On Some Recurrent Properties of Three Dimensional K-Contact Manifolds

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ABSTRACT

In this paper we characterize some recurrent properties of three dimensional K-contact manifolds. Here we study Ricci η-recurrent, semi-generalized recurrent and locally generalized concircularly φ-recurrent conditions on three dimensional K-contact manifolds.

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1 Introduction

In 1950, Walker [17] introduced the notion of recurrent manifolds. In the last five decades, recurrent structures have played an important role in the geometry and the topology of manifolds. In [3], the authors De and Guha introduced the idea of generalized recurrent manifold with the non-zero 1-form $A$ and another non-zero associated 1-form $B$. If the associated 1-form $B$ becomes zero, then the manifold reduces to a recurrent manifold given by Ruse [11]. As a generalization of recurrency, Khan [6] introduced the notion of generalized recurrent Sasakian manifold. Semi-generalized recurrent manifolds were first introduced and studied by Prasad [10]. The notion of recurrency in a Riemannian manifold has been weakened by many authors in several ways to different extent viz., [1, 8, 12] etc.,

A $K$-contact manifold is a differentiable manifold with a contact metric structure such that $\xi$ is a Killing vector field [2, 13]. These are studied by several authors like [3, 9, 14, 15] and many others. It is well known that every Sasakian manifold is $K$-contact, but the converse is not true, in general. However a three-dimensional $K$-contact manifold is Sasakian [5].

Motivated by the above studies, in this study we consider some recurrent properties of three dimensional $K$-contact manifolds. The paper is organized in the following way: In Section 2, we give the definitions and some results concerning the $K$-contact manifolds that will be needed hereafter. In Section 3, we discuss the Ricci $\eta$-recurrent property of three dimensional $K$-contact manifold. In particular, we obtain the 1-form $A$ is $\eta$ parallel and give the expression for Ricci tensor. The Section 4 is devoted to three dimensional semi-generalized recurrent $K$-contact manifolds. Here we prove some interesting results, such as the facts that a specific linear combination of the 1-forms $A$ and $B$ is always zero and that the manifold is Einstein. In Section 5, we consider three dimensional locally generalized concircularly $\phi$-recurrent $K$-contact manifolds. In this case the manifold is a space of constant curvature.

2 Preliminaries

A Riemannian manifold $M$ is said to admit an almost contact metric structure $(\phi, \xi, \eta, g)$ if it carries a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$, 1-form $\eta$ and compatible Riemannian metric $g$ on $M$, such that

\begin{align}
\phi^2 X &= -X + \eta(X)\xi, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \\
\eta(\xi) &= 1, \quad g(X, \xi) = \eta(X), \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\
g(\phi X, Y) &= -g(X, \phi Y), \quad g(\phi X, X) = 0.
\end{align}
If moreover $\xi$ is Killing vector field, then $M$ is called a $K$-contact manifold [2, 13]. A $K$-contact manifold is called Sasakian [2], if the relation

$$\nabla_X \phi(Y) = g(X, Y) \xi - \eta(Y) X,$$

holds on $M$, where $\nabla$ denotes the operator of covariant differentiation with respect of metric $g$.

In a $K$-contact manifold, the following relations hold:

$$\nabla_X \xi = -\phi X,$$

$$\nabla_X \eta(Y) = g(\nabla_X \xi, Y).$$

Also in a three dimensional $K$-contact manifold, the curvature tensor is given by

$$R(X, Y) Z = \frac{r-4}{2} [g(Y, Z) X - g(X, Z) Y] - \frac{r-6}{2} [g(Y, Z) \eta(X) \xi] + g(X, Z) \eta(Y) \eta(X) \eta(Z),$$

$$S(X, Y) = \frac{1}{2} [(r-2) g(X, Y) - (r-6) \eta(X) \eta(Y)],$$

$$QX = \frac{1}{2} [(r-2) X - (r-6) \eta(X) \xi],$$

$$S(\phi X, \phi Y) = S(X, Y) - 2 \eta(X) \eta(Y),$$

where $r$, $S$ and $Q$ are the scalar curvature, Ricci tensor and Ricci operator respectively.

**Definition 1.** A $K$-contact manifold is said to be Einstein if the Ricci tensor $S$ is of the form

$$S(X, Y) = a g(X, Y),$$

where $a$ is constant.

**3 On three dimensional Ricci $\eta$-recurrent $K$-contact manifold**

**Definition 2.** The Ricci tensor of an three dimensional $K$-contact manifold is said to be $\eta$-recurrent if its Ricci tensor satisfies the following:

$$\nabla_X S(\phi(Y), \phi(Z)) = A(X) S(\phi(Y), \phi(Z)),$$

for all vector fields $X, Y, Z \in TM$, where $A(X) = g(X, \rho)$, $\rho$ is called the associated vector field of 1-form $A$.

In particular, if the 1-form $A$ vanishes then the Ricci tensor is said to be $\eta$-parallel and this notion for Sasakian manifold was first introduced by Kon [18].
Now consider three dimensional Ricci $\eta$-Recurrent $K$-contact manifold. From (3.1), it follows that
\[
\nabla_Z S(\phi(X), \phi(Y)) - S(\nabla_Z \phi X, \phi Y) - S(\phi X, \nabla_Z \phi Y) = A(Z) S(\phi(X), \phi(Y)).
\] (3.2)

By using (2.5), (2.6) and (2.11) in (3.2), yields
\[
(\nabla_Z S)(X, Y) = -\eta(X)[2g(\phi Z, Y) + S(Z, \phi Y)] - \eta(Y)[2g(\phi Z, X) + S(\phi X, Z)]
\] (3.3)
\[+ A(Z) [S(X, Y) - 2\eta(X)\eta(Y)].
\]

Hence we can state the following:

**Theorem 3.1.** In a three dimensional $K$-contact manifold, the Ricci tensor is $\eta$-recurrent if and only if (3.3) holds.

By virtue of (3.3), let $\{e_i\}$ is an local orthonormal basis of the tangent space at each point of the manifold and taking summation over $i$, $1 \leq i \leq 3$, we have
\[
dr(Z) = [r - 2]A(Z).
\] (3.4)

If the manifold has a constant scalar curvature $r$ ($r \neq 2$ because the 1-form $A$ is definite), then from (3.4) it follows that
\[
A(Z) = 0, \ \forall \ Z.
\]

This leads to the following:

**Theorem 3.2.** In a three dimensional Ricci $\eta$-recurrent $K$-contact manifold $M$ if the scalar curvature is constant then the 1-form $A$ is $\eta$-parallel.

Again putting $X = Z = e_i$ in (3.3), and taking summation over $i$, $1 \leq i \leq 3$, we get
\[
\frac{1}{2} \dr(Y) + \mu \eta(Y) = S(Y, \rho) - 2\eta(\rho)\eta(Y),
\] (3.5)
where $\mu = \Sigma_{i=1}^3 S(\phi e_i, e_i)$. By using (3.4) in (3.5), we obtain
\[
\frac{1}{2} A(Y)[r - 2] + \mu \eta(Y) = S(Y, \rho) - 2\eta(\rho)\eta(Y),
\] (3.6)

Putting $Y = \xi$ in (3.6), yields
\[
\mu = \left(1 - \frac{r}{2}\right) \eta(\rho).
\] (3.7)

Considering (3.7) in (3.6), we get
\[
S(Y, \rho) = \left(\frac{r}{2} - 1\right) g(Y, \rho) + \left(3 - \frac{r}{2}\right) \eta(\rho)\eta(Y).
\] (3.8)

Thus we have the following result:
Theorem 3.3. *If the Ricci tensor in a three dimensional K-contact manifold is η-recurrent, then its Ricci tensor along the associated vector field of the 1-form is given by (3.8).*

Substituting $Y = \phi Y$ in (3.8) and by virtue of (2.1), we obtain

$$S(Y, L) = Kg(Y, L),$$

(3.9)

where $L = \phi \rho$, $K = \frac{r}{2} - 1$.

Hence we can state the following:

Theorem 3.4. *If the Ricci tensor in a three dimensional K-contact manifold is η-recurrent, then $K = \frac{r}{2} - 1$ is an eigen value of the Ricci tensor corresponding to the eigen vector $\phi \rho$.*

4 **On three dimensional semi-generalized recurrent K-contact manifolds**

Definition 3. A Riemannian manifold is said to be semi-generalized recurrent manifold if its curvature tensor $R$ satisfies the relation

$$\langle \nabla_X R \rangle(Y, Z)W = A(X)R(Y, Z)W + B(X)g(Z, W)Y,$$

(4.1)

where $A$ and $B$ are two 1-forms, $B$ is non-zero, $\rho_1$ and $\rho_2$ are two vector fields such that

$$g(X, \rho_1) = A(X), \quad g(X, \rho_2) = B(X),$$

(4.2)

for any vector field $X$ and $\nabla$ be the covariant differentiation operator with respect to the metric $g$.

Definition 4. A Riemannian manifold $M$ is said to be three dimensional semi-generalized Ricci recurrent manifold if:

$$\langle \nabla_X S \rangle(Y, Z) = A(X)S(Y, Z) + 3B(X)g(Y, Z).$$

(4.3)

Taking cyclic sum of (4.1) with respect to $X, Y, Z$, and using second Bianchi’s identity, we get

$$0 = A(X)R(Y, Z)W + A(Y)R(Z, X)W + A(Z)R(X, Y)W + B(X)g(Z, W)Y + B(Y)g(X, W)Z + B(Z)g(Y, W)X.$$ (4.4)

On contracting above equation with respect to $Y$, yields

$$0 = A(X)S(Z, W) - g[R(Z, X)\rho_1, W] - A(Z)S(X, W) + 3B(X)g(Z, W) + g(X, W)g(\rho_2, Z) + B(Z)g(X, W).$$ (4.5)

Again putting $Z = W = e_i$ in (4.5), and taking summation over $i$, $1 \leq i \leq 3$, we obtain

$$\tau A(X) + 11B(X) - 2S(X, \rho_1) = 0.$$ (4.6)
Putting $X = \xi$ in (4.6) and by virtue of (4.2) and (2.11), we get

$$r = \frac{1}{\eta(\rho_1)}[4\eta(\rho_1) - 11\eta(\rho_2)].$$

(4.7)

Since for a contact metric manifold $\eta(\rho_1) \neq 0$. Hence we can state the following:

**Theorem 4.1.** In a three dimensional semi-generalized recurrent $K$-contact manifold, the scalar curvature $r$ takes the form (4.7).

Again taking $Z = \xi$ in (4.3), we get

$$\nabla_X S(Y, \xi) = A(X)S(Y, \xi) + 3B(X)g(Y, \xi).$$

(4.8)

Left hand side of the above equation can be written as

$$(\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi).$$

(4.9)

In view of (2.2), (2.9) and (4.9) in (4.8), gives

$$-2g(\phi X, Y) + S(\phi X, Y) = 2A(X)\eta(Y) + 3B(X)\eta(Y).$$

(4.10)

Plugging $Y = \xi$ in (4.10), we obtain

$$2A(X) + 3B(X) = 0.$$

This leads to the following:

**Theorem 4.2.** In a three dimensional semi-generalized Ricci recurrent $K$-contact manifold, the linear combination $2A + 3B$ is always zero.

Replace $Y$ by $\phi Y$ in (4.10), we get

$$S(X, Y) = 2g(X, Y).$$

Thus we have the following result:

**Theorem 4.3.** A three dimensional semi-generalized Ricci recurrent $K$-contact manifold is Einstein manifold.

## 5 On three dimensional locally generalized concircularly $\phi$-recurrent $K$-contact manifolds

**Definition 5.** A three dimensional $K$-contact manifold is called the locally generalized concircularly $\phi$-recurrent if its concircular curvature tensor $\tilde{C}$

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{6}[g(Y, Z)X - g(X, Z)Y],$$

(5.1)
satisfies the condition

$$\phi^2((\nabla_W \tilde{\phi})(X, Y)Z) = A(W)\tilde{\phi}(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y],$$

(5.2)

for all $X, Y, Z$ and $W$ orthogonal to $\xi$.

Taking covariant differentiation of (2.8) with respect to $W$, we get

$$\nabla_W \tilde{\phi}(X, Y)Z = \frac{dr(W)}{2} [g(Y, Z)X - g(X, Z)Y] - \frac{r - 6}{2} [g(Y, Z)g(\phi Y, W)\xi - g(X, Z)g(\phi X, W)\xi].$$

(5.3)

From above equation it follows that

$$\phi^2((\nabla_W \tilde{\phi})(X, Y)Z) = \frac{dr(W)}{2} [g(X, Z)Y - g(Y, Z)X].$$

(5.5)

Taking covariant differentiation of (5.2) with respect to $W$, we get

$$\nabla_W \tilde{\phi}(X, Y)Z = (\nabla_W \tilde{\phi})(X, Y)Z - \frac{dr(W)}{6} [g(X, Z)X - g(X, Z)Y],$$

(5.6)

from which it follows that

$$\phi^2((\nabla_W \tilde{\phi})(X, Y)Z) = \phi^2((\nabla_W \tilde{\phi})(X, Y)Z) - \frac{dr(W)}{6} [g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y].$$

(5.7)

By virtue of (2.1), (5.2), (5.5) in (5.7), yields

$$R(X, Y)Z = \left[ \frac{r}{6} - \left( \frac{B(W)}{A(W)} + \frac{dr(W)}{6A(W)} \right) \right] [g(Y, Z)X - g(X, Z)Y].$$

(5.8)

Since in a locally generalized concircularly $\phi$-recurrant K-contact manifold $A(W) \neq 0$. On contracting above equation over $W$, we get

$$R(X, Y)Z = \mu [g(Y, Z)X - g(X, Z)Y],$$

(5.9)

where $\mu = \frac{r}{6} - \left( \frac{B(W)}{A(W)} + \frac{dr(W)}{6A(W)} \right)$ is a scalar. Then by Schur’s theorem \cite{7} $\mu$ will be constant on the manifold.

Thus we have the following result:

**Theorem 5.1.** A three dimensional locally generalized concircularly $\phi$-recurrant K-contact manifold is a space of constant curvature.
References


