsilon, \delta}(x; q^2) = \frac{\mu([\frac{\epsilon}{2}, \frac{\delta}{2}])}{x^{2\alpha+2}}, \quad x > 0 \] (76)

and

\[ K_{\epsilon, \delta}(\lambda; q^2) = \int_{\epsilon}^{\delta} \mathcal{F}_{\alpha,q}(\mu)(qa\lambda) \frac{d_q a}{a}, \lambda \geq 0. \] (77)

Then \(G_{\epsilon, \delta} \in L^1_{\alpha,q}(\mathbb{R}_{q,+})\) and

\[ \mathcal{F}_{\alpha,q}(G_{\epsilon, \delta})(\lambda; q^2) = K_{\epsilon, \delta}(\lambda; q^2) - \mu([0]) \log_q \left( \frac{\delta}{\epsilon} \right), \] (78)

where \(\log_q\) is given by \((22)\).

**Proof.** We have by (25) and (29),

\[ | \int_{0}^{\infty} G_{\epsilon, \delta}(x; q^2)x^{2\alpha+1} d_q x | \leq \int_{0}^{\infty} \left( \int_{\frac{\epsilon}{2}}^{\frac{\delta}{2}} d_q |\mu(y)| \right) \frac{d_q x}{x} \]

\[ = \int_{0}^{\infty} \left( \int_{\frac{\epsilon}{2}}^{\frac{\delta}{2}} d_q |\mu(y)| - \int_{\frac{\epsilon}{2}}^{\frac{\delta}{2}} d_q |\mu(y)| \right) \frac{d_q x}{x} \]

= \int_{0}^{\infty} \left[ \int_{q\epsilon y}^{q\delta y} \frac{d_q x}{x} - \int_{q\delta y}^{\infty} \frac{d_q x}{x} \right] d_q |\mu(y)| \]

\[ = \int_{0}^{\infty} \log_q \left( \frac{\epsilon}{\delta} \right) d_q |\mu(y)| \]

\[ = |\mu(\mathbb{R}_{q,+})| \log_q \left( \frac{\epsilon}{\delta} \right) < \infty. \]
Using again relation (29) and q-Fubini's theorem we obtain

\[ F_{\alpha,q}(G_{\epsilon,\delta})(\lambda) = \int_0^{\infty} \int_{q^\delta}^{q^\epsilon} j_\alpha(\lambda x; q^2) \frac{d_q x}{x} d_q \mu(y) \]

\[ = \int_0^{\infty} \int_{q^\epsilon}^{q^\delta} j_\alpha(\lambda x; q^2) \frac{d_q x}{x} d_q \mu(y) \]

\[ = \int_0^{q^\delta} j_\alpha(\lambda xy; q^2) d_q \mu(y) \frac{d_q x}{x} \]

\[ = \int_0^{q^\delta} \mathcal{F}_{\alpha,q} \mu(\lambda x) - \mu(\{0\}) \frac{d_q x}{x} \]

\[ = \int_0^{q^\delta} \mathcal{F}_{\alpha,q} \mu(q\lambda x) - \mu(\{0\}) \frac{d_q x}{x} \]

\[ = K_{\epsilon,\delta}(\lambda; q^2) - \mu(\{0\}) \log q(\frac{\delta}{\epsilon}). \]

\[ \blacksquare \]

**Lemma 4.2.** Let \( \mu \in M^+(\mathbb{R}_{q,+}) \), then for \( f \in L^p_{\alpha,q}(\mathbb{R}_{q,+}), p = 1,2 \) and \( 0 < \epsilon < \delta < \infty \), the function

\[ f^{\epsilon,\delta}(x; q^2) = \int_{\epsilon}^{\delta} f \ast_{\alpha,q} \mu_a(x; q^2) \frac{d_q a}{a} \]

(79)

belongs to \( L^p_{\alpha,q}(\mathbb{R}_{q,+}) \) and has the form

\[ f^{\epsilon,\delta}(x; q^2) = f \ast_{\alpha,q} G_{\epsilon,\delta}(x; q^2) + \mu(\{0\}) f(x) \log q(\frac{\delta}{\epsilon}). \]

(80)

where \( G_{\epsilon,\delta} \) is given by (4.7).

**Proof.** Applying q-Fubini's theorem we get

\[ f^{\epsilon,\delta}(x) = \int_{\epsilon}^{\delta} \int_0^{\infty} T_{q,\alpha}^x f(ay) d_q \mu(y) \frac{d_q a}{a} \]

\[ = \int_0^{\infty} \int_{\epsilon}^{\delta} T_{q,\alpha}^x f(x) \frac{d_q a}{a} d_q \mu(y) \]

\[ = f(x) \mu(\{0\}) \log q(\frac{\delta}{\epsilon}) + \int_{\mathbb{R}_{q,+}}^{\delta} T_{q,\alpha}^x f(a) \frac{d_q a}{a} d_q \mu(y) \]

\[ = f(x) \mu(\{0\}) \log q(\frac{\delta}{\epsilon}) + \int_{\mathbb{R}_{q,+}}^{\delta} T_{q,\alpha}^x f(a)(\int_a^{\infty} \frac{d_q a}{a} d_q \mu(y) \]

\[ = f(x) \mu(\{0\}) \log q(\frac{\delta}{\epsilon}) + f \ast_{\alpha,q} G_{\epsilon,\delta}(x). \]
From this relation, inequality (58) and Lemma 4.1 we deduce that $f_{\epsilon,\delta}$ belongs to $L^p_{\alpha,q}(\mathbb{R}^q_{+})$. □

**Lemma 4.3.** Let $\mu \in \mathcal{M}'(\mathbb{R}^q_{+})$, then for $f \in L^2_{\alpha,q}(\mathbb{R}^q_{+})$, we have

$$F_{\alpha,q}(f^{\epsilon,\delta})(\lambda; q^2) = F_q(f)(\lambda; q^2)K_{\epsilon,\delta}(\lambda; q^2),$$

where $K_{\epsilon,\delta}$, is the function defined in (67).

**Proof.** This follows from (59), (67) and (80). □

**Theorem 4.4.** Let $\mu \in \mathcal{M}'(\mathbb{R}^q_{+})$, be such that the $q$-integral

$$c_{\mu,\alpha,q} = \int_0^\infty F_{\alpha,q}(\mu)(\lambda) \frac{dq\lambda}{\lambda},$$

be finite. Then for all $f \in L^2_{\alpha,q}(\mathbb{R}^q_{+})$, we have

$$\lim_{\epsilon \to 0} \lim_{\delta \to \infty} \|f^{\epsilon,\delta} - c_{\mu,\alpha,q}f\|_{2,\alpha,q} = 0.$$  

**Proof.** By identity (81) and Theorem 2.4 we have

$$\|f^{\epsilon,\delta} - c_{\mu,\alpha,q}f\|_{2,\alpha,q} = \|F_{\alpha,q}(f^{\epsilon,\delta}) - c_{\mu,\alpha,q}F_{\alpha,q}(f)\|_{2,\alpha,q} = \|F_{\alpha,q}(f)|K_{\epsilon,\delta} - c_{\mu,\alpha,q}||_{2,\alpha,q}.$$

Or $\lim_{\epsilon \to 0} \lim_{\delta \to \infty} K_{\epsilon,\delta}(\lambda) = c_{\mu,\alpha,q}$, for all $\lambda > 0$ the result follows from the dominate convergence theorem.

**Lemma 4.5.** Let $\mu \in \mathcal{M}'(\mathbb{R}^q_{+})$, be such that the $q$-integral

$$\int_0^\infty |\mu([0,y])| \frac{dqy}{y},$$

be finite. Then the $q$-integral $c_{\mu,\alpha,q}$ is finite and admits the representation

$$c_{\mu,\alpha,q} = \int_0^\infty \mu([0,y]) \frac{dqy}{y}.$$  

**Proof.** From (70) we have

$$G_{\epsilon,\delta} = \frac{\mu([\frac{y}{\delta}, \frac{y}{\delta}])}{\chi^{2\alpha+2}} = G_{\epsilon} - G_{\delta},$$
where
\[ G(y) = \frac{\mu([0,y])}{y^{2\alpha+2}} \]  
(87)
and \( G_\varepsilon, G_\delta \) the dilated function of \( G \). Since \( G \in L^1_{\alpha,q}(\mathbb{R}_q^q) \), we deduce from (82) and (78) that

\[ F_{\alpha,q} G_\varepsilon, \delta (\lambda) = \int_{\varepsilon \lambda}^{\delta \lambda} F_{\alpha,q} \mu(a) \frac{da}{a} - \mu([0]) \log q \left( \frac{\varepsilon}{\lambda} \right) \]  
(88)

for all \( \lambda > 0 \). Or (84) implies necessarily \( \mu([0]) = 0 \). Hence when \( \varepsilon = 1 \) and \( \delta \to \infty \), a combination of (88) and (57) gives

\[ F_{\alpha,q} G(\lambda) = \int_{\lambda}^{\infty} F_{\alpha,q} \mu(a) \frac{da}{a}, \text{ for all } \lambda > 0. \]  
(89)

Now the result follows from Formula (84) by using the continuity of \( F_{\alpha,q}(\mu) \).

\[ \square \]

**Theorem 4.6.** Let \( \mu \in \mathcal{M}'(\mathbb{R}_q^q) \) such that

\[ \int_0^{\infty} |\mu([0,y])| \frac{dy}{y} \]  
(90)
is finite and \( f \in L^2_{\alpha,q}(\mathbb{R}_q^q) \). Then

\[ \lim_{\varepsilon \to 0} \lim_{\delta \to \infty} \| f^{\varepsilon,\delta} - c_{\mu,\alpha,q} f \|_{2,\alpha,q} = 0. \]  
(91)

**Proof.** By (80) and (86) we have

\[ f^{\varepsilon,\delta} = f *_{\alpha,q} G_\varepsilon - f *_{\alpha,q} G_\delta, \]  
(92)
where \( G \) is as in (57). Equation (91) is now a consequence of Proposition 2.5.

\[ \square \]

**References**


