S-paracompactness modulo an ideal

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ABSTRACT

The notion of S-paracompactness modulo an ideal was introduced and studied in [15]. In this paper, we introduce and investigate the notion of αS-paracompact subset modulo an ideal which is a generalization of the notions of αS-paracompact set [1] and α-paracompact set modulo an ideal [7].

RESUMEN

La noción de S-paracompacidad módulo un ideal fue introducida y estudiada en [15]. En este artículo, introducimos e investigamos la noción de un subconjunto αS-paracompacto módulo un ideal, que es una generalización de las nociones de conjunto αS-paracompacto [1] y conjunto α-paracompacto módulo un ideal [7].

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1 Introduction

The concept of $\alpha$-paracompact subset modulo an ideal was defined and investigated by Ergun and Noiri [7]. The notions of $S$-paracompact spaces and $\alpha S$-paracompact subsets were introduced in 2006 by Al-Zoubi [1] and also have been studied by Li and Song [13]. Very recently, Sanabria, Rosas, Carpintero, Salas and García [15] have introduced and investigated the concept of $S$-paracompact space with respect to an ideal as a generalization of the $S$-paracompact spaces. In this paper, we introduce the notion of $\alpha S$-paracompact subset modulo an ideal which is a generalization of both $\alpha S$-paracompact subset [1] and $\alpha$-paracompact subset modulo an ideal.

2 Preliminaries

Throughout this paper, $(X, \tau)$ always means a topological space on which no separation axioms are assumed unless explicitly stated. If $A$ is a subset of $(X, \tau)$, we denote the closure of $A$ and the interior of $A$ by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. Also, we denote by $\varphi(X)$ the class of all subset of $X$. A subset $A$ of $(X, \tau)$ is said to be semi-open [11] (resp. semi-preopen [2]) if $A \subseteq \text{Cl}(|\text{Int}(A)|)$ (resp. $A \subseteq \text{Cl}(|\text{Cl}(A)|)$). The complement of a semi-open set is called a semi-closed set. The semi-closure of $A$, denoted by $\text{sCl}(A)$, is defined by the intersection of all semi-closed sets containing $A$. The collection of all semi-open sets of a topological space $(X, \tau)$ is denoted by $\text{SO}(X, \tau)$. A collection $\mathcal{V}$ of subsets of a space $(X, \tau)$ is said to be locally finite, if for each $x \in X$ there exists $U_x \in \tau$ containing $x$ and $U_x$ intersects at most finitely many members of $\mathcal{V}$. A space $(X, \tau)$ is said to be paracompact (resp. $S$-paracompact [1]), if every open cover of $X$ has a locally finite open (resp. semi-open) refinement which covers to $X$ (we do not require a refinement to be a cover).

Lemma 2.1. Let $(X, \tau)$ be a space. Then, the following properties hold:

(1) If $(A, \tau_A)$ is a subspace of $(X, \tau)$, $B \subseteq A$ and $B \in \text{SO}(X, \tau)$, then $B \in \text{SO}(A, \tau_A)$ [11].

(2) If $A \in \tau$ and $B \in \text{SO}(X, \tau)$, then $A \cap B \in \text{SO}(X, \tau)$ [4].

(3) If $(A, \tau_A)$ is an open subspace of $(X, \tau)$, $B \subseteq A$ and $B \in \text{SO}(A, \tau_A)$, then $B \in \text{SO}(X, \tau)$ [5].

An ideal $\mathcal{I}$ on a nonempty set $X$ is a nonempty collection of subset of $X$ which satisfies the following two properties:

(1) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$;

(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

In this paper, the triplet $(X, \tau, \mathcal{I})$ denote a topological space $(X, \tau)$ together with an ideal $\mathcal{I}$ on $X$ and will simply called a space. Given a space $(X, \tau, \mathcal{I})$, a set operator $(.)^\ast : \varphi(X) \rightarrow \varphi(X)$, called the local function [10] of $A$ with respect to $\tau$ and $\mathcal{I}$, is defined as follows: for $A \subseteq X$,
\[ A^*(I, \tau) = \{ x \in X : U \cap A \notin I \text{ for every } U \in \tau(x) \}, \text{ where } \tau(x) = \{ U \in \tau : x \in U \}. \] When there is no chance for confusion, we will simply write \( A^* \) for \( A^*(I, \tau) \). In general, \( X^* \) is a proper subset of \( X \). The hypothesis \( X = X^* \) is equivalent to the hypothesis \( \tau \cap I = \emptyset \). According to [14], we call the ideals which satisfy this hypothesis \( \tau \)-boundary ideals. Note that \( \text{Cl}^*(A) = A \cup A^* \) defines a Kuratowski closure for a topology \( \tau^*(I) \), finer than \( \tau \). A basis \( \beta(I, \tau) \) for \( \tau^*(I) \) can be described as follows: \( \beta(I, \tau) = \{ V \setminus J : V \in \tau \text{ and } J \in I \} \). When there is no chance for confusion, we will simply write \( \tau^* \) for \( \tau^*(I) \) and \( \beta \) for \( \beta(I, \tau) \). In the sequel, the ideal of nowhere dense (resp. meager) subsets of \( (X, \tau) \) is denoted by \( \mathcal{N} \) (resp. \( \mathcal{M} \)).

### 3 \( \alpha S \)-paracompactness modulo an ideal

In this section, we shall introduce and study the \( \alpha S \)-paracompact subsets modulo an ideal \( I \), which is a natural generalization of \( \alpha S \)-paracompact subsets. First recall some notions of paracompactness.

**Definition 3.1.** A subset \( A \) of a space \( (X, \tau) \) is said to be \( \alpha \)-paracompact [3] (resp. \( \alpha \)-almost paracompact [9]) if for any open cover \( U \) of \( A \), there exists a locally finite collection \( V \) of open sets such that \( V \) refines \( U \) and \( A \subset \bigcup \{ V : V \in V \} \) (resp. \( A \subset \bigcup \{ \text{Cl}(V) : V \in V \} \) ). A space \( (X, \tau) \) is said to be paracompact (resp. almost-paracompact) if \( X \) is \( \alpha \)-paracompact (resp. \( \alpha \)-almost paracompact).

**Definition 3.2.** A subset \( A \) of a space \( (X, \tau, I) \) is said to be \( \alpha \)-paracompact modulo \( I \) [7] (briefly \( \alpha \)-paracompact (mod \( I \))), if for any open cover \( U \) of \( A \), there exist \( I \in I \) and a locally finite collection \( V \) of open sets such that \( V \) refines \( U \) and \( A \subset \bigcup \{ V : V \in V \} \cup I \).

A space \( (X, \tau, I) \) is said to be \( I \)-paracompact or paracompact with respect to \( I \) [16], if \( X \) is \( \alpha \)-paracompact modulo \( I \). In the present, it is called paracompact modulo \( I \) (or briefly paracompact (mod \( I \))).

**Definition 3.3.** A subset \( A \) of a space \( (X, \tau) \) is said to be \( \alpha S \)-paracompact [1] if for any open cover \( U \) of \( A \), there exists a locally finite collection \( V \) of open sets such that \( V \) refines \( U \) and \( A \subset \bigcup \{ V : V \in V \} \). A space \( (X, \tau) \) is said to be \( S \)-paracompact if \( X \) is \( \alpha S \)-paracompact.

Now, we give the definition of \( \alpha S \)-paracompact subset modulo an ideal \( I \).

**Definition 3.4.** A subset \( A \) of a space \( (X, \tau, I) \) is said to be \( \alpha S \)-paracompact modulo \( I \) (briefly \( \alpha S \)-paracompact (mod \( I \))), if for any open cover \( U \) of \( A \), there exist \( I \in I \) and a locally finite collection \( V \) of semi-open sets such that \( V \) refines \( U \) and \( A \subset \bigcup \{ V : V \in V \} \cup I \).

A space \( (X, \tau, I) \) is said to be \( I \)-S-paracompact or \( S \)-paracompact with respect to \( I \) [15], if \( X \) is \( \alpha S \)-paracompact modulo \( I \). In the present, it is called \( S \)-paracompact modulo \( I \) (or briefly \( S \)-paracompact (mod \( I \))). We say that \( A \) is \( S \)-paracompact (mod \( I \)) if \( (A, \tau_\alpha, I_\alpha) \) is \( S \)-paracompact (mod \( I_\alpha \)) as a subspace, where \( \tau_\alpha \) is the relative topology induced on \( A \) by \( \tau \) and \( I_\alpha = \{ I \cap A : I \in I \} \).
Proposition 3.1. Let $A$ be a subset of a space $(X, \tau)$ and $\mathcal{I}$ an ideal on $(X, \tau)$. Then, the following properties hold:

1. If $A$ is $\alpha$-paracompact (mod $\mathcal{I}$), then $A$ is $\alpha S$-paracompact (mod $\mathcal{I}$).

2. Every $I \in \mathcal{I}$ is an $\alpha S$-paracompact (mod $\mathcal{I}$).

3. $(X, \tau, \mathcal{I})$ is $S$-paracompact (mod $\mathcal{I}$) if there exists $I \in \mathcal{I}$ such that $X - I$ is $\alpha S$-paracompact (mod $\mathcal{I}$).

4. $A$ is $\alpha S$-paracompact if and only if it is $\alpha S$-paracompact (mod $\{\emptyset\}$).

Proof. (1) Follows from the fact that every open set is semi-open.

(2) Suppose that there exists $I \in \mathcal{I}$ such that $I$ is not $\alpha S$-paracompact (mod $\mathcal{I}$). Then, there exists an open cover $\mathcal{U}$ of $I$ such that $I \not\subseteq \bigcup\{V : V \in \mathcal{V}\} \cup J$ for every $J \in \mathcal{I}$ and every locally finite collection $\mathcal{V}$ which refines $\mathcal{U}$. This is a contradiction, because $I \in \mathcal{I}$ and $I \subseteq \bigcup\{V : V \in \mathcal{V}\} \cup J$. Thus, $(X - I) \cup I \subseteq \bigcup\{V : V \in \mathcal{V}\} \cup (J \cup I)$ and as $J \cup I \in \mathcal{I}$, we have $(X, \tau, \mathcal{I})$ is $S$-paracompact (mod $\mathcal{I}$).

(3) Suppose that there exists $I \in \mathcal{I}$ such that $X - I$ is $\alpha S$-paracompact (mod $\mathcal{I}$) and let $\mathcal{U}$ be an open cover of $X$. Then, $\mathcal{U}$ is an open cover of $X - I$ and hence there exist $J \in \mathcal{I}$ and a locally finite collection $\mathcal{V}$ of semi-open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $X - I \subseteq \bigcup\{V : V \in \mathcal{V}\} \cup J$. Thus, $X = (X - I) \cup I \subseteq \bigcup\{V : V \in \mathcal{V}\} \cup (J \cup I)$ and as $J \cup I \in \mathcal{I}$, we have $(X, \tau, \mathcal{I})$ is $S$-paracompact (mod $\mathcal{I}$).

(4) It is obvious. □

Now, we give some comments related with the Proposition 3.1.

Remark 3.1. According to Proposition 3.1(1), every $\alpha$-paracompact (mod $\mathcal{I}$) (resp. $\alpha S$-paracompact) subset is $\alpha S$-paracompact (mod $\mathcal{I}$), and from this point of view, the notion of $\alpha S$-paracompact (mod $\mathcal{I}$) subset is a natural generalization of the notion of $\alpha$-paracompact (mod $\mathcal{I}$) (resp. $\alpha S$-paracompact) subset. On the other hand, in Example 2.11 of [13], it is shows that there exists a semiregular Hausdorff space $X$ and a regular closed subset $M$ of $X$ such that $M$ is an $\alpha S$-paracompact (mod $\{\emptyset\}$) subset of $X$, but $M$ is not $\alpha$-paracompact (mod $\{\emptyset\}$). Thus, the converse of Proposition 3.1(1) in general is not true.

Proposition 3.2. Let $A$ be a subset of a space $(X, \tau)$ and $\mathcal{I}$ an ideal on $(X, \tau)$. Then, the following properties hold:

1. If $A$ is a semi-open and $\alpha S$-paracompact (mod $\mathcal{I}$) set and $\mathcal{I}$ is $\tau$-boundary, then $A$ is $\alpha$-almost paracompact.

2. A semi-preopen set $A$ is $\alpha S$-paracompact (mod $\mathcal{N}$) if and only if it is $\alpha$-almost paracompact.

Proof. (1) Let $\mathcal{U}$ be any open cover of $A$. Then there exist $I \in \mathcal{I}$ and a locally finite collection $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of semi-open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $A \subseteq \bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I$. Since $A$ is
semi-open, $A \subset \text{Cl}(\text{Int}(A))$ and as $I$ is $\tau$-boundary, $\text{Int}(I) = \emptyset$. Now, by the locally finiteness of $V$, the collection $V' = \{\text{Int}(V_\lambda) : \lambda \in \Lambda\}$ is also locally finite, it follows that

$$A \subset \text{Cl}(\text{Int}(A)) \subset \text{Cl}\left(\text{Int}\left(\bigcup_{\lambda \in \Lambda} V_\lambda \cup I\right)\right)$$

$$\subset \text{Cl}\left(\text{Int}\left(\bigcup_{\lambda \in \Lambda} \text{Cl}(\text{Int}(V_\lambda)) \cup I\right)\right)$$

$$= \text{Cl}\left(\text{Int}\left(\bigcup_{\lambda \in \Lambda} \text{Cl}(\text{Int}(V_\lambda)) \cup \text{Int}(I)\right)\right)$$

$$= \text{Cl}\left(\text{Int}\left(\bigcup_{\lambda \in \Lambda} \text{Cl}(\text{Int}(V_\lambda))\right)\right)$$

$$\subset \text{Cl}\left(\bigcup_{\lambda \in \Lambda} \text{Cl}(\text{Int}(V_\lambda))\right) = \bigcup_{\lambda \in \Lambda} \text{Cl}(\text{Int}(V_\lambda)).$$

If $W_\lambda = \text{Int}(V_\lambda)$, then $A \subset \bigcup_{\lambda \in \Lambda} \text{Cl}(W_\lambda)$. Observe that $W_\lambda$ is open for each $\lambda \in \Lambda$ and $W_\lambda \subset V_\lambda \subset U$ for some $U \in \mathcal{U}$, hence $W = \{W_\lambda : \lambda \in \Lambda\}$ is a locally finite open refinement of $\mathcal{U}$. Therefore, $A$ is $\alpha$-almost paracompact.

(2) Similar to the proof of (1), if $A$ is semi-preopen, then

$$A \subset \text{Cl}(\text{Int}(\text{Cl}(A))) \subset \text{Cl}\left(\text{Int}\left(\bigcup_{\lambda \in \Lambda} V_\lambda \cup I\right)\right)$$

$$= \text{Cl}\left(\text{Int}\left(\bigcup_{\lambda \in \Lambda} V_\lambda \cup \text{Cl}(I)\right)\right)$$

$$= \text{Cl}\left(\text{Int}\left(\bigcup_{\lambda \in \Lambda} V_\lambda \cup \text{Int}(\text{Cl}(I))\right)\right)$$

$$= \text{Cl}\left(\text{Int}\left(\bigcup_{\lambda \in \Lambda} \text{Cl}(V_\lambda)\right)\right)$$

$$\subset \text{Cl}\left(\text{Int}\left(\bigcup_{\lambda \in \Lambda} \text{Cl}(\text{Int}(V_\lambda))\right)\right)$$

$$= \text{Cl}\left(\text{Int}\left(\bigcup_{\lambda \in \Lambda} \text{Int}(V_\lambda)\right)\right)$$

$$\subset \text{Cl}\left(\bigcup_{\lambda \in \Lambda} \text{Cl}(\text{Int}(V_\lambda))\right) = \bigcup_{\lambda \in \Lambda} \text{Cl}(\text{Int}(V_\lambda)).$$
Therefore, the proof follows. \hfill \square

As a consequence of Proposition 3.2, we obtain the following result.

**Corollary 3.1.** (Sanabria et al. [15]) Let \( I \) be an ideal on a space \((X, \tau)\). Then, the following properties hold:

1. If \( I \) is \( \tau \)-boundary and \((X, \tau)\) is \( S \)-paracompact (mod \( I \)), then \((X, \tau)\) is almost-paracompact.
2. \((X, \tau)\) is \( S \)-paracompact (mod \( N \)) if and only if it is almost-paracompact.

**Theorem 3.1.** If every open subset of a space \((X, \tau, I)\) is \( \alpha S \)-paracompact (mod \( I \)), then every subspace of \((X, \tau, I)\) is \( S \)-paracompact (mod \( I \)).

**Proof.** Suppose that \( A \) is any subspace of \((X, \tau, I)\) and let \( U = \{ U_\mu : \mu \in \Delta \} \) be a \( \tau_\Delta \)-open cover of \( A \). For every \( \mu \in \Delta \) there exists \( V_\mu \in \tau \) such that \( U_\mu = V_\mu \cap A \). Put \( V = \bigcup\{ V_\mu : \mu \in \Delta \} \), then \( V \in \tau \) and \( V = \{ V_\mu : \mu \in \Delta \} \) is a \( \tau \)-open cover of \( V \). By hypothesis, there exist \( I \in I \) and a \( \tau \)-locally finite collection \( W = \{ W_\lambda : \lambda \in \Lambda \} \) of \( \tau \)-semi-open sets such that \( W \) refines \( V \) and \( V \subseteq \bigcup\{ W_\lambda : \lambda \in \Lambda \} \cup I \). Then, we have

\[
A = \bigcup_{\mu \in \Delta} U_\mu = \bigcup_{\mu \in \Delta} (V_\mu \cap A) = \left( \bigcup_{\mu \in \Delta} V_\mu \right) \cap A = V \cap A \subseteq \left( \bigcup_{\lambda \in \Lambda} W_\lambda \cup I \right) \cap A = \bigcup_{\lambda \in \Lambda} (W_\lambda \cap A) \cup I_A,
\]

where \( I_A = I \cap A \in I_A \). If \( x \in A \), then there exists \( G_x \in \tau \) containing \( x \) such that \( W_\lambda \cap G_x = \emptyset \) for all \( \lambda \neq \lambda_1, \lambda_2, \ldots, \lambda_n \) and so \( (W_\lambda \cap G_x) \cap A = \emptyset \) for all \( \lambda \neq \lambda_1, \lambda_2, \ldots, \lambda_n \). It follows that \( (W_\lambda \cap A) \cap (G_x \cap A) = \emptyset \) for all \( \lambda \neq \lambda_1, \lambda_2, \ldots, \lambda_n \) and hence, the collection \( H = \{ W_\lambda \cap A : \lambda \in \Lambda \} \) is \( \tau_\lambda \)-locally finite. If \( W_\lambda \cap A \in H \), then \( W_\lambda \in W \) and since \( W \) refines \( \tau \), \( W_\lambda \subseteq V_\mu \) for some \( V_\mu \in V \), which implies that \( W_\lambda \cap A \subseteq V_\mu \cap A = U_\mu \in U \). Therefore, \( H \) refines \( U \). This shows that \( H = \{ W_\lambda \cap A : \lambda \in \Lambda \} \) is a \( \tau_\lambda \)-locally finite collection of \( \tau_\lambda \)-semi-open sets which refines \( U \) such that \( A \subseteq \bigcup\{ H : H \in H \} \cup I_A \). Thus, every subspace of \((X, \tau, I)\) is \( S \)-paracompact (mod \( I \)). \hfill \square

The following result is an immediate consequence of Theorem 3.2.

**Corollary 3.2.** If every open subset of a space \((X, \tau, I)\) is \( \alpha S \)-paracompact (mod \( I \)), then \((X, \tau, I)\) is \( S \)-paracompact (mod \( I \)).

Recall that a subset \( A \) of a space \((X, \tau)\) is said to be \( g \)-closed [12] if \( \text{Cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \in \tau \).

**Theorem 3.2.** If \((X, \tau, I)\) is \( S \)-paracompact (mod \( I \)) and \( A \) is a \( g \)-closed subset of \( X \), then \( A \) is \( \alpha S \)-paracompact (mod \( I \)).
Proof. Suppose that $A$ is a $g$-closed subset of an $S$-paracompact (mod $\mathcal{I}$) space $(X, \tau, \mathcal{I})$. Let $\mathcal{U} = \{U_\mu : \mu \in \Delta\}$ be an open cover of $A$. Since $A$ is $g$-closed and $A \subset \bigcup \{U_\mu : \mu \in \Delta\}$, then $s\text{Cl}(A) \subset \bigcup \{U_\mu : \mu \in \Delta\}$. For each $\mu \notin \text{Cl}(A)$ there exists a $\tau$-open set $G_x$ containing $x$ such that $A \cap G_x = \emptyset$. Put $\mathcal{U}' = \{U_\mu : \mu \in \Delta\} \cup \{G_x : x \notin \text{Cl}(A)\}$. Then $\mathcal{U}'$ is an open cover of the $S$-paracompact (mod $\mathcal{I}$) space $X$ and so, there exist $I \in \mathcal{I}$ and a locally finite collection $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of semi-open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $X = \bigcup \{V_\lambda : \lambda \in \Lambda\} \cup I$. For each $\lambda \in \Lambda$, either $V_\lambda \subset \bigcup U_\mu$ for some $\mu(\lambda) \in \Delta$ or $V_\lambda \subset G_x(\lambda)$ for some $x(\lambda) \notin \text{Cl}(A)$. Now, put $\Lambda_0 = \{\lambda \in \Lambda : V_\lambda \subset \bigcup U_\mu(\lambda)\}$. Then $\mathcal{V}' = \{V_\lambda : \lambda \in \Lambda_0\}$ is a collection of semi-open sets which is locally finite and refines $\mathcal{U}$. Also,

$$X - \bigcup_{\lambda \in \Lambda_0} V_\lambda = \left( \bigcup_{\lambda \in \Lambda} V_\lambda \cup I \right) - \bigcup_{\lambda \notin \Lambda_0} V_\lambda = \bigcup_{\lambda \notin \Lambda_0} V_\lambda \cup I$$

$$\subset \bigcup_{\lambda \notin \Lambda_0} G_x(\lambda) \cup I \subset (X - A) \cup I = X - (A - I),$$

which implies $A - I \subset \bigcup_{\lambda \in \Lambda_0} V_\lambda$ and hence $A \subset \bigcup_{\lambda \in \Lambda_0} V_\lambda \cup I$. This shows that $A$ is $\alpha S$-paracompact (mod $\mathcal{I}$).

Theorem 3.3. Let $(X, \tau, \mathcal{I})$ be a space. Then, the following properties hold:

1. If $A$ is an open $\alpha S$-paracompact (mod $\mathcal{I}$) subset of $(X, \tau, \mathcal{I})$, then $A$ is $S$-paracompact (mod $\mathcal{I}$).

2. If $A$ is a clopen subset of $(X, \tau, \mathcal{I})$, then $A$ is $\alpha S$-paracompact (mod $\mathcal{I}$) if and only if it is $S$-paracompact (mod $\mathcal{I}$).

Proof. (1) Let $A$ be an open $\alpha S$-paracompact (mod $\mathcal{I}$) subset of $(X, \tau, \mathcal{I})$. Let $\mathcal{U} = \{U_\mu : \mu \in \Delta\}$ be a $\tau_\alpha$-open cover of $A$. Since $A$ is $\tau$-open, we have $\mathcal{U}$ is a $\tau$-open cover of $A$ and hence, there exist $I \in \mathcal{I}$ and a $\tau$-locally finite collection $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of $\tau$-semi-open sets which refines $\mathcal{U}$ such that $A \subset \bigcup \{V_\lambda : \lambda \in \Lambda\} \cup I$. It follows that $A \subset \bigcup \{V_\lambda \cap A : \lambda \in \Lambda\} \cup (I \cap A)$ and so, the collection $\mathcal{V}_A = \{V_\lambda \cap A : \lambda \in \Lambda\}$ is a $\tau_\alpha$-locally finite $\tau_\alpha$-semi-open refinement of $\mathcal{U}$ and is an $\mathcal{I}_\alpha$-cover of $A$. Therefore, $A$ is $S$-paracompact (mod $\mathcal{I}$).

(2) If $A$ is a clopen and $\alpha S$-paracompact (mod $\mathcal{I}$) subset of $(X, \tau, \mathcal{I})$, then from (1) we obtain that $A$ is $S$-paracompact (mod $\mathcal{I}$). Conversely, let $\mathcal{U} = \{U_\mu : \mu \in \Delta\}$ be a $\tau$-open cover of $A$. The collection $\mathcal{V} = \{A \cap U_\mu : \mu \in \Delta\}$ is a $\tau_\alpha$-open cover of the $S$-paracompact (mod $\mathcal{I}$) subspace $(A, \tau_\alpha, \mathcal{I}_\alpha)$ and hence, there exist $I_A \in \mathcal{I}_\alpha$ and a $\tau_\alpha$-locally finite $\tau_\alpha$-semi-open refinement $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ of $\mathcal{V}$ such that $A = \bigcup \{W_\lambda : \lambda \in \Lambda\} \cup I_A$. It is easy to see that $\mathcal{W}$ refines $\mathcal{U}$ and by Lemma 2.1(3), we have that $W_\lambda \in \text{SO}(X, \tau)$ for each $\lambda \in \Lambda$. To show $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ is $\tau$-locally finite, let $x \in X$. Since $x \in A$, then there exists $O_x \in \tau_\alpha \subset \tau$ containing $x$ such that $O_x$ intersects at most finitely many members of $\mathcal{W}$. Otherwise $X \setminus A$ is a $\tau$-open set containing $x$ which intersects no member of $\mathcal{W}$. Therefore, $\mathcal{W}$ is $\tau$-locally finite and such that
\[ A = \bigcup \{ W_\lambda : \lambda \in \Lambda \} \cup I_\lambda \subset \bigcup \{ W_\lambda : \lambda \in \Lambda \} \cup I \text{ for some } I \in \mathcal{I}. \text{ Thus, } A \text{ is } \alpha\text{-S-paracompact (mod } \mathcal{I}) \].

As a consequence of Theorem 3.3, we obtain the following result.

**Corollary 3.3.** Every clopen subspace of a S-paracompact (mod \( \mathcal{I} \)) space is S-paracompact (mod \( \mathcal{I} \)).

**Lemma 3.1.** Let \( A \) be a subset of a space \((X, \tau, \mathcal{I})\). If every open cover of \( A \) has a locally finite closed refinement \( \mathcal{V} \) such that \( A \subset \bigcup \{ V : V \in \mathcal{V} \} \cup I \) for some \( I \in \mathcal{I} \), then \( \mathcal{V} \) has a locally finite open refinement \( \mathcal{W} \) such that \( A \subset \bigcup \{ W : W \in \mathcal{W} \} \cup I \).

**Proof.** Let \( \mathcal{U} \) be an open cover of \( A \). By hypothesis, there exist \( I \in \mathcal{I} \) and a locally finite closed refinement \( \mathcal{V} = \{ V_\lambda : \lambda \in \Lambda \} \) of \( \mathcal{U} \) such that \( A \subset \bigcup \{ V_\lambda : \lambda \in \Lambda \} \cup I \). For each \( \lambda \in \Lambda \), there exists an open set \( G_\lambda \) containing \( \lambda \) such that \( \mathcal{G} = \{ G_\lambda : \lambda \in \Lambda \} \) is an open cover of \( A \) and again by hypothesis, there exist \( J \in \mathcal{I} \) and a locally finite closed refinement \( \mathcal{H} = \{ H_\mu : \mu \in \Delta \} \) of \( \mathcal{G} \) such that \( A \subset \bigcup \{ H_\mu : \mu \in \Delta \} \cup J \). Now, as \( \{ H_\mu : H_\mu \cap V_\lambda = \emptyset \} \subset \mathcal{H} \), then the collection \( \{ H_\mu : H_\mu \cap V_\lambda = \emptyset \} \) is locally finite and \( \bigcup \{ H_\mu : H_\mu \cap V_\lambda = \emptyset \} = \bigcup \{ \text{Cl}(H_\mu) : H_\mu \cap V_\lambda = \emptyset \} = \text{Cl}(\bigcup \{ H_\mu : H_\mu \cap V_\lambda = \emptyset \}) \), it follows that \( O_\lambda = X - \bigcup \{ H_\mu : H_\mu \cap V_\lambda = \emptyset \} \) is an open set and \( V_\lambda \subset O_\lambda \), for each \( \lambda \in \Lambda \). For each \( \mu \in \Delta \) and \( \lambda \in \Lambda \), we have

\[ H_\mu \cap O_\lambda \neq \emptyset \iff H_\mu \cap V_\lambda \neq \emptyset. \quad (\ast) \]

Since \( \mathcal{V} \) refines \( \mathcal{U} \), for every \( \lambda \in \Lambda \) there exists \( U(\lambda) \in \mathcal{U} \) such that \( V_\lambda \subset U(\lambda) \). Put \( W_\lambda = O_\lambda \cap U(\lambda) \), then the collection \( \mathcal{W} = \{ W_\lambda : \lambda \in \Lambda \} \) is an open refinement of \( \mathcal{U} \). Furthermore, if \( x \in A \) there exists an open set \( D_x \) such that \( D_x \) intersects at most finitely many members of \( \mathcal{H} \), it follows from \((\ast)\) that \( \mathcal{W} \) is locally finite. Also, \( A \subset \bigcup \{ V_\lambda : \lambda \in \Lambda \} \cup I \subset \bigcup \{ O_\lambda \cap U(\lambda) : \lambda \in \Lambda \} \cup I = A \subset \bigcup \{ W_\lambda : \lambda \in \Lambda \} \cup I \).

The following theorem shows that, in the presence of the axiom of regularity, the notions of \( \alpha \)-paracompact (mod \( \mathcal{I} \)) and \( \alpha\text{-S-paracompact (mod } \mathcal{I}) \) subsets are equivalent.

**Theorem 3.4.** Let \( \mathcal{I} \) be an ideal on a regular space \((X, \tau)\) and \( A \) be a subset of \( X \). Then, \( A \) is \( \alpha \)-paracompact (mod \( \mathcal{I} \)) if and only if it is \( \alpha\text{-S-paracompact (mod } \mathcal{I}) \).

**Proof.** Necessity is obvious from the definitions. To show sufficiency, assume \( A \) is an \( \alpha\text{-S-paracompact (mod } \mathcal{I}) \) subset of \((X, \tau, \mathcal{I})\) and let \( \mathcal{U} = \{ U_\mu : \mu \in \Delta \} \) be an open cover of \( A \). For each \( x \in A \), there exists \( U(x) \in \mathcal{U} \) such that \( x \in U_\mu(x) \) and since \((X, \tau, \mathcal{I})\) is a regular space, there exists an open set \( V_x \) such that \( x \in V_x \subset \text{Cl}(V_x) \subset U_\mu(x) \). Thus, \( \mathcal{V} = \{ V_x : x \in A \} \) is an open cover of \( A \) and because \( A \) is \( \alpha\text{-S-paracompact (mod } \mathcal{I}) \), there exist \( I \in \mathcal{I} \) and a locally finite semi-open refinement \( \mathcal{W} = \{ W_\lambda : \lambda \in \Lambda \} \) of \( \mathcal{V} \) such that \( A \subset \bigcup \{ W_\lambda : \lambda \in \Lambda \} \cup I \). Since \( \mathcal{W} \) refines \( \mathcal{V} \), then for each \( \lambda \in \Lambda \) there exists \( x(\lambda) \in X \) such that \( W_\lambda \subset V_{x(\lambda)} \) and so, \( W_\lambda \subset \text{Cl}(W_\lambda) \subset \text{Cl}(V_{x(\lambda)}) \subset U_\mu(x(\lambda)). \) Obviously the collection \( \{ \text{Cl}(W_\lambda) : \lambda \in \Lambda \} \) is a locally finite closed refinement of \( \mathcal{U} \) such that
A ⊂ ∪(Cl(Wλ) : λ ∈ Λ) ∪ I. By Lemma 3.1, the open cover U of A has a locally finite open refinement H such that A ⊂ ∪(H : H ∈ H) ∪ I. Therefore, A is an α-paracompact (mod I) subset of (X, τ, I).

Proposition 3.3. If A is an αS-paracompact (mod I) subset of a space (X, τ, I) and B is a subset of X, there exist A ∩ Cl(B) in I, then A ∩ Cl(B) is αS-paracompact (mod I).

Proof. Let U be an open cover of A ∩ Cl(B). Then U′ = U ∪ {X − Cl(B)} is an open cover of A and so, there exist I ∈ I and a locally finite semi-open refinement V = (Vλ : λ ∈ Λ) of U′ such that A ⊂ ∪(Vλ : λ ∈ Λ) ∪ I. Then, ∂(Cl(B)) ⊂ ∂(B) ∈ I and

\[ A ∩ Cl(B) ⊂ \bigcup_{λ ∈ Λ} Vλ ∩ Int(Cl(B)) ∪ J, \]

where J = [(∪(Vλ : λ ∈ Λ) ∩ ∂(Cl(B))) ∪ (∁ ∩ Cl(B))] ∈ I. Thus, the collection V′ = (Vλ ∩ Int(Cl(B)) : λ ∈ Λ) is a locally finite semi-open refinement of U such that A ∩ Cl(B) ⊂ ∪(V : V ∈ V′) ∪ J. Therefore, A ∩ Cl(B) is αS-paracompact (mod I).

The following result follows from Proposition 3.3 and the fact that the topological frontier of a semi-open (resp. semi-closed) set is nowhere dense.

Corollary 3.4. If A is an αS-paracompact (mod N) subset of a space (X, τ, I) and B is either semi-open or semi-closed, then A ∩ Cl(B) is αS-paracompact (mod N).

Remark 3.2. If \{Vλ : λ ∈ Λ\} is a locally finite collection of subsets of a space (X, τ), then the collection \{∂(Vλ) : λ ∈ Λ\} is locally finite.

According to [7], if I is an ideal on a space (X, τ) and \(\mathfrak{G}\) is the collection of all closed sets of (X, τ), then the collection \{A ⊂ X : Cl(A) ∈ I\} is an ideal contained in I. The ideal generated by the collection of whole closed sets in I is denoted by \(⟨I ∩ \mathfrak{G}\rangle\). It is clear that \(⟨I ∩ \mathfrak{G}\rangle = \{A ⊂ X : Cl(A) ∈ I\} \).

Proposition 3.4. Let A be a subset of a space (X, τ, I). If A is αS-paracompact (mod ⟨I ∩ \mathfrak{G}\rangle) and N ⊂ I, then Cl(A) is αS-paracompact (mod N).

Proof. Let U be an open cover of Cl(A). By hypothesis, there exist IA ∈ ⟨I ∩ \mathfrak{G}\rangle and a locally finite collection V = (Vλ : λ ∈ Λ) of semi-open sets such that V refines U and A ⊂ ∪(Vλ : λ ∈ Λ) ∪ IA. Then,

\[ Cl(A) ⊂ \bigcup_{λ ∈ Λ} Cl(Vλ) ∪ Cl(IA) = \left( \bigcup_{λ ∈ Λ} Vλ \right) ∪ \left( \bigcup_{λ ∈ Λ} ∂(Vλ) \right) ∪ Cl(IA). \]

By Remark 3.2, the collection \{∂(Vλ) : λ ∈ Λ\} is locally finite and ∂(Vλ) ∈ N for each λ ∈ Λ. Thus, by [6, Lemma 2.1], we have ∪(∂(Vλ) : λ ∈ Λ) ∈ N ∩ I. Put I = ∪(∂(Vλ) : λ ∈ Λ) ∪ Cl(IA), then I ∈ I and Cl(A) ⊂ ∪(Vλ ∪ I). Therefore, Cl(A) is αS-paracompact (mod I).
Since $\mathcal{N}$ is the ideal of nowhere dense subsets of $(X,\tau)$, $A \in \mathcal{N}$ if and only if $\text{Cl}(A) \in \mathcal{N}$. In the case that $\mathcal{I} = \mathcal{N}$, then $(\mathcal{I} \cap \mathcal{F}) = \mathcal{N}$. The following corollary is a direct consequence of Proposition 3.4.

**Corollary 3.5.** If $A$ is an $\alpha S$-paracompact (mod $\mathcal{N}$) subset of a space $(X,\tau,\mathcal{I})$, then $\text{Cl}(A)$ is $\alpha S$-paracompact (mod $\mathcal{N}$).

**Lemma 3.2.** [7] If $\{A_\lambda : \lambda \in \Lambda\}$ is a locally finite collection of meager sets of a space $(X,\tau)$, then $\bigcup\{A_\lambda : \lambda \in \Lambda\}$ is meager.

**Theorem 3.5.** If $\{A_\lambda : \lambda \in \Lambda\}$ is a locally finite collection of $\alpha S$-paracompact (mod $\mathcal{M}$) subsets of a space $(X,\tau)$, then $\bigcup\{A_\lambda : \lambda \in \Lambda\}$ is $\alpha S$-paracompact (mod $\mathcal{M}$).

**Proof.** Let $\mathcal{U}$ be an open cover of $\bigcup\{A_\lambda : \lambda \in \Lambda\}$ and put $\mathcal{U}_\lambda = \{U \in \mathcal{U} : U \cap A_\lambda \neq \emptyset\}$ for each $\lambda \in \Lambda$. By the hypothesis, there exist $M_\lambda \in \mathcal{M}$ and a locally finite collection $\mathcal{V}_\lambda$ of semi-open sets such that $\mathcal{V}_\lambda$ refines $\mathcal{U}_\lambda$ and $A_\lambda \subset \bigcup\{V : V \in \mathcal{V}_\lambda\} \cup M_\lambda$. Then, we have

$$A_\lambda \subset \bigcup_{V \in \mathcal{V}_\lambda} (V \cap \text{Int}(\text{Cl}(A_\lambda))) \cup \bigcup_{V \in \mathcal{V}_\lambda} (V \cap \partial(\text{Cl}(A_\lambda))) \cup M_\lambda.$$

For each $V \in \mathcal{V}_\lambda$ and each $\lambda \in \Lambda$, $V \cap \partial(\text{Cl}(A_\lambda))$ is nowhere dense and the collection $\{V \cap \partial(\text{Cl}(A_\lambda)) : V \in \mathcal{V}_\lambda, \lambda \in \Lambda\}$ is locally finite, so by [6, Lemma 2.1], the union of all elements of $\{V \cap \partial(\text{Cl}(A_\lambda)) : V \in \mathcal{V}_\lambda, \lambda \in \Lambda\}$ is a nowhere dense set. By Lemma 3.2, we obtain $\bigcup\{M_\lambda : \lambda \in \Lambda\} \in \mathcal{M}$ and

$$\mathcal{M} = \bigcup_{\lambda \in \Lambda} \bigcup_{V \in \mathcal{V}_\lambda} V \cap \partial(\text{Cl}(A_\lambda)) \cup \bigcup_{\lambda \in \Lambda} M_\lambda \in \mathcal{M}.$$

Now, the collection $\{V \cap \text{Int}(\text{Cl}(A_\lambda)) : V \in \mathcal{V}_\lambda, \lambda \in \Lambda\}$ of semi-open sets is locally finite and refines $\mathcal{U}$ and also

$$\bigcup_{\lambda \in \Lambda} A_\lambda \subset \bigcup_{\lambda \in \Lambda} \bigcup_{V \in \mathcal{V}_\lambda} V \cap \text{Int}(\text{Cl}(A_\lambda)) \cup \mathcal{M}.$$

Therefore, $\bigcup\{A_\lambda : \lambda \in \Lambda\}$ is $\alpha S$-paracompact (mod $\mathcal{M}$).

**References**


