Right General Fractional Monotone Approximation

GEORGE A. ANASTASSIOU
Department of Mathematical Sciences, University of Memphis,
Memphis, TN 38152, U.S.A.,
ganastss@memphis.edu

ABSTRACT

Here is introduced a right general fractional derivative Caputo style with respect to a base absolutely continuous strictly increasing function $g$. We give various examples of such right fractional derivatives for different $g$. Let $f$ be $p$-times continuously differentiable function on $[a, b]$, and let $L$ be a linear right general fractional differential operator such that $L(f)$ is non-negative over a critical closed subinterval $J$ of $[a, b]$. We can find a sequence of polynomials $Q_n$ of degree less-equal $n$ such that $L(Q_n)$ is non-negative over $J$, furthermore $f$ is approximated uniformly by $Q_n$ over $[a, b]$.

The degree of this constrained approximation is given by an inequality using the first modulus of continuity of $f^{(p)}$. We finish we applications of the main right fractional monotone approximation theorem for different $g$.

RESUMEN

Aquí introducimos una derivada fraccional derecha general al estilo de Caputo con respecto a una base de funciones absolutamente continuas estrictamente crecientes $g$. Damos varios ejemplos de dichas derivadas fraccionales derechas para diferentes $g$. Sea $f$ una función $p$-veces continuamente diferenciable en $[a, b]$, y sea $L$ un operador diferencial fraccional derecho general tal que $L(f)$ es no-negativo en un subintervalo cerrado crítico $J$ de $[a, b]$. Podemos encontrar una sucesión de polinomios $Q_n$ de grado menor o igual a $n$ tal que $L(Q_n)$ es no-negativo en $J$, más aún $f$ es aproximada uniformemente por $Q_n$ en $[a, b]$. El grado de esta aproximación restringida es dada por una desigualdad usando el primer módulo de continuidad de $f^{(p)}$. Concluimos con aplicaciones del teorema principal de aproximación monótona fraccional derecha para diferentes $g$.

Keywords and Phrases: Right Fractional Monotone Approximation, general right fractional derivative, linear general right fractional differential operator, modulus of continuity.

2010 AMS Mathematics Subject Classification: 26A33, 41A10, 41A17, 41A25, 41A29.
1 Introduction and Preparation

The topic of monotone approximation started in [11] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer \(k\), approximate a given function whose \(k\)th derivative is \(\geq 0\) by polynomials having this property.

In [4] the authors replaced the \(k\)th derivative with a linear ordinary differential operator of order \(k\).

Furthermore in [1], the author generalized the result of [4] for linear right fractional differential operators.

To describe the motivating result here we need

**Definition 1.** ([5]) Let \(\alpha > 0\) and \([\alpha] = m\) (\([\cdot]\) ceiling of the number). Consider \(f \in C^m([-1, 1])\).

We define the right Caputo fractional derivative of \(f\) of order \(\alpha\) as follows:

\[
(D_1^\alpha f)(x) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^1 (t-x)^{m-\alpha-1} f^{(m)}(t) \, dt,
\]

for any \(x \in [-1, 1]\), where \(\Gamma\) is the gamma function \(\Gamma(\nu) = \int_0^\infty e^{-t}t^{\nu-1} \, dt\), \(\nu > 0\).

We set

\[
D_1^\alpha - f(x) = f(x),
\]

\[
D_1^\alpha + f(x) = (-1)^m f^{(m)}(x), \quad \forall x \in [-1, 1].
\]

In [1] we proved

**Theorem 1.1.** Let \(h, k, p\) be integers, \(h\) is even, \(0 \leq h \leq k \leq p\) and let \(f\) be a real function, \(f^{(p)}\) continuous in \([-1, 1]\) with modulus of continuity \(\omega_1(f^{(p)}; \delta), \delta > 0\), there. Let \(\alpha_j(x), j = h, h + 1, \ldots, k\) be real functions, defined and bounded on \([-1, 1]\) and assume for \(x \in [-1, 0]\) that \(\alpha_h(x)\) is either \(\geq\) some number \(\alpha > 0\) or \(\leq\) some number \(\beta < 0\). Let the real numbers \(\alpha_0 = 0 < \alpha_1 < 1 < \alpha_2 < 2 < \ldots < \alpha_p < p\). Here \(D_1^\alpha\) stands for the right Caputo fractional derivative of \(f\) of order \(\alpha\) anchored at 1. Consider the linear right fractional differential operator

\[
L := \sum_{j=h}^{k} \alpha_j(x) \left[D_1^{\alpha_j}\right]
\]

and suppose, throughout \([-1, 0]\),

\[
L(f) \geq 0.
\]

Then, for any \(n \in \mathbb{N}\), there exists a real polynomial \(Q_n(x)\) of degree \(\leq n\) such that

\[
L(Q_n) \geq 0 \quad \text{throughout} \quad [-1, 0],
\]

and

\[
\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq Cn^{k-p} \omega_1 \left(f^{(p)}; \frac{1}{n}\right),
\]

where \(C\) is independent of \(n\) or \(f\).
Notice above that the monotonicity property is only true on \([-1, 0]\), see (5), (6). However the approximation property (7) it is true over the whole interval \([-1, 1]\).

In this article we extend Theorem 1.1 to much more general linear right fractional differential operators.

We use here the following right generalised fractional integral.

**Definition 2.** (see also [8, p. 99]) The right generalised fractional integral of a function \(f\) with respect to given function \(g\) is defined as follows:

Let \(a, b \in \mathbb{R}, a < b, \alpha > 0\). Here \(g \in AC ([a, b])\) (absolutely continuous functions) and is strictly increasing, \(f \in L_\infty ([a, b])\). We set

\[
(I_{a+}^{-\alpha} f)(x) = \frac{1}{\Gamma (\alpha)} \int_{x}^{b} \frac{(g(t) - g(x))^{\alpha-1}}{g'(t)} f(t) dt, \quad x \leq b, \quad (8)
\]

Clearly \((I_{a+}^{-\alpha} f)(b) = 0\).

When \(g\) is the identity function \(id\), we get that \((I_{a+}^{-\alpha} f)(b) = I_{a+}^{-\alpha},\) the ordinary right Riemann-Liouville fractional integral, where

\[
(I_{a+}^{-\alpha} f)(x) = \frac{1}{\Gamma (\alpha)} \int_{x}^{b} (t-x)^{\alpha-1} f(t) dt, \quad x \leq b, \quad (9)
\]

\((I_{a+}^{-\alpha} f)(b) = 0\).

When \(g(x) = \ln x\) on \([a, b]\), \(0 < a < b < \infty\), we get

**Definition 3.** ([8, p. 110]) Let \(0 < a < b < \infty, \alpha > 0\). The right Hadamard fractional integral of order \(\alpha\) is given by

\[
(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma (\alpha)} \int_{x}^{b} \frac{(\ln \frac{y}{x})^{\alpha-1} f(y)}{y} dy, \quad x \leq b, \quad (10)
\]

where \(f \in L_\infty ([a, b])\).

We mention

**Definition 4.** The right fractional exponential integral is defined as follows: Let \(a, b \in \mathbb{R}, a < b, \alpha > 0, f \in L_\infty ([a, b])\). We set

\[
(I_{a+}^{\alpha} e^f)(x) = \frac{1}{\Gamma (\alpha)} \int_{x}^{b} (e^t - e^x)^{\alpha-1} e^t f(t) dt, \quad x \leq b. \quad (11)
\]

**Definition 5.** Let \(a, b \in \mathbb{R}, a < b, \alpha > 0, f \in L_\infty ([a, b]), A > 1\). We introduce the right fractional integral

\[
(I_{a+}^{\alpha} A^f)(x) = \frac{\ln A}{\Gamma (\alpha)} \int_{x}^{b} (A^t - A^x)^{\alpha-1} A^t f(t) dt, \quad x \leq b. \quad (12)
\]
We also give

**Definition 6.** Let $\alpha, \sigma > 0$, $0 \leq a < b < \infty$, $f \in L_\infty ([a, b])$. We set

$$
(K_{b-a}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t^\sigma - x^\sigma)^{\alpha-1} f(t) \alpha t^{\alpha-1} dt, \quad x \leq b. \quad (13)
$$

We introduce the following general right fractional derivative.

**Definition 7.** Let $\alpha > 0$ and $[\alpha] = m$, $([\cdot])$ ceiling of the number. Consider $f \in AC^m ([a, b])$ (space of functions $f$ with $f^{(m-1)} \in AC ([a, b])$). We define the right general fractional derivative of $f$ of order $\alpha$ as follows

$$
(D_{b-a}^\alpha f)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) f^{(m)}(t) dt,
$$

for any $x \in [a, b]$, where $\Gamma$ is the gamma function.

We set

$$
D_{b-a}^m f(x) = (-1)^m f^{(m)}(x), \quad (15)
$$

$$
D_{b-a}^0 f(x) = f(x), \quad \forall x \in [a, b]. \quad (16)
$$

When $g = \text{id}$, then $D_{b-a}^\alpha f = D_{b-a}^\alpha id f$ is the right Caputo fractional derivative.

So we have the specific general right fractional derivatives.

**Definition 8.**

$$
D_{b-a}^\alpha \ln f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b \ln \frac{y}{x}^{m-\alpha-1} f^{(m)}(y) \frac{dy}{y}, \quad 0 < a \leq x \leq b, \quad (17)
$$

$$
D_{b-a}^\alpha e^s f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (e^t - e^x)^{m-\alpha-1} e^t f^{(m)}(t) dt, \quad a \leq x \leq b, \quad (18)
$$

and

$$
D_{b-a}^\alpha A^s f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (A^t - A^x)^{m-\alpha-1} A^t f^{(m)}(t) dt, \quad a \leq x \leq b, \quad (19)
$$

$$
(D_{b-a}^\alpha t^s f)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (t^\sigma - x^\sigma)^{m-\alpha-1} \sigma t^{\alpha-1} f^{(m)}(t) dt, \quad 0 \leq a \leq x \leq b. \quad (20)
$$

We mention

**Theorem 1.2.** (Trigub, [12, 13]) Let $g \in C^p ([0, 1])$, $p \in \mathbb{N}$. Then there exists real polynomial $q_n(x)$ of degree $n \leq n$, $x \in [-1, 1]$, such that

$$
\max_{-1 \leq x \leq 1} |g^{(j)}(x) - q_n^{(j)}(x)| \leq R_p n^{j-p} \omega_1 \left( \frac{g^{(p)}(1)}{n} \right), \quad (21)
$$

where $R_p$ is independent of $n$ or $g$. 

$$
j = 0, 1, ..., p, \quad (21)$$

In [2], based on Theorem 1.2 we proved the following useful here result

**Theorem 1.3.** Let \( f \in C^p ([a, b]) \), \( p \in \mathbb{N} \). Then there exist real polynomials \( Q_n^* (x) \) of degree \( \leq n \in \mathbb{N} \), \( x \in [a, b] \), such that
\[
\max_{a \leq x \leq b} \left| f^{(j)} (x) - Q_n^{(j)} (x) \right| \leq R_p \left( \frac{b - a}{2n} \right)^{p-j} \omega_1 \left( f^{(p)} \left( \frac{b - a}{2n} \right) \right),
\]
(22)\( j = 0, 1, \ldots, p \), where \( R_p \) is independent of \( n \) or \( g \).

**Remark 1.4.** Here \( g \in AC ([a, b]) \) (absolutely continuous functions), \( g \) is increasing over \([a, b] \), \( \alpha > 0 \).

Let \( g(a) = c \), \( g(b) = d \). We want to calculate
\[
I = \int_a^b (g(t) - g(a))^{\alpha-1} g'(t) \, dt.
\]
(23)

Consider the function
\[
f(y) = (y - g(a))^{\alpha-1} = (y - c)^{\alpha-1}, \quad \forall \, y \in [c, d].
\]
(24)

We have that \( f(y) \geq 0 \), it may be \( +\infty \) when \( y = c \) and \( 0 < \alpha < 1 \), but \( f \) is measurable on \([c, d] \).

By [9], Royden, p. 107, exercise 13 d, we get that
\[
(f \circ g)(t) g'(t) = (g(t) - g(a))^{\alpha-1} g'(t)
\]
is measurable on \([a, b] \), and
\[
I = \int_c^d (y - c)^{\alpha-1} \, dy = \frac{(d - c)^\alpha}{\alpha}
\]
(26)

(notice that \( (y - c)^{\alpha-1} \) is Riemann integrable).

That is
\[
I = \frac{(g(b) - g(a))^\alpha}{\alpha}.
\]
(27)

Similarly it holds
\[
\int_a^b (g(t) - g(x))^{\alpha-1} g'(t) \, dt = \frac{(g(b) - g(x))^\alpha}{\alpha}, \quad \forall \, x \in [a, b].
\]
(28)

Finally we will use

**Theorem 1.5.** Let \( \alpha > 0 \), \( N \ni m = [\alpha] \), and \( f \in C^m ([a, b]) \). Then \( (D_x^{\alpha}) f(x) \) is continuous in \( x \in [a, b] \), \( -\infty < a < b < \infty \).

**Proof.** By [3], Apostol, p. 78, we get that \( g^{-1} \) exists and it is strictly increasing on \([g(a), g(b)] \). Since \( g \) is continuous on \([a, b] \), it implies that \( g^{-1} \) is continuous on \([g(a), g(b)] \). Hence \( f^{(m)} \circ g^{-1} \) is a continuous function on \([g(a), g(b)] \).
If \( \alpha = m \in \mathbb{N} \), then the claim is trivial.

We treat the case of \( 0 < \alpha < m \).

It holds that

\[
(D_{b-g}^\alpha f)(x) = (-1)^m \frac{1}{\Gamma(m - \alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) f^{(m)}(t) \, dt =
\]

\[
\frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) \left( f^{(m)} \circ g^{-1} \right)(g(t)) \, dt =
\]

\[
\frac{(-1)^m}{\Gamma(m - \alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{m-\alpha-1} \left( f^{(m)} \circ g^{-1} \right)(z) \, dz.
\]

An explanation follows.

The function

\[
G(z) = (z - g(x))^{m-\alpha-1} \left( f^{(m)} \circ g^{-1} \right)(z)
\]

is integrable on \([g(x), g(b)]\), and by assumption \( g \) is absolutely continuous : \([a, b] \rightarrow [g(a), g(b)]\).

Since \( g \) is monotone (strictly increasing here) the function

\[
(g(t) - g(x))^{m-\alpha-1} g'(t) \left( f^{(m)} \circ g^{-1} \right)(g(t))
\]

is integrable on \([x, b]\) (see [7]). Furthermore it holds (see also [7]),

\[
\frac{(-1)^m}{\Gamma(m - \alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{m-\alpha-1} \left( f^{(m)} \circ g^{-1} \right)(z) \, dz =
\]

\[
\frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) \left( f^{(m)} \circ g^{-1} \right)(g(t)) \, dt
\]

\[
= (D_{b-g}^\alpha f)(x), \quad \forall \ x \in [a, b].
\]

And we can write

\[
(D_{b-g}^\alpha f)(x) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{m-\alpha-1} \left( f^{(m)} \circ g^{-1} \right)(z) \, dz,
\]

\[
(D_{b-g}^\alpha f)(y) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_{g(y)}^{g(b)} (z - g(y))^{m-\alpha-1} \left( f^{(m)} \circ g^{-1} \right)(z) \, dz.
\]

(31)

Here \( a \leq y \leq x \leq b \), and \( g(a) \leq g(y) \leq g(x) \leq g(b) \), and \( 0 \leq g(b) - g(x) \leq g(b) - g(y) \).

Let \( \lambda = z - g(x) \), then \( z = g(x) + \lambda \). Thus

\[
(D_{b-g}^\alpha f)(x) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_0^{g(b) - g(x)} \lambda^{m-\alpha-1} \left( f^{(m)} \circ g^{-1} \right)(g(x) + \lambda) \, d\lambda.
\]

(32)

Clearly, see that \( g(x) \leq z \leq g(b) \), and \( 0 \leq \lambda \leq g(b) - g(x) \).
Similarly
\[
(D_{b-}^{\alpha} f)(y) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_{0}^{g(y)} \lambda^{m-\alpha-1} \left( f^{(m)}(\lambda) \right) d\lambda.
\]
Hence it holds
\[
(D_{b-}^{\alpha} f)(y) - (D_{b-}^{\alpha} f)(x) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_{0}^{g(y)} \lambda^{m-\alpha-1} \left( f^{(m)}(\lambda) \right) d\lambda + \int_{g(y)}^{g(x)} \lambda^{m-\alpha-1} \left( f^{(m)}(\lambda) \right) d\lambda.
\]
Thus we obtain
\[
\left| (D_{b-}^{\alpha} f)(y) - (D_{b-}^{\alpha} f)(x) \right| \leq \frac{1}{\Gamma(m - \alpha)} \left[ (g(y) - g(x))^{m-\alpha} \omega_1 \left( f^{(m)} \right) \right] + \frac{\| f^{(m)} \|_{\infty, [a,b]}}{m - \alpha} \left[ (g(b) - g(y))^{m-\alpha} - (g(b) - g(x))^{m-\alpha} \right].
\]
As \( y \to x \), then \( g(y) \to g(x) \) (since \( g \in AC([a,b]) \)). So that \( (\xi) \to 0 \). As a result
\[
(D_{b-}^{\alpha} f)(y) \to (D_{b-}^{\alpha} f)(x),
\]
proving that \( (D_{b-}^{\alpha} f)(x) \) is continuous in \( x \in [a,b] \).

\[\square\]

2 Main Result

We present

**Theorem 2.1.** Here we assume that \( g(b) - g(a) > 0 \). Let \( h, k, p \) be integers, \( h \) is even, \( 0 \leq h \leq k \leq p \) and let \( f \in C^p([a,b]), \ a < b, \) with modulus of continuity \( \omega_1 \left( f^{(p)}, \delta \right), 0 < \delta \leq b - a \). Let \( \alpha_j(x), j = h, h + 1, ..., k \) be real functions, defined and bounded on \([a,b]\) and assume for \( x \in [a, g^{-1}(g(b) - 1)] \) that \( \alpha_h(x) \) is either \( \geq \) some number \( \alpha^* > 0 \), or \( \leq \) some number \( \beta^* < 0 \). Let the real numbers \( \alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < ... < \alpha_p \leq p \). Consider the linear right general fractional differential operator
\[
L = \sum_{j=h}^{k} \alpha_j(x) \left[ D_{b-}^{\alpha_j} \right],
\]
and suppose, throughout \([a, g^{-1}(g(b) - 1)],\)
\[
L(f) \geq 0.
\]
Then, for any \( n \in \mathbb{N} \), there exists a real polynomial \( Q_n(x) \) of degree \( \leq n \) such that

\[
L(Q_n) \geq 0 \quad \text{throughout} \quad [a, g^{-1}(g(b)) - 1],
\]

and

\[
\max_{x \in [a, b]} |f(x) - Q_n(x)| \leq Cn^k - p \omega_1 \left( f^{(p)}, \frac{b - a}{2n} \right),
\]

where \( C \) is independent of \( n \) or \( f \).

**Proof.** of Theorem 2.1.

Here \( h, k, p \in \mathbb{Z}_+ \), \( 0 \leq h \leq k \leq p \). Let \( \alpha_j > 0 \), \( j = 1, ..., p \), such that \( 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3 < ... < \alpha_p \leq p \). That is \( \lfloor \alpha_j \rfloor = j \), \( j = 1, ..., p \).

Let \( Q_n^j(x) \) be as in Theorem 1.3.

We have that

\[
\left( D_{b - g}^{\alpha_j} \right)(x) = \left( \frac{(-1)^j}{\Gamma(j - \alpha_j)} \right) \int_x^b (g(t) - g(x))^{j - \alpha_j - 1} g'(t) f^{(j)}(t) dt,
\]

and

\[
\left( D_{b - g}^{\alpha_j} Q_n^j \right)(x) = \left( \frac{(-1)^j}{\Gamma(j - \alpha_j)} \right) \int_x^b (g(t) - g(x))^{j - \alpha_j - 1} g'(t) Q_n^j(t) dt,
\]

\( j = 1, ..., p \).

Also it holds

\[
\left( D_{b - g}^{\alpha_j} f \right)(x) = (-1)^j f^{(j)}(x), \quad \left( D_{b - g}^{\alpha_j} Q_n^j \right)(x) = (-1)^j Q_n^j(x), \quad j = 1, ..., p.
\]

By [10], we get that there exists \( g' \) a.e., and \( g' \) is measurable and non-negative.

We notice that

\[
\left| \left( D_{b - g}^{\alpha_j} f \right)(x) - \left( D_{b - g}^{\alpha_j} Q_n^j \right)(x) \right| =
\]

\[
\frac{1}{\Gamma(j - \alpha_j)} \left| \int_x^b (g(x) - g(t))^{j - \alpha_j - 1} g'(t) \left( f^{(j)}(t) - Q_n^j(t) \right) dt \right| \leq
\]

\[
\frac{1}{\Gamma(j - \alpha_j + 1)} \left( \int_x^b (g(b) - g(t))^{j - \alpha_j} g'(t) dt \right) R_p \left( \frac{b - a}{2n} \right)^{p - j} \omega_1 \left( f^{(p)}, \frac{b - a}{2n} \right)
\]

\[
\leq \frac{28}{\Gamma(j - \alpha_j + 1)} R_p \left( \frac{b - a}{2n} \right)^{p - j} \omega_1 \left( f^{(p)}, \frac{b - a}{2n} \right) \leq
\]

\[
\frac{(b - g(x))^{j - \alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left( \frac{b - a}{2n} \right)^{p - j} \omega_1 \left( f^{(p)}, \frac{b - a}{2n} \right).
\]

(44)
Hence $\forall x \in [a, b]$, it holds
\[ \left| \left( D_{b_{-i}g}^{\alpha_j} f \right)(x) - D_{b_{-i}g}^{\alpha_j} Q_n^*(x) \right| \leq \frac{(g(b) - g(a))^{1-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left( \frac{b - a}{2n} \right)^{p-j} \omega_1 \left( \frac{f(p)}{2n} \right) \] (45)
and
\[ \max_{x \in [a, b]} \left| \left( D_{b_{-i}g}^{\alpha_j} f \right)(x) - D_{b_{-i}g}^{\alpha_j} Q_n^*(x) \right| \leq \frac{(g(b) - g(a))^{1-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left( \frac{b - a}{2n} \right)^{p-j} \omega_1 \left( \frac{f(p)}{2n} \right), \] (46)
j = 0, 1, ..., p.

Above we set $D_{b_{-i}g}^0 f(x) = f(x)$, $D_{b_{-i}g}^0 Q_n^*(x) = Q_n^*(x)$, $\forall x \in [a, b]$, and $\alpha_0 = 0$, i.e. $[\alpha_0] = 0$.

Put
\[ s_j = \sup_{a \leq x \leq b} \left| \alpha_h^{-1}(x) \alpha_j(x) \right|, \quad j = h, ..., k, \] (47)
and
\[ \eta_n = R_p \omega_1 \left( \frac{f(p)}{2n} \right) \left( \sum_{j=h}^{k} \frac{s_j (g(b) - g(a))^{1-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \left( \frac{b - a}{2n} \right)^{p-j} \right) \] (48)

I. Suppose, throughout $[a, g^{-1}(g(b) - 1)]$ holds
\[ \alpha_h(x) \geq \alpha^* > 0. \]
Let $Q_n(x)$, $x \in [a, b]$, be a real polynomial of degree $\leq n$, according to Theorem 1.3 and (46), so that
\[ \max_{x \in [a, b]} \left| D_{b_{-i}g}^{\alpha_j} \left( f(x) + \eta_n (h!)^{-1} x^h \right) - D_{b_{-i}g}^{\alpha_j} Q_n(x) \right| \leq \] (49)
\[ \frac{(g(b) - g(a))^{1-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left( \frac{b - a}{2n} \right)^{p-j} \omega_1 \left( \frac{f(p)}{2n} \right), \]
j = 0, 1, ..., p.

In particular ($j = 0$) holds
\[ \max_{x \in [a, b]} \left| f(x) + \eta_n (h!)^{-1} x^h \right| - Q_n(x) \left| \leq R_p \left( \frac{b - a}{2n} \right)^p \omega_1 \left( \frac{f(p)}{2n} \right), \] (50)
and
\[ \max_{x \in [a, b]} \left| f(x) - Q_n(x) \right| \leq \eta_n (h!)^{-1} (\max(|a|, |b|))^h + R_p \left( \frac{b - a}{2n} \right)^p \omega_1 \left( \frac{f(p)}{2n} \right) \] (51)
\[ = \eta_n (h!)^{-1} \max(|a|^h, |b|^h) + R_p \left( \frac{b - a}{2n} \right)^p \omega_1 \left( \frac{f(p)}{2n} \right) \]
\[ = \eta_n (h!)^{-1} \max(|a|^h, |b|^h) + R_p \left( \frac{b - a}{2n} \right)^p \omega_1 \left( \frac{f(p)}{2n} \right). \]
+\rho_p \left( \frac{b-a}{2n} \right)^p \omega_1 \left( f^{(p)}, \frac{b-a}{2n} \right) \leq \\
\rho_p \omega_1 \left( f^{(p)}, \frac{b-a}{2n} \right) n^{k-p}.

\left[ \sum_{j=h}^{k} s_j \left( \frac{g(b) - g(a)}{\Gamma(j - \alpha_j + 1)} \left( \frac{b-a}{2} \right)^{p-j} \right) (h!)^{-1} \max \left( |a|^h, |b|^h \right) + \left( \frac{b-a}{2} \right)^p \right]. \tag{52}

We have found that

\max_{x \in [a,b]} |f(x) - Q_n(x)| \leq \rho_p \left[ \left( \frac{b-a}{2} \right)^p + (h!)^{-1} \max \left( |a|^h, |b|^h \right) \right] \left[ \sum_{j=h}^{k} s_j \left( \frac{g(b) - g(a)}{\Gamma(j - \alpha_j + 1)} \left( \frac{b-a}{2} \right)^{p-j} \right) \right] n^{k-p} \omega_1 \left( f^{(p)}, \frac{b-a}{2n} \right), \tag{53}

proving (40).

Notice for \( j = h + 1, \ldots, k \), that

\left( D_{b^{-g}}^{\alpha_j} x^h \right) = \frac{(-1)^j}{\Gamma(j - \alpha_j)} \int_x^b (g(t) - g(x))^{j-\alpha_j} g'(t) (t^h)^{(j)} \, dt = 0. \tag{54}

Here

\[ L = \sum_{j=h}^{k} \alpha_j \] \[ D_{b^{-g}}^{\alpha_j} \] \[ x^h \]

and suppose, throughout \([a, g^{-1}(g(b) - 1)]\), \( L \rho_0 \geq 0 \). So over \( a \leq x \leq g^{-1}(g(b) - 1) \), we get

\[ \alpha_n^{-1}(x) L(Q_n(x)) \overset{(54)}{=} \alpha_n^{-1}(x) L(f(x)) + \frac{\eta_n}{h!} \left( D_{b^{-g}}^{\alpha_n} x^h \right) + \]

\[ \sum_{j=h}^{k} \alpha_n^{-1}(x) \left[ D_{b^{-g}}^{\alpha_j} Q_n(x) - D_{b^{-g}}^{\alpha_j} f(x) - \frac{\eta_n}{h!} D_{b^{-g}}^{\alpha_j} x^h \right] \overset{(49)}{\geq} \]

\[ \frac{\eta_n}{h!} \left( D_{b^{-g}}^{\alpha_n} x^h \right) - \left( \sum_{j=h}^{k} \frac{\eta_n}{h!} \left( \frac{D_{b^{-g}}^{\alpha_j} x^h}{h!} \right) \right) \overset{(48)}{=} \]

\[ \frac{\eta_n}{h!} \left( D_{b^{-g}}^{\alpha_n} x^h \right) - \eta_n = \eta_n \left( \frac{D_{b^{-g}}^{\alpha_n} x^h}{h!} - 1 \right) = \]

\[ \eta_n \left( \frac{1}{\Gamma(h - \alpha_n)} \right) \int_x^b (g(t) - g(x))^{h-\alpha_n-1} g'(t) (t^h)^{(h)} \, dt - 1 = \]

\[ \eta_n \left( \frac{h!}{h! \Gamma(h - \alpha_n)} \right) \int_x^b (g(t) - g(x))^{h-\alpha_n-1} g'(t) \, dt - 1 \overset{(28)}{=} \]
\[ \eta_n \left( \frac{(g(b) - g(x))^{h-\alpha_n}}{\Gamma(h - \alpha_n + 1)} - 1 \right) = \] 

\[ \eta_n \left( \frac{(g(b) - g(x))^{h-\alpha_n} - \Gamma(h - \alpha_n + 1)}{\Gamma(h - \alpha_n + 1)} \right) \geq \] 

\[ \eta_n \left( \frac{1 - \Gamma(h - \alpha_n + 1)}{\Gamma(h - \alpha_n + 1)} \right) \geq 0. \] 

Clearly here \( g(b) - g(x) \geq 1. \)

Hence

\[ L(Q_n(x)) \geq 0, \text{ for } x \in [a, g^{-1}(g(b) - 1)]. \] 

A further explanation follows: We know \( \Gamma(1) = 1, \Gamma(2) = 1, \) and \( \Gamma \) is convex and positive on \((0,\infty).\) Here \( 0 \leq h - \alpha_k < 1 \) and \( 1 \leq h - \alpha_k + 1 < 2. \) Thus

\[ \Gamma(h - \alpha_k + 1) \leq 1 \text{ and } 1 - \Gamma(h - \alpha_k + 1) \geq 0. \] 

II. Suppose, throughout \([a, g^{-1}(g(b) - 1)], \alpha_k(x) \leq \beta^* < 0.\)

Let \( Q_n(x), x \in [a, b] \) be a real polynomial of degree \( \leq n, \) according to Theorem 1.3 and (46), so that

\[ \max_{x \in [a,b]} \left| D_{b-g}^{\alpha_j} \left( f(x) - \eta_n(h!)^{-1} x^h \right) - \left( D_{b-g}^{\alpha_j} Q_n \right)(x) \right| \leq \] 

\[ \frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \omega_1 \left( \frac{f^{(p)}, b-a}{2n} \right), \] 

\( j = 0, 1, \ldots, p. \)

In particular \((j = 0)\) holds

\[ \max_{x \in [a,b]} \left| \left( f(x) - \eta_n(h!)^{-1} x^h \right) - Q_n(x) \right| \leq R_p \left( \frac{b-a}{2n} \right)^p \omega_1 \left( \frac{f^{(p)}, b-a}{2n} \right), \] 

and

\[ \max_{x \in [a,b]} |f(x) - Q_n(x)| \leq \eta_n(h!)^{-1} \left( \max(|a|, |b|) \right)^h + R_p \left( \frac{b-a}{2n} \right)^p \omega_1 \left( \frac{f^{(p)}, b-a}{2n} \right) \] 

\[ = \eta_n(h!)^{-1} \max(|a|^h, |b|^h) + R_p \left( \frac{b-a}{2n} \right)^p \omega_1 \left( \frac{f^{(p)}, b-a}{2n} \right), \] 

e tc.

We find again that

\[ \max_{x \in [a,b]} |f(x) - Q_n(x)| \leq R_p \left( \frac{b-a}{2} \right)^p + (h!)^{-1} \max(|a|^h, |b|^h). \]
Reproving (40).

Here again

\[
L = \sum_{j=h}^{k} \alpha_j(x) \left[ D_{b^{-g}}^\alpha \right],
\]

and suppose, throughout \([a, g^{-1}(b-1)]\), \(Lf \geq 0\). So over \(a \leq x \leq g^{-1}(b-1)\), we get

\[
\alpha_n^{-1}(x) L(Q_n(x)) \overset{(54)}{=} \alpha_n^{-1}(x) L(f(x)) - \eta_n \left( D_{b^{-g}}^\alpha (x^h) \right) + \sum_{j=h}^{k} \alpha_j^{-1}(x) \alpha_j(x) \left[ D_{b^{-g}}^\alpha Q_n(x) - D_{b^{-g}}^\alpha f(x) + \eta_n \left( D_{b^{-g}}^\alpha x^h \right) \right] \overset{(62)}{\leq} \]

\[
- \eta_n \left( D_{b^{-g}}^\alpha (x^h) \right) + \sum_{j=h}^{k} \left( \frac{1}{j!} \sum_{i=1}^{j-1} \frac{\alpha_i}{i!} \right) \left( \frac{a}{2} \right)^{-p-j} R_p \left( f^{(p)}(x), b-a \right)
\]

\[
\overset{(48)}{=} - \eta_n \left( D_{b^{-g}}^\alpha (x^h) \right) + \eta_n = \eta_n \left( 1 - \frac{D_{b^{-g}}^\alpha (x^h)}{h!} \right) \overset{(48)}{=} \]

\[
\eta_n \left( 1 - \frac{h!}{h!(h - \alpha_h)} \right) \int_x^b (g(t) - g(x))^{h-\alpha_h-1} g'(t) \left( \frac{t}{h} \right)^{\alpha_h} dt \overset{(28)}{=} \]

\[
\eta_n \left( 1 - \frac{h!}{h!(h - \alpha_h)} \right) \int_x^b (g(t) - g(x))^{h-\alpha_h-1} g'(t) dt \overset{(28)}{=} \]

\[
\eta_n \left( 1 - \frac{(g(b) - g(x))^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} \right) \overset{(69)}{=} \]

\[
\eta_n \left( \frac{\Gamma(h - \alpha_h + 1) - (g(b) - g(x))^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} \right) \overset{(61)}{\leq} \]

\[
\eta_n \left( \frac{1 - (g(b) - g(x))^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} \right) \leq 0. \]

Hence again

\[
L(Q_n(x)) \geq 0, \ \forall \ x \in [a, g^{-1}(b-1)].
\]

The case of \(\alpha_h = h\) is trivially concluded from the above. The proof of the theorem is now over. \(\square\)

We make
Remark 2.2. By Theorem 1.5 we have that $D_{b^{-\ln x}}^{\alpha_j} f$ are continuous functions, $j = 0, 1, \ldots, p$. Suppose that $\alpha_h (x), \ldots, \alpha_k (x)$ are continuous functions on $[a, b]$, and $L (f) \geq 0$ on $[a, g^{-1} (g(b) - 1)]$ is replaced by $L (f) > 0$ on $[a, g^{-1} (g(b) - 1)]$. Disregard the assumption made in the main theorem on $\alpha_h (x)$. For $n \in \mathbb{N}$, let $Q_n (x)$ be the $Q_n^*$ of Theorem 1.3, and $f$ as in Theorem 1.3 (same as in Theorem 2.1). Then $Q_n (x)$ converges to $f(x)$ at the Jackson rate $\frac{1}{n^p}$ ([6], p. 18, Theorem VIII) and at the same time, since $L (Q_n)$ converges uniformly to $L(f)$ on $[a, b]$, $L (Q_n) > 0$ on $[a, g^{-1} (g(b) - 1)]$ for all $n$ sufficiently large.

3 Applications (to Theorem 2.1)

1) When $g (x) = \ln x$ on $[a, b]$, $0 < a < b < \infty$.

Here we would assume that $b > ac$, $\alpha_h (x)$ restriction true on $[a, \frac{b}{c}]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j (x) \left[D_{b^{-\ln x}}^{\alpha_j} f \right] \geq 0,$$

throughout $[a, \frac{b}{c}]$.

Then $L (Q_n) \geq 0$ on $[a, \frac{b}{c}]$.

2) When $g (x) = e^x$ on $[a, b]$, $0 < b < \infty$.

Here we assume that $b > \ln (1 + e^a)$, $\alpha_h (x)$ restriction true on $[a, \ln (e^b - 1)]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j (x) \left[D_{b^{-e^x}}^{\alpha_j} f \right] \geq 0,$$

throughout $[a, \ln (e^b - 1)]$.

Then $L (Q_n) \geq 0$ on $[a, \ln (e^b - 1)]$.

3) When, $A > 1$, $g (x) = A^x$ on $[a, b]$, $0 < b < \infty$.

Here we assume that $b > \log_A (1 + A^a)$, $\alpha_h (x)$ restriction true on $[a, \log_A (A^b - 1)]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j (x) \left[D_{b^{-A^x}}^{\alpha_j} f \right] \geq 0,$$

throughout $[a, \log_A (A^b - 1)]$.

Then $L (Q_n) \geq 0$ on $[a, \log_A (A^b - 1)]$.

4) When $\sigma > 0$, $g (x) = x^\sigma$, $0 \leq a < b < \infty$.

Here we assume that $b > (1 + a^\sigma)^\frac{1}{\sigma}$, $\alpha_h (x)$ restriction true on $[a, (b^\sigma - a^\sigma)^\frac{1}{\sigma}]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j (x) \left[D_{b^{-x^\sigma}}^{\alpha_j} f \right] \geq 0$$

(75)
throughout \([a, (b^\sigma - 1)^{1/\sigma}]\).

Then \(L(Q_n) \geq 0\) on \([a, (b^\sigma - 1)^{1/\sigma}]\).

Received: April 2015. Accepted: July 2015.

References


