Right General Fractional Monotone Approximation

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ABSTRACT

Here is introduced a right general fractional derivative Caputo style with respect to a base absolutely continuous strictly increasing function g . We give various examples of such right fractional derivatives for different g . Let f be p -times continuously differentiable function on $[a, b]$, and let L be a linear right general fractional differential operator such that $L(f)$ is non-negative over a critical closed subinterval J of $[a, b]$. We can find a sequence of polynomials Q_n of degree less-equal n such that $L(Q_n)$ is non-negative over J , furthermore f is approximated uniformly by Q_n over $[a, b]$.

The degree of this constrained approximation is given by an inequality using the first modulus of continuity of $f^{(p)}$. We finish we applications of the main right fractional monotone approximation theorem for different g .

RESUMEN

Aquí introducimos una derivada fraccional derecha general al estilo de Caputo con respecto a una base de funciones absolutamente continuas estrictamente crecientes g . Damos varios ejemplos de dichas derivadas fraccionales derechas para diferentes g . Sea f una función p -veces continuamente diferenciable en $[a, b]$, y sea L un operador diferencial fraccional derecho general tal que $L(f)$ es no-negativo en un subintervalo cerrado crítico J de $[a, b]$. Podemos encontrar una sucesión de polinomios $L(Q_n)$ de grado menor o igual a n tal que $L(Q_n)$ es no-negativo en J , más aún f es aproximada uniformemente por Q_n en $[a, b]$. El grado de esta aproximación restringida es dada por una desigualdad usando el primer módulo de continuidad de $f^{(p)}$. Concluimos con aplicaciones del teorema principal de aproximación monótona fraccional derecha para diferentes g .

Keywords and Phrases: Right Fractional Monotone Approximation, general right fractional derivative, linear general right fractional differential operator, modulus of continuity.

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1 Introduction and Preparation

The topic of monotone approximation started in [11] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k , approximate a given function whose k th derivative is ≥ 0 by polynomials having this property.

In [4] the authors replaced the k th derivative with a linear ordinary differential operator of order k .

Furthermore in [1], the author generalized the result of [4] for linear right fractional differential operators.

To describe the motivating result here we need

Definition 1. ([5]) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$, ($\lceil \cdot \rceil$ ceiling of the number). Consider $f \in C^m([-1, 1])$. We define the right Caputo fractional derivative of f of order α as follows:

$$(D_{1-}^{\alpha} f)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^1 (t-x)^{m-\alpha-1} f^{(m)}(t) dt, \quad (1)$$

for any $x \in [-1, 1]$, where Γ is the gamma function $\Gamma(v) = \int_0^{\infty} e^{-t} t^{v-1} dt$, $v > 0$.

We set

$$D_{1-}^0 f(x) = f(x), \quad (2)$$

$$D_{1-}^m f(x) = (-1)^m f^{(m)}(x), \quad \forall x \in [-1, 1]. \quad (3)$$

In [1] we proved

Theorem 1.1. Let h, k, p be integers, h is even, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1, 1]$ with modulus of continuity $\omega_1(f^{(p)}, \delta)$, $\delta > 0$, there. Let $\alpha_j(x)$, $j = h, h+1, \dots, k$ be real functions, defined and bounded on $[-1, 1]$ and assume for $x \in [-1, 0]$ that $\alpha_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$. Let the real numbers $\alpha_0 = 0 < \alpha_1 < 1 < \alpha_2 < 2 < \dots < \alpha_p < p$. Here $D_{1-}^{\alpha_j} f$ stands for the right Caputo fractional derivative of f of order α_j anchored at 1. Consider the linear right fractional differential operator

$$L := \sum_{j=h}^k \alpha_j(x) [D_{1-}^{\alpha_j}] \quad (4)$$

and suppose, throughout $[-1, 0]$,

$$L(f) \geq 0. \quad (5)$$

Then, for any $n \in \mathbb{N}$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1, 0], \quad (6)$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1\left(f^{(p)}, \frac{1}{n}\right), \quad (7)$$

where C is independent of n or f .

Notice above that the monotonicity property is only true on $[-1, 0]$, see (5), (6). However the approximation property (7) it is true over the whole interval $[-1, 1]$.

In this article we extend Theorem 1.1 to much more general linear right fractional differential operators.

We use here the following right generalised fractional integral.

Definition 2. (see also [8, p. 99]) *The right generalised fractional integral of a function f with respect to given function g is defined as follows:*

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here $g \in AC([a, b])$ (absolutely continuous functions) and is strictly increasing, $f \in L_\infty([a, b])$. We set

$$(I_{b-;g}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) f(t) dt, \quad x \leq b, \quad (8)$$

clearly $(I_{b-;g}^\alpha f)(b) = 0$.

When g is the identity function id , we get that $I_{b-;\text{id}}^\alpha = I_{b-}^\alpha$, the ordinary right Riemann-Liouville fractional integral, where

$$(I_{b-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t) dt, \quad x \leq b, \quad (9)$$

$(I_{b-}^\alpha f)(b) = 0$.

When $g(x) = \ln x$ on $[a, b]$, $0 < a < b < \infty$, we get

Definition 3. ([8, p. 110]) *Let $0 < a < b < \infty$, $\alpha > 0$. The right Hadamard fractional integral of order α is given by*

$$(J_{b-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{y}{x}\right)^{\alpha-1} \frac{f(y)}{y} dy, \quad x \leq b, \quad (10)$$

where $f \in L_\infty([a, b])$.

We mention

Definition 4. *The right fractional exponential integral is defined as follows: Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a, b])$. We set*

$$(I_{b-;e^x}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (e^t - e^x)^{\alpha-1} e^t f(t) dt, \quad x \leq b. \quad (11)$$

Definition 5. *Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a, b])$, $A > 1$. We introduce the right fractional integral*

$$(I_{b-;A^x}^\alpha f)(x) = \frac{\ln A}{\Gamma(\alpha)} \int_x^b (A^t - A^x)^{\alpha-1} A^t f(t) dt, \quad x \leq b. \quad (12)$$

We also give

Definition 6. Let $\alpha, \sigma > 0$, $0 \leq a < b < \infty$, $f \in L_\infty([a, b])$. We set

$$(\mathcal{K}_{b-;x^\sigma}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t^\sigma - x^\sigma)^{\alpha-1} f(t) \sigma t^{\sigma-1} dt, \quad x \leq b. \quad (13)$$

We introduce the following general right fractional derivative.

Definition 7. Let $\alpha > 0$ and $[\alpha] = m$, ($[\cdot]$ ceiling of the number). Consider $f \in AC^m([a, b])$ (space of functions f with $f^{(m-1)} \in AC([a, b])$). We define the right general fractional derivative of f of order α as follows

$$(D_{b-;g}^\alpha f)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) f^{(m)}(t) dt, \quad (14)$$

for any $x \in [a, b]$, where Γ is the gamma function.

We set

$$D_{b-;g}^m f(x) = (-1)^m f^{(m)}(x), \quad (15)$$

$$D_{b-;g}^0 f(x) = f(x), \quad \forall x \in [a, b]. \quad (16)$$

When $g = \text{id}$, then $D_{b-}^\alpha f = D_{b-;\text{id}}^\alpha f$ is the right Caputo fractional derivative.

So we have the specific general right fractional derivatives.

Definition 8.

$$D_{b-;\ln x}^\alpha f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b \left(\ln \frac{y}{x}\right)^{m-\alpha-1} \frac{f^{(m)}(y)}{y} dy, \quad 0 < a \leq x \leq b, \quad (17)$$

$$D_{b-;e^x}^\alpha f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (e^t - e^x)^{m-\alpha-1} e^t f^{(m)}(t) dt, \quad a \leq x \leq b, \quad (18)$$

and

$$D_{b-;A^x}^\alpha f(x) = \frac{(-1)^m \ln A}{\Gamma(m-\alpha)} \int_x^b (A^t - A^x)^{m-\alpha-1} A^t f^{(m)}(t) dt, \quad a \leq x \leq b, \quad (19)$$

$$(\mathcal{D}_{b-;x^\sigma}^\alpha f)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (t^\sigma - x^\sigma)^{m-\alpha-1} \sigma t^{\sigma-1} f^{(m)}(t) dt, \quad 0 \leq a \leq x \leq b. \quad (20)$$

We mention

Theorem 1.2. (Trigub, [12], [13]) Let $g \in C^p([-1, 1])$, $p \in \mathbb{N}$. Then there exists real polynomial $q_n(x)$ of degree $\leq n$, $x \in [-1, 1]$, such that

$$\max_{-1 \leq x \leq 1} |g^{(j)}(x) - q_n^{(j)}(x)| \leq R_p n^{j-p} \omega_1\left(g^{(p)}, \frac{1}{n}\right), \quad (21)$$

$j = 0, 1, \dots, p$, where R_p is independent of n or g .

In [2], based on Theorem 1.2 we proved the following useful here result

Theorem 1.3. *Let $f \in C^p([a, b])$, $p \in \mathbb{N}$. Then there exist real polynomials $Q_n^*(x)$ of degree $\leq n \in \mathbb{N}$, $x \in [a, b]$, such that*

$$\max_{a \leq x \leq b} \left| f^{(j)}(x) - Q_n^{*(j)}(x) \right| \leq R_p \left(\frac{b-a}{2n} \right)^{p-j} \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right), \quad (22)$$

$j = 0, 1, \dots, p$, where R_p is independent of n or g .

Remark 1.4. *Here $g \in AC([a, b])$ (absolutely continuous functions), g is increasing over $[a, b]$, $\alpha > 0$.*

Let $g(a) = c$, $g(b) = d$. We want to calculate

$$I = \int_a^b (g(t) - g(a))^{\alpha-1} g'(t) dt. \quad (23)$$

Consider the function

$$f(y) = (y - g(a))^{\alpha-1} = (y - c)^{\alpha-1}, \quad \forall y \in [c, d]. \quad (24)$$

We have that $f(y) \geq 0$, it may be $+\infty$ when $y = c$ and $0 < \alpha < 1$, but f is measurable on $[c, d]$. By [9], Royden, p. 107, exercise 13 d, we get that

$$(f \circ g)(t) g'(t) = (g(t) - g(a))^{\alpha-1} g'(t) \quad (25)$$

is measurable on $[a, b]$, and

$$I = \int_c^d (y - c)^{\alpha-1} dy = \frac{(d - c)^\alpha}{\alpha} \quad (26)$$

(notice that $(y - c)^{\alpha-1}$ is Riemann integrable).

That is

$$I = \frac{(g(b) - g(a))^\alpha}{\alpha}. \quad (27)$$

Similarly it holds

$$\int_x^b (g(t) - g(x))^{\alpha-1} g'(t) dt = \frac{(g(b) - g(x))^\alpha}{\alpha}, \quad \forall x \in [a, b]. \quad (28)$$

Finally we will use

Theorem 1.5. *Let $\alpha > 0$, $\mathbb{N} \ni m = [\alpha]$, and $f \in C^m([a, b])$. Then $(D_{b-;g}^\alpha f)(x)$ is continuous in $x \in [a, b]$, $-\infty < a < b < \infty$.*

Proof. By [3], Apostol, p. 78, we get that g^{-1} exists and it is strictly increasing on $[g(a), g(b)]$. Since g is continuous on $[a, b]$, it implies that g^{-1} is continuous on $[g(a), g(b)]$. Hence $f^{(m)} \circ g^{-1}$ is a continuous function on $[g(a), g(b)]$.

If $\alpha = m \in \mathbb{N}$, then the claim is trivial.

We treat the case of $0 < \alpha < m$.

It holds that

$$\begin{aligned}
 (D_{b-;g}^\alpha f)(x) &= \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) f^{(m)}(t) dt = \\
 &= \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) \left(f^{(m)} \circ g^{-1} \right)(g(t)) dt = \\
 &= \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{m-\alpha-1} \left(f^{(m)} \circ g^{-1} \right)(z) dz.
 \end{aligned} \tag{29}$$

An explanation follows.

The function

$$G(z) = (z - g(x))^{m-\alpha-1} \left(f^{(m)} \circ g^{-1} \right)(z)$$

is integrable on $[g(x), g(b)]$, and by assumption g is absolutely continuous: $[a, b] \rightarrow [g(a), g(b)]$.

Since g is monotone (strictly increasing here) the function

$$(g(t) - g(x))^{m-\alpha-1} g'(t) \left(f^{(m)} \circ g^{-1} \right)(g(t))$$

is integrable on $[x, b]$ (see [7]). Furthermore it holds (see also [7]),

$$\begin{aligned}
 &\frac{(-1)^m}{\Gamma(m-\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{m-\alpha-1} \left(f^{(m)} \circ g^{-1} \right)(z) dz = \\
 &\frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) \left(f^{(m)} \circ g^{-1} \right)(g(t)) dt \\
 &= (D_{b-;g}^\alpha f)(x), \quad \forall x \in [a, b].
 \end{aligned} \tag{30}$$

And we can write

$$\begin{aligned}
 (D_{b-;g}^\alpha f)(x) &= \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{m-\alpha-1} \left(f^{(m)} \circ g^{-1} \right)(z) dz, \\
 (D_{b-;g}^\alpha f)(y) &= \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{g(y)}^{g(b)} (z - g(y))^{m-\alpha-1} \left(f^{(m)} \circ g^{-1} \right)(z) dz.
 \end{aligned} \tag{31}$$

Here $a \leq y \leq x \leq b$, and $g(a) \leq g(y) \leq g(x) \leq g(b)$, and $0 \leq g(b) - g(x) \leq g(b) - g(y)$.

Let $\lambda = z - g(x)$, then $z = g(x) + \lambda$. Thus

$$(D_{b-;g}^\alpha f)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_0^{g(b)-g(x)} \lambda^{m-\alpha-1} \left(f^{(m)} \circ g^{-1} \right)(g(x) + \lambda) d\lambda. \tag{32}$$

Clearly, see that $g(x) \leq z \leq g(b)$, and $0 \leq \lambda \leq g(b) - g(x)$.

Similarly

$$(D_{b-,g}^\alpha f)(y) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_0^{g(b)-g(y)} \lambda^{m-\alpha-1} \left(f^{(m)} \circ g^{-1} \right) (g(y) + \lambda) d\lambda. \quad (33)$$

Hence it holds

$$\begin{aligned} (D_{b-,g}^\alpha f)(y) - (D_{b-,g}^\alpha f)(x) &= \frac{(-1)^m}{\Gamma(m-\alpha)} \\ &\left[\int_0^{g(b)-g(x)} \lambda^{m-\alpha-1} \left(\left(f^{(m)} \circ g^{-1} \right) (g(y) + \lambda) - \left(f^{(m)} \circ g^{-1} \right) (g(x) + \lambda) \right) d\lambda + \right. \\ &\left. \int_{g(b)-g(x)}^{g(b)-g(y)} \lambda^{m-\alpha-1} \left(f^{(m)} \circ g^{-1} \right) (g(y) + \lambda) d\lambda \right]. \end{aligned} \quad (34)$$

Thus we obtain

$$\begin{aligned} |(D_{b-,g}^\alpha f)(y) - (D_{b-,g}^\alpha f)(x)| &\leq \frac{1}{\Gamma(m-\alpha)} \\ &\left[\frac{(g(b) - g(x))^{m-\alpha}}{m-\alpha} \omega_1 \left(f^{(m)} \circ g^{-1}, |g(y) - g(x)| \right) + \right. \\ &\left. \frac{\|f^{(m)} \circ g^{-1}\|_{\infty, [g(a), g(b)]}}{m-\alpha} \left((g(b) - g(y))^{m-\alpha} - (g(b) - g(x))^{m-\alpha} \right) \right] =: (\xi). \end{aligned} \quad (35)$$

As $y \rightarrow x$, then $g(y) \rightarrow g(x)$ (since $g \in AC([a, b])$). So that $(\xi) \rightarrow 0$. As a result

$$(D_{b-,g}^\alpha f)(y) \rightarrow (D_{b-,g}^\alpha f)(x), \quad (36)$$

proving that $(D_{b-,g}^\alpha f)(x)$ is continuous in $x \in [a, b]$. $\square$

2 Main Result

We present

Theorem 2.1. Here we assume that $g(b) - g(a) > 1$. Let h, k, p be integers, h is even, $0 \leq h \leq k \leq p$ and let $f \in C^p([a, b])$, $a < b$, with modulus of continuity $\omega_1(f^{(p)}, \delta)$, $0 < \delta \leq b - a$. Let $\alpha_j(x)$, $j = h, h+1, \dots, k$ be real functions, defined and bounded on $[a, b]$ and assume for $x \in [a, g^{-1}(g(b) - 1)]$ that $\alpha_h(x)$ is either $\geq$ some number $\alpha^* > 0$, or $\leq$ some number $\beta^* < 0$. Let the real numbers $\alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \dots < \alpha_p \leq p$. Consider the linear right general fractional differential operator

$$L = \sum_{j=h}^k \alpha_j(x) \left[D_{b-,g}^{\alpha_j} \right], \quad (37)$$

and suppose, throughout $[a, g^{-1}(g(b) - 1)]$,

$$L(f) \geq 0. \quad (38)$$

Then, for any $n \in \mathbb{N}$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [a, g^{-1}(g(b) - 1)], \quad (39)$$

and

$$\max_{x \in [a, b]} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right), \quad (40)$$

where C is independent of n or f .

Proof. of Theorem 2.1.

Here $h, k, p \in \mathbb{Z}_+$, $0 \leq h \leq k \leq p$. Let $\alpha_j > 0$, $j = 1, \dots, p$, such that $0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3 \dots < \dots < \alpha_p \leq p$. That is $\lceil \alpha_j \rceil = j$, $j = 1, \dots, p$.

Let $Q_n^*(x)$ be as in Theorem 1.3.

We have that

$$\left(D_{b-,g}^{\alpha_j} f\right)(x) = \frac{(-1)^j}{\Gamma(j - \alpha_j)} \int_x^b (g(t) - g(x))^{j-\alpha_j-1} g'(t) f^{(j)}(t) dt, \quad (41)$$

and

$$\left(D_{b-,g}^{\alpha_j} Q_n^*\right)(x) = \frac{(-1)^j}{\Gamma(j - \alpha_j)} \int_x^b (g(t) - g(x))^{j-\alpha_j-1} g'(t) Q_n^{*(j)}(t) dt, \quad (42)$$

$j = 1, \dots, p$.

Also it holds

$$\left(D_{b-,g}^j f\right)(x) = (-1)^j f^{(j)}(x), \quad \left(D_{b-,g}^j Q_n^*\right)(x) = (-1)^j Q_n^{*(j)}(x), \quad j = 1, \dots, p. \quad (43)$$

By [10], we get that there exists g' a.e., and g' is measurable and non-negative.

We notice that

$$\begin{aligned} & \left| \left(D_{b-,g}^{\alpha_j} f\right)(x) - D_{b-,g}^{\alpha_j} Q_n^*(x) \right| = \\ & \frac{1}{\Gamma(j - \alpha_j)} \left| \int_x^b (g(x) - g(t))^{j-\alpha_j-1} g'(t) \left(f^{(j)}(t) - Q_n^{*(j)}(t)\right) dt \right| \leq \\ & \frac{1}{\Gamma(j - \alpha_j)} \int_x^b (g(x) - g(t))^{j-\alpha_j-1} g'(t) \left| f^{(j)}(t) - Q_n^{*(j)}(t) \right| dt \stackrel{(22)}{\leq} \\ & \frac{1}{\Gamma(j - \alpha_j)} \left(\int_x^b (g(x) - g(t))^{j-\alpha_j-1} g'(t) dt \right) R_p \left(\frac{b-a}{2n} \right)^{p-j} \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right) \\ & \stackrel{(28)}{=} \frac{(g(b) - g(x))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left(\frac{b-a}{2n} \right)^{p-j} \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right) \leq \\ & \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left(\frac{b-a}{2n} \right)^{p-j} \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right). \end{aligned} \quad (44)$$

Hence $\forall x \in [a, b]$, it holds

$$\left| \left(D_{b-;g}^{\alpha_j} f \right) (x) - D_{b-;g}^{\alpha_j} Q_n^* (x) \right| \leq \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left(\frac{b-a}{2n} \right)^{p-j} \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right), \quad (45)$$

and

$$\max_{x \in [a, b]} \left| D_{b-;g}^{\alpha_j} f(x) - D_{b-;g}^{\alpha_j} Q_n^* (x) \right| \leq \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left(\frac{b-a}{2n} \right)^{p-j} \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right), \quad (46)$$

$j = 0, 1, \dots, p$.

Above we set $D_{b-;g}^0 f(x) = f(x)$, $D_{b-;g}^0 Q_n^*(x) = Q_n^*(x)$, $\forall x \in [a, b]$, and $\alpha_0 = 0$, i.e. $[\alpha_0] = 0$.

Put

$$s_j = \sup_{a \leq x \leq b} \left| \alpha_n^{-1}(x) \alpha_j(x) \right|, \quad j = h, \dots, k, \quad (47)$$

and

$$\eta_n = R_p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right) \left(\sum_{j=h}^k s_j \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left(\frac{b-a}{2n} \right)^{p-j} \right). \quad (48)$$

I. Suppose, throughout $[a, g^{-1}(g(b) - 1)]$, $\alpha_n(x) \geq \alpha^* > 0$. Let $Q_n(x)$, $x \in [a, b]$, be a real polynomial of degree $\leq n$, according to Theorem 1.3 and (46), so that

$$\max_{x \in [a, b]} \left| D_{b-;g}^{\alpha_j} \left(f(x) + \eta_n (h!)^{-1} x^h \right) - \left(D_{b-;g}^{\alpha_j} Q_n \right) (x) \right| \leq \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left(\frac{b-a}{2n} \right)^{p-j} \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right), \quad (49)$$

$j = 0, 1, \dots, p$.

In particular ($j = 0$) holds

$$\max_{x \in [a, b]} \left| \left(f(x) + \eta_n (h!)^{-1} x^h \right) - Q_n(x) \right| \leq R_p \left(\frac{b-a}{2n} \right)^p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right), \quad (50)$$

and

$$\begin{aligned} \max_{x \in [a, b]} |f(x) - Q_n(x)| &\leq \eta_n (h!)^{-1} (\max(|a|, |b|))^h + R_p \left(\frac{b-a}{2n} \right)^p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right) \\ &= \eta_n (h!)^{-1} \max(|a|^h, |b|^h) + R_p \left(\frac{b-a}{2n} \right)^p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right) = \\ &R_p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right) \left(\sum_{j=h}^k s_j \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left(\frac{b-a}{2n} \right)^{p-j} \right) (h!)^{-1} \max(|a|^h, |b|^h) \end{aligned} \quad (51)$$

$$\begin{aligned}
& + R_p \left(\frac{b-a}{2n} \right)^p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right) \leq \\
& R_p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right) n^{k-p}. \\
& \left[\left(\sum_{j=h}^k s_j \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left(\frac{b-a}{2} \right)^{p-j} \right) (h!)^{-1} \max(|a|^h, |b|^h) + \left(\frac{b-a}{2} \right)^p \right]. \quad (52)
\end{aligned}$$

We have found that

$$\begin{aligned}
& \max_{x \in [a, b]} |f(x) - Q_n(x)| \leq R_p \left[\left(\frac{b-a}{2} \right)^p + (h!)^{-1} \max(|a|^h, |b|^h) \right. \\
& \left. \left(\sum_{j=h}^k s_j \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left(\frac{b-a}{2} \right)^{p-j} \right) \right] n^{k-p} \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right), \quad (53)
\end{aligned}$$

proving (40).

Notice for $j = h + 1, \dots, k$, that

$$(D_{b-;g}^{\alpha_j} x^h) = \frac{(-1)^j}{\Gamma(j - \alpha_j)} \int_x^b (g(t) - g(x))^{j-\alpha_j-1} g'(t) (t^h)^{(j)} dt = 0. \quad (54)$$

Here

$$L = \sum_{j=h}^k \alpha_j(x) [D_{b-;g}^{\alpha_j}],$$

and suppose, throughout $[a, g^{-1}(g(b) - 1)]$, $Lf \geq 0$. So over $a \leq x \leq g^{-1}(g(b) - 1)$, we get

$$\begin{aligned}
& \alpha_h^{-1}(x) L(Q_n(x)) \stackrel{(54)}{=} \alpha_h^{-1}(x) L(f(x)) + \frac{\eta_n}{h!} (D_{b-;g}^{\alpha_h}(x^h)) + \\
& \sum_{j=h}^k \alpha_h^{-1}(x) \alpha_j(x) \left[D_{b-;g}^{\alpha_j} Q_n(x) - D_{b-;g}^{\alpha_j} f(x) - \frac{\eta_n}{h!} D_{b-;g}^{\alpha_j} x^h \right] \stackrel{(49)}{\geq} \quad (55)
\end{aligned}$$

$$\frac{\eta_n}{h!} (D_{b-;g}^{\alpha_h}(x^h)) - \left(\sum_{j=h}^k s_j \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left(\frac{b-a}{2n} \right)^{p-j} \right) R_p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right) \quad (56)$$

$$\stackrel{(48)}{=} \frac{\eta_n}{h!} (D_{b-;g}^{\alpha_h}(x^h)) - \eta_n = \eta_n \left(\frac{D_{b-;g}^{\alpha_h}(x^h)}{h!} - 1 \right) = \quad (57)$$

$$\begin{aligned}
& \eta_n \left(\frac{1}{\Gamma(h - \alpha_h) h!} \int_x^b (g(t) - g(x))^{h-\alpha_h-1} g'(t) (t^h)^{(h)} dt - 1 \right) = \\
& \eta_n \left(\frac{h!}{h! \Gamma(h - \alpha_h)} \int_x^b (g(t) - g(x))^{h-\alpha_h-1} g'(t) dt - 1 \right) \stackrel{(28)}{=}
\end{aligned}$$

$$\eta_n \left(\frac{(g(b) - g(x))^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} - 1 \right) = \quad (58)$$

$$\eta_n \left(\frac{(g(b) - g(x))^{h-\alpha_h} - \Gamma(h - \alpha_h + 1)}{\Gamma(h - \alpha_h + 1)} \right) \geq \quad (59)$$

$$\eta_n \left(\frac{1 - \Gamma(h - \alpha_h + 1)}{\Gamma(h - \alpha_h + 1)} \right) \geq 0.$$

Clearly here $g(b) - g(x) \geq 1$.

Hence

$$L(Q_n(x)) \geq 0, \text{ for } x \in [a, g^{-1}(g(b) - 1)]. \quad (60)$$

A further explanation follows: We know $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0, \infty)$. Here $0 \leq h - \alpha_h < 1$ and $1 \leq h - \alpha_h + 1 < 2$. Thus

$$\Gamma(h - \alpha_h + 1) \leq 1 \text{ and } 1 - \Gamma(h - \alpha_h + 1) \geq 0. \quad (61)$$

II. Suppose, throughout $[a, g^{-1}(g(b) - 1)]$, $\alpha_h(x) \leq \beta^* < 0$.

Let $Q_n(x)$, $x \in [a, b]$ be a real polynomial of degree $\leq n$, according to Theorem 1.3 and (46), so that

$$\max_{x \in [a, b]} \left| D_{b-,g}^{\alpha_j} \left(f(x) - \eta_n (h!)^{-1} x^h \right) - \left(D_{b-,g}^{\alpha_j} Q_n \right) (x) \right| \leq \quad (62)$$

$$\frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left(\frac{b-a}{2n} \right)^{p-j} \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right),$$

$j = 0, 1, \dots, p$.

In particular ($j = 0$) holds

$$\max_{x \in [a, b]} \left| \left(f(x) - \eta_n (h!)^{-1} x^h \right) - Q_n(x) \right| \leq R_p \left(\frac{b-a}{2n} \right)^p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right), \quad (63)$$

and

$$\begin{aligned} \max_{x \in [a, b]} |f(x) - Q_n(x)| &\leq \eta_n (h!)^{-1} (\max(|a|, |b|))^h + R_p \left(\frac{b-a}{2n} \right)^p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right) \\ &= \eta_n (h!)^{-1} \max(|a|^h, |b|^h) + R_p \left(\frac{b-a}{2n} \right)^p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right), \end{aligned} \quad (64)$$

etc.

We find again that

$$\max_{x \in [a, b]} |f(x) - Q_n(x)| \leq R_p \left[\left(\frac{b-a}{2} \right)^p + (h!)^{-1} \max(|a|^h, |b|^h) \right].$$

$$\left(\sum_{j=h}^k s_j \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left(\frac{b-a}{2} \right)^{p-j} \right) \Bigg] n^{k-p} \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right), \quad (65)$$

reproving (40).

Here again

$$L = \sum_{j=h}^k \alpha_j(x) \left[D_{b-;g}^{\alpha_j} \right],$$

and suppose, throughout $[a, g^{-1}(g(b) - 1)]$, $Lf \geq 0$. So over $a \leq x \leq g^{-1}(g(b) - 1)$, we get

$$\begin{aligned} \alpha_h^{-1}(x) L(Q_n(x)) &\stackrel{(54)}{=} \alpha_h^{-1}(x) L(f(x)) - \frac{\eta_n}{h!} \left(D_{b-;g}^{\alpha_h}(x^h) \right) + \\ &\sum_{j=h}^k \alpha_j^{-1}(x) \alpha_j(x) \left[D_{b-;g}^{\alpha_j} Q_n(x) - D_{b-;g}^{\alpha_j} f(x) + \frac{\eta_n}{h!} D_{b-;g}^{\alpha_j} x^h \right] \stackrel{(62)}{\leq} \end{aligned} \quad (66)$$

$$- \frac{\eta_n}{h!} \left(D_{b-;g}^{\alpha_h}(x^h) \right) + \left(\sum_{j=h}^k s_j \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left(\frac{b-a}{2n} \right)^{p-j} \right) R_p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right) \quad (67)$$

$$\stackrel{(48)}{=} - \frac{\eta_n}{h!} \left(D_{b-;g}^{\alpha_h}(x^h) \right) + \eta_n = \eta_n \left(1 - \frac{D_{b-;g}^{\alpha_h}(x^h)}{h!} \right) = \quad (68)$$

$$\eta_n \left(1 - \frac{1}{\Gamma(h - \alpha_h) h!} \int_x^b (g(t) - g(x))^{h-\alpha_h-1} g'(t) (t^h)^{(h)} dt \right) =$$

$$\eta_n \left(1 - \frac{h!}{h! \Gamma(h - \alpha_h)} \int_x^b (g(t) - g(x))^{h-\alpha_h-1} g'(t) dt \right) \stackrel{(28)}{=} \quad (69)$$

$$\eta_n \left(1 - \frac{(g(b) - g(x))^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} \right) = \quad (69)$$

$$\eta_n \left(\frac{\Gamma(h - \alpha_h + 1) - (g(b) - g(x))^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} \right) \stackrel{(61)}{\leq}$$

$$\eta_n \left(\frac{1 - (g(b) - g(x))^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} \right) \leq 0. \quad (70)$$

Hence again

$$L(Q_n(x)) \geq 0, \quad \forall x \in [a, g^{-1}(g(b) - 1)].$$

The case of $\alpha_h = h$ is trivially concluded from the above. The proof of the theorem is now over. $\square$

We make

Remark 2.2. By Theorem 1.5 we have that $D_{b-,g}^{\alpha_j} f$ are continuous functions, $j = 0, 1, \dots, p$. Suppose that $\alpha_h(x), \dots, \alpha_k(x)$ are continuous functions on $[a, b]$, and $L(f) \geq 0$ on $[a, g^{-1}(g(b) - 1)]$ is replaced by $L(f) > 0$ on $[a, g^{-1}(g(b) - 1)]$. Disregard the assumption made in the main theorem on $\alpha_h(x)$. For $n \in \mathbb{N}$, let $Q_n(x)$ be the $Q_n^*(x)$ of Theorem 1.3, and f as in Theorem 1.3 (same as in Theorem 2.1). Then $Q_n(x)$ converges to $f(x)$ at the Jackson rate $\frac{1}{n^{p+1}}$ ([6], p. 18, Theorem VIII) and at the same time, since $L(Q_n)$ converges uniformly to $L(f)$ on $[a, b]$, $L(Q_n) > 0$ on $[a, g^{-1}(g(b) - 1)]$ for all n sufficiently large.

3 Applications (to Theorem 2.1)

1) When $g(x) = \ln x$ on $[a, b]$, $0 < a < b < \infty$.

Here we would assume that $b > ae$, $\alpha_h(x)$ restriction true on $[a, \frac{b}{e}]$, and

$$Lf = \sum_{j=h}^k \alpha_j(x) [D_{b-; \ln x}^{\alpha_j} f] \geq 0, \quad (72)$$

throughout $[a, \frac{b}{e}]$.

Then $L(Q_n) \geq 0$ on $[a, \frac{b}{e}]$.

2) When $g(x) = e^x$ on $[a, b]$, $a < b < \infty$.

Here we assume that $b > \ln(1 + e^a)$, $\alpha_h(x)$ restriction true on $[a, \ln(e^b - 1)]$, and

$$Lf = \sum_{j=h}^k \alpha_j(x) [D_{b-; e^x}^{\alpha_j} f] \geq 0, \quad (73)$$

throughout $[a, \ln(e^b - 1)]$.

Then $L(Q_n) \geq 0$ on $[a, \ln(e^b - 1)]$.

3) When, $A > 1$, $g(x) = A^x$ on $[a, b]$, $a < b < \infty$.

Here we assume that $b > \log_A(1 + A^a)$, $\alpha_h(x)$ restriction true on $[a, \log_A(A^b - 1)]$, and

$$Lf = \sum_{j=h}^k \alpha_j(x) [D_{b-; A^x}^{\alpha_j} f] \geq 0, \quad (74)$$

throughout $[a, \log_A(A^b - 1)]$.

Then $L(Q_n) \geq 0$ on $[a, \log_A(A^b - 1)]$.

4) When $\sigma > 0$, $g(x) = x^\sigma$, $0 \leq a < b < \infty$.

Here we assume that $b > (1 + a^\sigma)^{\frac{1}{\sigma}}$, $\alpha_h(x)$ restriction true on $[a, (b^\sigma - 1)^{\frac{1}{\sigma}}]$, and

$$Lf = \sum_{j=h}^k \alpha_j(x) [D_{b-; x^\sigma}^{\alpha_j} f] \geq 0 \quad (75)$$

throughout $\left[a, (b^\sigma - 1)^{\frac{1}{\sigma}}\right]$.

Then $L(Q_n) \geq 0$ on $\left[a, (b^\sigma - 1)^{\frac{1}{\sigma}}\right]$.

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