W_2 -Curvature Tensor on Generalized Sasakian Space Forms

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ABSTRACT

In this paper, we study W_2 -pseudosymmetric, W_2 -locally symmetric, W_2 -locally ϕ -symmetric and W_2 - ϕ -recurrent generalized Sasakian space form. Further, illustrative examples are given.

RESUMEN

En este artículo, estudiamos formas espaciales Sasakianas generalizadas W_2 -seudosimétricas, W_2 -localmente ϕ -simétricas y W_2 - ϕ -recurrentes. Ejemplos ilustrativos son dados.

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1 Introduction

The nature of a Riemannian manifold depends on the curvature tensor R of the manifold. It is well known that the sectional curvatures of a manifold determine its curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as a real space form and its curvature tensor is given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}.$$

Representation for these spaces are hyperbolic spaces ($c < 0$), spheres ($c > 0$) and Euclidean spaces ($c = 0$).

The ϕ -sectional curvature of a Sasakian space form is defined by Sasakian manifold and it has a specific form of its curvature tensor. Same notion also holds for Kenmotsu and cosymplectic space forms. In order to generalize such space forms in a common frame Alegre, Blair and Carriazo [1] introduced and studied generalized Sasakian space forms.

A generalized Sasakian space form is an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, whose curvature tensor is given by

$$\begin{aligned} R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y \\ &\quad - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ &\quad - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \end{aligned} \quad (1.1)$$

The Riemannian curvature tensor of a generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ is simply given by

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3,$$

where f_1, f_2, f_3 are differential functions on $M^{2n+1}(f_1, f_2, f_3)$ and

$$\begin{aligned} R_1(X, Y)Z &= g(Y, Z)X - g(X, Z)Y, \\ R_2(X, Y)Z &= g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z, \quad \text{and} \\ R_3(X, Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi, \end{aligned}$$

where $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$. Here c denotes the constant ϕ -sectional curvature. The properties of generalized Sasakian space form was studied by many geometers such as [2, 9, 10, 14, 17, 18, 19, 21, 26]. The concept of local symmetry of a Riemannian manifold has been studied by many authors in several ways to a different extent. The locally ϕ -symmetry of Sasakian manifold was introduced by Takahashi in [28]. De and et al generalize this to the notion of ϕ -symmetry and then introduced the notion of ϕ -recurrent Sasakian manifold in [11]. Further ϕ -recurrent condition was studied on Kenmotsu manifold [8], LP-Sasakian manifold [29] and $(LCS)_n$ -manifold [20].

In [16], Pokhariyal and Mishra have defined the W_2 -curvature tensor, given by

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{2n}\{g(X, Z)QY - g(Y, Z)QX\}, \quad (1.2)$$

here R and Q are the Riemannian curvature tensor and Ricci operator of Riemannian manifold respectively.

In a generalized Sasakian space forms, the W_2 -curvature tensor satisfies the condition

$$\eta(W_2(X, Y)Z) = 0. \quad (1.3)$$

Many Geometers studied the W_2 curvature tensor studied on different manifolds such as generalized Sasakian space forms [13], Lorentzian para Sasakian manifolds [30] and Kenmotsu manifolds [25]

Motivated by these ideas, we made an attempt to study the properties of generalized Sasakian space form. The present paper is organized as follows: In section 2, we review some preliminary results. In section 3, we study W_2 -pseudosymmetric generalized Sasakian space form. Section 4, deals with the W_2 -locally symmetric generalized Sasakian space forms and it is shown that a generalized Sasakian space form of dimension greater than three is W_2 -locally symmetric if and only if it is conformally flat. Section 5, is devoted to the study of W_2 -locally ϕ -symmetric generalized Sasakian space forms. Finally in last section, we discuss the W_2 - ϕ -recurrent generalized Sasakian space form and found to be Einstein manifold.

2 Generalized Sasakian space-forms

The Riemannian manifold M^{2n+1} is called an almost contact metric manifold if the following result holds [5, 6]:

$$\phi^2X = -X + \eta(X)\xi, \quad (2.1)$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(X, \xi) = \eta(X), \quad (2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.3)$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0 \quad (2.4)$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y), \quad \forall X, Y \in (T_p M). \quad (2.5)$$

A almost contact metric manifold is said to be Sasakian if and only if [5, 23]

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.6)$$

$$\nabla_X \xi = -\phi X. \quad (2.7)$$

Again we know that [1] in $(2n+1)$ -dimensional generalized Sasakian space form:

$$\begin{aligned} S(X, Y) &= (2nf_1 + 3f_2 - f_3)g(X, Y) \\ &- (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \end{aligned} \quad (2.8)$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n(f_1 - f_3)\eta(X)\eta(Y), \quad (2.9)$$

$$\begin{aligned} QX &= (2nf_1 + 3f_2 - f_3)X \\ &- (3f_2 + (2n-1)f_3)\eta(X)\xi, \end{aligned} \quad (2.10)$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \quad (2.11)$$

$$R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \quad (2.12)$$

$$R(\xi, X)Y = (f_1 - f_3)\{g(X, Y)\xi - \eta(Y)X\}, \quad (2.13)$$

$$\eta(R(X, Y)Z) = (f_1 - f_3)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \quad (2.14)$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X). \quad (2.15)$$

Here R , S , Q and r are the Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature tensor of generalized Sasakian space forms in that order.

3 W_2 -pseudosymmetric generalized Sasakian space forms

The concept of a pseudosymmetric manifold was introduced by Chaki [7] and Deszcz [12]. In this article we shall study properties of pseudosymmetric manifold according to Deszcz. Semisymmetric manifolds satisfies the condition $R \cdot R = 0$ and were categorized by Szabo in [27]. Every pseudosymmetric manifold is semisymmetric but semisymmetric manifold need not be pseudosymmetric.

An $(2n+1)$ -dimensional Riemannian manifold M^{2n+1} is said to be pseudosymmetric, if

$$(R(X, Y) \cdot R)(U, V)W = L_R\{((X \wedge Y) \cdot R)(U, V)W\}. \quad (3.1)$$

where L_R is some smooth function on $U_R = \{x \in M^{2n+1} | R - \frac{r}{n(n-1)}G \neq 0 \text{ at } x\}$, where G is the $(0, 4)$ -tensor defined by $G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$ and $(X \wedge Y)Z$ is the endomorphism and it is defined as,

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y \quad (3.2)$$

An $(2n+1)$ -dimensional generalized Sasakian space form M^{2n+1} is said to be W_2 -pseudosymmetric, if

$$(R(X, Y) \cdot W_2)(U, V)Z = L_{W_2}\{(X \wedge Y) \cdot W_2)(U, V)Z\}, \quad (3.3)$$

holds on the set $U_{W_2} = \{x \in M^{2n+1} | W_2 \neq 0 \text{ at } x\}$, where L_{W_2} is some function on U_{W_2} .

Suppose that generalized Sasakian space form is W_2 -pseudosymmetric.

Now the left- hand side of (3.3) is

$$\begin{aligned} &R(\xi, Y)W_2(U, V)Z - W_2(R(\xi, Y)U, V)Z \\ &- W_2(U, R(\xi, Y)V)Z - W_2(U, V)R(\xi, Y)Z = 0. \end{aligned} \quad (3.4)$$

In the view of (2.12) the above expression becomes

$$\begin{aligned} & (f_1 - f_3)\{g(Y, W_2(U, V)Z)\xi - \eta(W_2(U, V)Z)Y \\ & - g(Y, U)W_2(\xi, V)Z + \eta(U)W_2(Y, V)Z \\ & - g(Y, V)W_2(U, \xi)Z + \eta(V)W_2(U, Y)Z \\ & - g(Y, Z)W_2(U, V)\xi + \eta(Z)W_2(U, V)Y\} = 0. \end{aligned} \quad (3.5)$$

Next the right hand side of (3.3) is

$$\begin{aligned} & L_{W_2}\{(\xi \wedge Y)W_2(U, V)Z - W_2((\xi \wedge Y)U, V)Z \\ & - W_2(U, (\xi \wedge Y)V)Z - W_2(U, V)(\xi \wedge Y)Z\} = 0. \end{aligned} \quad (3.6)$$

By virtue of (3.2), (3.6) becomes

$$\begin{aligned} & L_{W_2}\{g(Y, W_2(U, V)Z)\xi - \eta(W_2(U, V)Z)Y \\ & - g(Y, U)W_2(\xi, V)Z + \eta(U)W_2(Y, V)Z \\ & - g(Y, V)W_2(U, \xi)Z + \eta(V)W_2(U, Y)Z \\ & - g(Y, Z)W_2(U, V)\xi + \eta(Z)W_2(U, V)Y\} = 0. \end{aligned} \quad (3.7)$$

Using the expressions (3.5) and (3.7) in (3.3) and taking inner product with ξ , we obtain

$$\begin{aligned} & \{L_{W_2} - (f_1 - f_3)\}\{W_2(U, V, Z, Y) - \eta(W_2(U, V)Z)\eta(Y) \\ & - g(Y, U)\eta(W_2(\xi, V)Z) + \eta(U)\eta(W_2(Y, V)Z) \\ & - g(Y, V)\eta(W_2(U, \xi)Z) + \eta(V)\eta(W_2(U, Y)Z) \\ & - g(Y, Z)\eta(W_2(U, V)\xi) + \eta(Z)\eta(W_2(U, V)Z)\} = 0, \end{aligned} \quad (3.8)$$

where $W_2(U, V, Z, Y) = g(Y, W_2(U, V)Z)$ and using (1.3) we get either

$$L_{W_2} = (f_1 - f_3) \text{ or } W_2(U, V, Z, Y) = 0. \quad (3.9)$$

Thus we have following:

Theorem 3.1. *If $M^{2n+1}(f_1, f_2, f_3)$ is W_2 -pseudosymmetric generalized Sasakian space form, then $M^{2n+1}(f_1, f_2, f_3)$ is either W_2 -flat, or $L_{W_2} = (f_1 - f_3)$ if $(f_1 \neq f_3)$.*

Also in a generalized Sasakian space form, Singh and Pandey [24] proved the following,

Theorem 3.2. *A $(2n+1)$ -dimensional $(n > 1)$ generalized Sasakian space form satisfying $W_2 = 0$ is an η -Einstein manifolds.*

In view of theorem (3.1) and theorem (3.2) we can state the following corollary.

Corolary 1. *If $M^{2n+1}(f_1, f_2, f_3)$ is a W_2 -pseudosymmetric generalized Sasakian space forms then M^{2n+1} is either η -Einstein manifold or $L_{W_2} = (f_1 - f_3)$ if $(f_1 \neq f_3)$.*

4 W_2 -locally symmetric generalized Sasakian space forms

Definition 1. A $(2n+1)$ dimensional ($n > 1$) generalized Sasakian space form is called projectively locally symmetric if it satisfies [18].

$$(\nabla_W P)(X, Y)Z = 0.$$

for all vector fields X, Y, Z orthogonal to ξ and an arbitrary vector field W .

Analogous to this definition, we define a $(2n+1)$ dimensional ($n > 1$) W_2 -locally symmetric generalized Sasakian space form if

$$(\nabla_W W_2)(X, Y)Z = 0,$$

for all vector fields X, Y, Z orthogonal to ξ and an arbitrary vector field W .

From (1.1) and (1.2), we have

$$\begin{aligned} W_2(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &+ f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ &- g(Y, Z)\eta(X)\xi\} + \frac{1}{2n}\{g(X, Z)QY - g(Y, Z)QX\}. \end{aligned} \quad (4.1)$$

Taking covariant differentiation of (4.1) with respect to an arbitrary vector field W , we get

$$\begin{aligned} (\nabla_W W_2)(X, Y)Z &= df_1(W)\{g(Y, Z)X - g(X, Z)Y\} \\ &+ df_2(W)\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X \\ &+ 2g(X, \phi Y)\phi Z\} + f_2\{g(X, \phi Z)(\nabla_W \phi)Y \\ &+ g(X, (\nabla_W \phi)Z)\phi Y - g(Y, \phi Z)(\nabla_W \phi)X \\ &- g(Y, (\nabla_W \phi)Z)\phi X + 2g(X, \phi Y)(\nabla_W \phi)Z \\ &+ 2g(X, (\nabla_W \phi)Y)\phi Z\} + df_3(W)\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \\ &+ f_3\{(\nabla_W \eta)(X)\eta(Z)Y + \eta(X)(\nabla_W \eta)(Z)Y \\ &- (\nabla_W \eta)(Y)\eta(Z)X - \eta(Y)(\nabla_W \eta)(Z)X \\ &+ g(X, Z)(\nabla_W \eta)(Y)\xi + g(X, Z)\eta(Y)\nabla_W \xi \\ &- g(Y, Z)(\nabla_W \eta)(X)\xi - g(Y, Z)\eta(X)\nabla_W \xi\} \\ &+ \frac{1}{2n}\{g(X, Z)(\nabla_W Q)(Y) - g(Y, Z)(\nabla_W Q)(X)\}. \end{aligned} \quad (4.2)$$

where ∇ denotes the Riemannian connection on the manifold.

Differentiating (2.10) covariantly with respect to a W , one can get

$$\begin{aligned} (\nabla_W Q)(Y) &= d(2nf_1 + 3f_2 - f_3)(W)Y - d(3f_2 + (2n-1)f_3)(W)\eta(Y)\xi \\ &- (3f_2 + (2n-1)f_3)[(\nabla_W \eta)(Y)\xi + \eta(Y)(\nabla_W \xi)]. \end{aligned} \quad (4.3)$$

In view of (4.3) and (4.2), it follows that

$$\begin{aligned}
 (\nabla_W W_2)(X, Y)Z &= df_1(W)\{g(Y, Z)X - g(X, Z)Y\} \\
 &+ df_2(W)\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X \\
 &+ 2g(X, \phi Y)\phi Z\} + f_2\{g(X, \phi Z)(\nabla_W \phi)Y \\
 &+ g(X, (\nabla_W \phi)Z)\phi Y - g(Y, \phi Z)(\nabla_W \phi)X \\
 &- g(Y, (\nabla_W \phi)Z)\phi X + 2g(X, \phi Y)(\nabla_W \phi)Z \\
 &+ 2g(X, (\nabla_W \phi)Y)\phi Z\} + df_3(W)\{\eta(X)\eta(Z)Y \\
 &- \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \\
 &+ f_3\{(\nabla_W \eta)(X)\eta(Z)Y + \eta(X)(\nabla_W \eta)(Z)Y \\
 &- (\nabla_W \eta)(Y)\eta(Z)X - \eta(Y)(\nabla_W \eta)(Z)X \\
 &+ g(X, Z)(\nabla_W \eta)(Y)\xi + g(X, Z)\eta(Y)\nabla_W \xi \\
 &- g(Y, Z)(\nabla_W \eta)(X)\xi - g(Y, Z)\eta(X)\nabla_W \xi\} \\
 &+ \frac{1}{2n}[g(X, Z)\{d(2nf_1 + 3f_2 - f_3)(W)Y - d(3f_2 \\
 &+ (2n - 1)f_3)(W)\eta(Y)\xi - (3f_2 + (2n - 1)f_3)[(\nabla_W \eta)(Y)\xi \\
 &+ \eta(Y)(\nabla_W \xi)]\} - g(Y, Z)\{d(2nf_1 + 3f_2 - f_3)(W)X \\
 &- d(3f_2 + (2n - 1)f_3)(W)\eta(X)\xi \\
 &- (3f_2 + (2n - 1)f_3)[(\nabla_W \eta)(X)\xi + \eta(X)(\nabla_W \xi)]\}]. \quad (4.4)
 \end{aligned}$$

Taking X, Y, Z orthogonal to ξ in (4.4) and then taking the inner product of the resultant equation with V , followed by setting $V = Z = e_i$ in the above equation, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i, i = 1, 2, \dots, 2n + 1$, we get

$$\begin{aligned}
 &f_2\{-g(\phi X, (\nabla_W \phi)Y) + \sum_{i=1}^n g(X, (\nabla_W \phi)e_i)g(\phi Y, e_i) \\
 &+ g(\phi Y, (\nabla_W \phi)X) - \sum_{i=1}^n g(Y, (\nabla_W \phi)e_i)g(\phi X, e_i) \\
 &+ 2 \sum_{i=1}^n g(X, \phi Y)g((\nabla_W \phi)e_i, e_i)\} = 0. \quad (4.5)
 \end{aligned}$$

For Levi Civita connection ∇ ,

$$(\nabla_W g)(X, Y) = 0,$$

which gives

$$(\nabla_W g)(X, Y) - g(\nabla_W X, Y) - g(X, \nabla_W Y) = 0.$$

Putting $X = e_i$ and $Y = \phi e_i$ in the above equation, we obtain

$$-g(\nabla_W e_i, \phi e_i) - g(e_i, (\nabla_W \phi)e_i) = 0,$$

which can be written as

$$g(e_i, \phi(\nabla_W e_i)) - g(e_i, (\nabla_W \phi)e_i) = 0.$$

Thus we have

$$g(e_i, (\nabla_W \phi)e_i) = 0. \quad (4.6)$$

By the virtue of (4.5) and (4.6) takes the form

$$\begin{aligned} f_2 \{ -g(\phi X, (\nabla_W \phi)Y) + \sum_{i=1} g(X, (\nabla_W \phi)e_i)g(\phi Y, e_i) \\ + g(\phi Y, (\nabla_W \phi)X) - \sum_{i=1} g(Y, (\nabla_W \phi)e_i)g(\phi X, e_i) \} = 0. \end{aligned} \quad (4.7)$$

The above equation yields $f_2 = 0$. It is known that a generalized Sasakian space form of dimension greater than three is conformally flat if and only if $f_2 = 0$ [14]. Hence the manifold under consideration is conformally flat. Conversely, suppose that the manifold is conformally flat. Then $f_2 = 0$. In addition, if we consider X, Y, Z orthogonal to ξ then (1.1) yields

$$R(X, Y)Z = f_1 \{ g(Y, Z)X - g(X, Z)Y \}.$$

The above equation gives,

$$r = 2n(2n + 1)f_1. \quad (4.8)$$

In view of (2.11) and (4.8), we obtain $f_3 = 0$. Hence from (4.4), we get

$$(\nabla_W W_2)(X, Y)Z = 0.$$

Therefore, the manifold is W_2 -locally symmetric.

Thus we have the following assertion.

Theorem 4.1. *A $(2n + 1)$ dimensional $(n > 1)$ generalized Sasakian space form is W_2 -locally symmetric if and only if it is conformally flat.*

or

Theorem 4.2. *A $(2n + 1)$ dimensional $(n > 1)$ generalized Sasakian space form is W_2 -locally symmetric if and only if f_1 is constant.*

5 W_2 -Locally ϕ -symmetric generalized Sasakian space forms

Definition 2. *A generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ of dimension greater than three is called W_2 -locally ϕ -symmetric if it satisfies*

$$\phi^2((\nabla_W W_2)(X, Y)Z) = 0, \quad (5.1)$$

for all vector fields X, Y, Z orthogonal to ξ , on M^{2n+1} . Let us consider a W_2 -locally ϕ -symmetric generalized Sasakian space form of dimension greater than three. Then from the definition and (2.1), we have

$$-((\nabla_W W_2)(X, Y)Z) + \eta(\nabla_W W_2)(X, Y)Z\xi = 0, \quad (5.2)$$

Taking the inner product g in both sides of the above equation with respect to W , we get

$$-g((\nabla_W W_2)(X, Y)Z, W) + \eta(\nabla_W W_2)(X, Y)Z\eta(W) = 0, \quad (5.3)$$

If we take orthogonal to W , then the above equation yields,

$$g((\nabla_W W_2)(X, Y)Z, W) = 0, \quad (5.4)$$

The above equation is true for all W orthogonal to ξ . If we choose $W \neq 0$ and not orthogonal to $(\nabla_W W_2)(X, Y)Z$, then it follows that

$$(\nabla_W W_2)(X, Y)Z = 0 \quad (5.5)$$

Hence, the manifold is W_2 -locally symmetric and hence by theorem 4.3, it is conformally flat. Conversely, let the manifold is conformally flat and hence $f_2 \neq 0$. Again, for X, Y, Z orthogonal to ξ , we have applying ϕ^2 on both side to equation (4.4), one can get

$$\begin{aligned} \phi^2(\nabla_W W_2)(X, Y)Z &= -df_2(W)\{g(X, \phi Z)\phi X - g(Y, \phi Z) + 2g(X, \phi Y)\phi Z\} \\ &\quad - \frac{1}{2n}\{d(3f_2 - f_3)(W)[g(X, Z)Y - g(Y, Z)X]\}. \end{aligned} \quad (5.6)$$

if $f_2 = f_3 = 0$, the above equation yields

$$\phi^2(\nabla_W W_2)(X, Y)Z = 0$$

for all X, Y, Z are orthogonal to ξ , therefore the manifold is W_2 -locally ϕ -symmetric.

Now we are in a position to state the following statement,

Theorem 5.1. *A $(2n + 1)$ -dimensional ($n > 1$) generalized Sasakian space form M^{2n+1} is W_2 -locally ϕ -symmetric if and only if it is conformally flat.*

6 W_2 - ϕ -recurrent generalized Sasakian Space form

Definition 3. *A generalized Sasakian space form is said to be ϕ -recurrent if there exists a non-zero 1-form A such that, (see[11])*

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z,$$

for arbitrary vector fields X, Y, Z, W . If the 1-form A vanishes, then the manifold reduces to a ϕ -symmetric manifold.

According to the definition of ϕ -recurrent generalized Sasakian space form, we define W_2 - ϕ -recurrent generalized sasakian space form by

$$\phi^2((\nabla_W W_2)(X, Y)Z) = A(W)W_2(X, Y)Z. \quad (6.1)$$

Then by (2.1) and (6.1), we have

$$-(\nabla_W W_2)(X, Y)Z + \eta((\nabla_W W_2)(X, Y)Z)\xi = A(W)W_2(X, Y)Z, \quad (6.2)$$

for arbitrary vector fields X, Y, Z, W . From the above equation it follows that

$$\begin{aligned} & -g((\nabla_W W_2)(X, Y)Z, U) + \eta((\nabla_W W_2)(X, Y)Z)\eta(U) \\ & = A(W)g(W_2(X, Y)Z, U). \end{aligned} \quad (6.3)$$

Let $\{e_i\}, i = 1, 2, \dots, 2n+1$, be an orthogonal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (6.3) and taking summation over $i, 1 \leq i \leq 2n+1$, we get

$$\begin{aligned} & -(\nabla_W S)(Y, Z) - \frac{1}{2n}[(\nabla_W S)(Y, Z)) - g(Y, Z)\text{dr}(W)] \\ & + \sum_{i=1}^{2n+1} \eta((\nabla_W W_2)(e_i, Y)Z)\eta(e_i) = A(W)\{(\nabla_W S)(Y, Z) \\ & - \frac{1}{2n}[(\nabla_W S)(Y, Z) - g(Y, Z)\text{dr}(W)]\}. \end{aligned} \quad (6.4)$$

Setting $Z = \xi$ in (6.4) then using (2.5), (2.13) and (2.7) and then replace Y by ϕY in (6.4), we get

$$S(Y, W) = 2n(f_1 - f_3)g(Y, W). \quad (6.5)$$

Hence we can state following theorem:

Theorem 6.1. *Let generalized Sasakian space forms M^{2n+1} is W_2 - ϕ -recurrent, then it is an Einstein manifold, provided $(f_1 - f_3) \neq 0$.*

7 Example

In [1], generalized complex space-form of dimension two is $N(a, b)$ and the warped product $M = R \times N$ endowed with the almost contact metric structure is a three dimensional generalized Sasakian-space-form whose smooth functions $f_1 = \frac{a-(f')^2}{f^2}$, $f_2 = \frac{b}{f^2}$ and $f_3 = \frac{a-(f')^2}{f^2} + \frac{f''}{f}$. Here $f = f(t)$, $t \in \mathbb{R}$ and f' indicates the derivative of f with respect to t . Suppose we set $a = 2$, $b = 0$ and $f(t) = t$ with $t \neq 0$, then $f_1 = \frac{1}{t^2}$, $f_2 = 0$ and $f_3 = \frac{1}{t^2}$, we have from (1.2)

$$\begin{aligned} W_2(X, Y)Z &= \frac{1}{t^2}\{g(Y, Z)X - g(X, Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} + \frac{1}{2t^2}\{g(X, Z)Y - g(Y, Z)X \\ &- g(X, Z)\eta(Y)\xi + g(Y, Z)\eta(X)\xi\}. \end{aligned} \quad (7.1)$$

Now differentiating covariantly with respect to W and taking X, Y, Z are orthogonal to ξ and then apply ϕ^2 on both side of the above equation

$$\phi^2(\nabla_W W_2(X, Y)Z) = -\frac{3}{2}d\left(\frac{1}{t^2}\right)\{g(X, Z)Y - g(Y, Z)X\}. \quad (7.2)$$

By the virtue of (7.2) we can easily say generalized Sasakian space forms is W_2 -locally ϕ -symmetric if and only if $\frac{1}{t^2}$ is constant or both f_1 and f_2 are constants.

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