Common Fixed Point Results in C^* -Algebra Valued b -Metric Spaces Via Digraphs

SUSHANTA KUMAR MOHANTA

Department of Mathematics,

West Bengal State University,,

Barasat, 24 Parganas (North), West Bengal, Kolkata 700126, India.

smwbes@yahoo.in

ABSTRACT

We discuss the existence and uniqueness of points of coincidence and common fixed points for a pair of self-mappings defined on a C^* -algebra valued b -metric space endowed with a graph. Our results extend and supplement several recent results in the literature. Strength of hypotheses made in the first result have been weighted through illustrative examples.

RESUMEN

Discutimos la existencia y unicidad de puntos de coincidencia y puntos fijos comunes para un par de aplicaciones definidas en un b -espacio métrico a valores en una álgebra C^* dotado de un grafo en sí mismo. Nuestros resultados extienden y suplementan diversos resultados recientes en la literatura. La fuerza de las hipótesis impuestas al primer resultado se evalúa a través de ejemplos ilustrativos.

Keywords and Phrases: C^* -algebra, C^* -algebra valued b -metric, directed graph, C^* -algebra valued G -contraction, common fixed point.

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1 Introduction

In 1922 [5], Polish mathematician S. Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle. This fundamental principle was largely applied in many branches of mathematics. Several authors generalized this interesting theorem in different ways (see [1, 2, 6, 13, 18, 25, 26, 27]). In this context, Bakhtin [4] and Czerwik [11] developed the notion of b -metric spaces and proved some fixed point theorems for single-valued and multi-valued mappings in the setting of b -metric spaces. In 2014, Z. Ma et.al. [22] introduced the concept of C^* -algebra valued metric spaces by using the set of all positive elements of a unital C^* -algebra instead of the set of real numbers. In [21], the authors introduced another new concept, known as C^* -algebra valued b -metric spaces as a generalization of C^* -algebra valued metric spaces and b -metric spaces.

In recent investigations, the study of fixed point theory endowed with a graph plays an important role in many aspects. In 2005, Echenique [15] studied fixed point theory by using graphs. After that, Espinola and Kirk [16] applied fixed point results in graph theory. Recently, combining fixed point theory and graph theory, a series of articles (see [3, 8, 9, 10, 20, 24] and references therein) have been dedicated to the improvement of fixed point theory.

The idea of common fixed point was initially given by Junck [19]. In fact, the author introduced the concept of weak compatibility and obtained a common fixed point result. Several authors have obtained coincidence points and common fixed points for various classes of mappings on a metric space by using this concept. Motivated by some recent works on the extension of Banach contraction principle to metric spaces with a graph, we reformulated some important common fixed point results in metric spaces to C^* -algebra valued b -metric spaces endowed with a graph. As some consequences of this study, we deduce several related results in fixed point theory. Finally, some examples are provided to illustrate the results.

2 Some basic concepts

We begin with some basic notations, definitions and properties of C^* -algebras. Let $\mathbb{A}$ be a unital algebra with the unit I . An involution on $\mathbb{A}$ is a conjugate linear map $a \mapsto a^*$ on $\mathbb{A}$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a $*$ -algebra. A Banach $*$ -algebra is a $*$ -algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ for all $a \in \mathbb{A}$. A C^* -algebra is a Banach $*$ -algebra such that $\|a^*a\| = \|a\|^2$ for all $a \in \mathbb{A}$. Let H be a Hilbert space and $B(H)$, the set of all bounded linear operators on H . Then, under the norm topology, $B(H)$ is a C^* -algebra.

Throughout this discussion, by $\mathbb{A}$ we always denote a unital C^* -algebra with the unit I and

the zero element θ . Set $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$, if $x \in \mathbb{A}_h$ and $\sigma(x) \subset [0, \infty)$, where $\sigma(x)$ is the spectrum of x . Using positive elements, one can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows:

$$x \preceq y \text{ if and only if } y - x \succeq \theta.$$

We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$.

From now on, by $\mathbb{A}_+$, we denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ and by $\mathbb{A}'$, we denote the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$.

Lemma 2.1. [14, 23] Suppose that $\mathbb{A}$ is an unital C^* -algebra with a unit I .

- (i) For any $x \in \mathbb{A}_+$, we have $x \preceq I \Leftrightarrow \|x\| \leq 1$.
- (ii) If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(I - a)^{-1}\| < 1$.
- (iii) Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$.
- (iv) Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$, and $I - a \in \mathbb{A}'_+$ is an invertible operator, then $(I - a)^{-1}b \succeq (I - a)^{-1}c$.

Remark 2.2. It is worth mentioning that $x \preceq y \Rightarrow \|x\| \leq \|y\|$ for $x, y \in \mathbb{A}_+$. In fact, it follows from Lemma 2.1 (i).

Definition 2.3. [22] Let X be a nonempty set. Suppose the mapping $d : X \times X \rightarrow \mathbb{A}$ satisfies:

- (i) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a C^* -algebra valued metric on X and $(X, \mathbb{A}, d)$ is called a C^* -algebra valued metric space.

Definition 2.4. [4] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \rightarrow \mathbb{R}^+$ is said to be a b -metric on X if the following conditions hold:

- (i) $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, y) \leq s(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b -metric space.

Definition 2.5. [21] Let X be a nonempty set and $A \in \mathbb{A}'_+$ such that $A \succeq I$. Suppose the mapping $d : X \times X \rightarrow \mathbb{A}$ satisfies:

- (i) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, y) \preceq A(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

Then d is called a C^* -algebra valued b -metric on X and $(X, \mathbb{A}, d)$ is called a C^* -algebra valued b -metric space.

It seems important to note that if $\mathbb{A} = \mathbb{C}, A = 1$, then the C^* -algebra valued b -metric spaces are just the ordinary metric spaces. Moreover, it is obvious that C^* -algebra valued b -metric spaces generalize the concepts of C^* -algebra valued metric spaces and b -metric spaces.

Definition 2.6. [26] Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space, $x \in X$ and (x_n) be a sequence in X . Then

- (i) (x_n) converges to x with respect to $\mathbb{A}$ if for any $\epsilon > 0$ there is n_0 such that for all $n > n_0$, $\|d(x_n, x)\| \leq \epsilon$. We denote it by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x (n \rightarrow \infty)$.
- (ii) (x_n) is Cauchy with respect to $\mathbb{A}$ if for any $\epsilon > 0$ there is n_0 such that for all $n, m > n_0$, $\|d(x_n, x_m)\| \leq \epsilon$.
- (iii) $(X, \mathbb{A}, d)$ is a complete C^* -algebra valued b -metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Example 2.7. If X is a Banach space, then $(X, \mathbb{A}, d)$ is a complete C^* -algebra valued b -metric space with $A = 2^{p-1}I$ if we set

$$d(x, y) = \|x - y\|^p I$$

where $p > 1$ is a real number. But $(X, \mathbb{A}, d)$ is not a C^* -algebra valued metric space because if $X = \mathbb{R}$, then $|x - y|^p \leq |x - z|^p + |z - y|^p$ is impossible for all $x > z > y$.

Definition 2.8. Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space with the coefficient $A \succeq I$. We call a mapping $f : X \rightarrow X$ a C^* -algebra valued contraction mapping on X if there exists $B \in \mathbb{A}$ with $\|B\|^2 < \frac{1}{\|A\|}$ such that

$$d(fx, fy) \preceq B^* d(x, y) B$$

for all $x, y \in X$.

Definition 2.9. Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space with the coefficient $A \succeq I$. A mapping $f : X \rightarrow X$ is called a C^* -algebra valued Fisher contraction if there exists $B \in \mathbb{A}'_+$ with $\|BA\| < \frac{1}{\|A\|+1}$ such that

$$d(fx, fy) \preceq B[d(fx, y) + d(fy, x)]$$

for all $x, y \in X$.

Definition 2.10. Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space with the coefficient $A \succeq I$. A mapping $f : X \rightarrow X$ is called a C^* -algebra valued Kannan operator if there exists $B \in \mathbb{A}'_+$ with $\|B\| < \frac{1}{\|A\|+1}$ such that

$$d(fx, fy) \preceq B [d(fx, x) + d(fy, y)]$$

for all $x, y \in X$.

Definition 2.11. [2] Let T and S be self mappings of a set X . If $y = Tx = Sx$ for some x in X , then x is called a coincidence point of T and S and y is called a point of coincidence of T and S .

Definition 2.12. [19] The mappings $T, S : X \rightarrow X$ are weakly compatible, if for every $x \in X$, the following holds:

$$T(Sx) = S(Tx) \text{ whenever } Sx = Tx.$$

Proposition 2.13. [2] Let S and T be weakly compatible selfmaps of a nonempty set X . If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T .

Definition 2.14. Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space with the coefficient $A \succeq I$. A mapping $f : X \rightarrow X$ is called C^* -algebra valued expansive if there exists $B \in \mathbb{A}$ with $0 < \|B\|^2 < \frac{1}{\|A\|}$ such that

$$B^* d(fx, fy) B \succeq d(x, y)$$

for all $x, y \in X$.

We next review some basic notions in graph theory.

Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space. Let G be a directed graph (digraph) with a set of vertices $V(G) = X$ and a set of edges $E(G)$ contains all the loops, i.e., $E(G) \supseteq \Delta$, where $\Delta = \{(x, x) : x \in X\}$. We also assume that G has no parallel edges and so we can identify G with the pair $(V(G), E(G))$. G may be considered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the graph obtained from G by reversing the direction of edges i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [7, 12, 17]. If x, y are vertices of the digraph G , then a path in G from x to y of length n ($n \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^n$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \dots, n$. A graph G is connected if there is a path between any two vertices of G . G is weakly connected if $\tilde{G}$ is connected.

Definition 2.15. Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space with the coefficient $A \succeq I$ and let $G = (V(G), E(G))$ be a graph. A mapping $f : X \rightarrow X$ is called a C^* -algebra valued G -contraction if there exists a $B \in \mathbb{A}$ with $\|B\|^2 < \frac{1}{\|A\|}$ such that

$$d(fx, fy) \preceq B^* d(x, y) B,$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

Any C^* -algebra valued contraction mapping on X is a G_0 -contraction, where G_0 is the complete graph defined by $(X, X \times X)$. But it is worth mentioning that a C^* -algebra valued G -contraction need not be a C^* -algebra valued contraction (see Remark 3.23).

Definition 2.16. Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space with the coefficient $A \succeq I$ and let $G = (V(G), E(G))$ be a graph. A mapping $f : X \rightarrow X$ is called C^* -algebra valued Fisher G -contraction if there exists $B \in \mathbb{A}'_+$ with $\|BA\| < \frac{1}{\|A\|+1}$ such that

$$d(fx, fy) \preceq B [d(fx, y) + d(fy, x)]$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

It is easy to observe that a C^* -algebra valued Fisher contraction is a C^* -algebra valued Fisher G_0 -contraction. But it is important to note that a C^* -algebra valued Fisher G -contraction need not be a C^* -algebra valued Fisher contraction. The following example supports the above remark.

Example 2.17. Let $X = [0, \infty)$ and $B(H)$ be the set of all bounded linear operators on a Hilbert space H . Define $d : X \times X \rightarrow B(H)$ by $d(x, y) = |x - y|^2 I$ for all $x, y \in X$. Then $(X, B(H), d)$ is a C^* -algebra valued b -metric space with the coefficient $A = 2I$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(3^t x, 3^t(x+1)) : x \in X \text{ with } x \geq 2, t = 0, 1, 2, \dots\}$.

Let $f : X \rightarrow X$ be defined by $fx = 3x$ for all $x \in X$.

For $x = 3^t z$, $y = 3^t(z+1)$, $z \geq 2$, we have

$$\begin{aligned} d(fx, fy) &= d(3^{t+1}z, 3^{t+1}(z+1)) \\ &= 3^{2t+2}I \\ &\preceq \frac{9}{58} 3^{2t}(8z^2 + 8z + 10)I \\ &= B [d(3^{t+1}z, 3^t(z+1)) + d(3^{t+1}(z+1), 3^t z)] \\ &= B [d(fx, y) + d(fy, x)], \end{aligned}$$

where $B = \frac{9}{58}I \in B(H)'_+$ with $\|BA\| < \frac{1}{\|A\|+1}$. Thus, f is a C^* -algebra valued Fisher G -contraction.

We now verify that f is not a C^* -algebra valued Fisher contraction. In fact, if $x = 3$, $y = 0$,

then for any arbitrary $B \in B(H)'_+$ with $\|BA\| < \frac{1}{\|A\|+1} = \frac{1}{3}$ (which implies $3BA \prec I$), we have

$$\begin{aligned} B[d(fx, y) + d(fy, x)] &= B[d(f3, 0) + d(f0, 3)] \\ &= 90BI \\ &= 45BA \\ &= \frac{5}{27}(3BA)(81I) \\ &< 81I \\ &= d(fx, fy). \end{aligned}$$

Definition 2.18. Let $(X, \mathbb{A}, d)$ be a C*-algebra valued b-metric space with the coefficient $A \succeq I$ and let $G = (V(G), E(G))$ be a graph. A mapping $f : X \rightarrow X$ is called C*-algebra valued G-Kannan if there exists $B \in \mathbb{A}'_+$ with $\|B\| < \frac{1}{\|A\|+1}$ such that

$$d(fx, fy) \preceq B[d(fx, x) + d(fy, y)]$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

Note that any C*-algebra valued Kannan operator is C*-algebra valued G_0 -Kannan. However, a C*-algebra valued G-Kannan operator need not be a C*-algebra valued Kannan operator (see Remark 3.28).

Remark 2.19. If f is a C*-algebra valued G-contraction (resp., G-Kannan or Fisher G-contraction), then f is both a C*-algebra valued G^{-1} -contraction (resp., G^{-1} -Kannan or Fisher G^{-1} -contraction) and a C*-algebra valued $\tilde{G}$ -contraction (resp., $\tilde{G}$ -Kannan or Fisher $\tilde{G}$ -contraction).

3 Main Results

In this section we always assume that $(X, \mathbb{A}, d)$ is a C*-algebra valued b-metric space with the coefficient $A \succeq I$ and G is a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$.

Let $f, g : X \rightarrow X$ be such that $f(X) \subseteq g(X)$. If $x_0 \in X$ is arbitrary, then there exists an element $x_1 \in X$ such that $fx_0 = gx_1$, since $f(X) \subseteq g(X)$. Proceeding in this way, we can construct a sequence (gx_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \dots$.

Definition 3.1. Let $(X, \mathbb{A}, d)$ be a C*-algebra valued b-metric space endowed with a graph G and $f, g : X \rightarrow X$ be such that $f(X) \subseteq g(X)$. We define $C_{g,f}$ the set of all elements x_0 of X such that $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \dots$ and for every sequence (gx_n) such that $gx_n = fx_{n-1}$.

If $g = I$, the identity map on X , then obviously $C_{g,f}$ becomes C_f which is the collection of all elements x of X such that $(f^n x, f^m x) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \dots$.

Theorem 3.2. *Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space endowed with a graph G and the mappings $f, g : X \rightarrow X$ be such that*

$$d(fx, fy) \preceq B^* d(gx, gy) B \quad (3.1)$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, where $B \in \mathbb{A}$ and $\|B\|^2 < \frac{1}{\|\mathbb{A}\|}$. Suppose $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X with the following property:

() If (gx_n) is a sequence in X such that $gx_n \rightarrow x$ and $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$. Then f and g have a point of coincidence in X if $C_{gf} \neq \emptyset$. Moreover, f and g have a unique point of coincidence in X if the graph G has the following property:*

(**) If x, y are points of coincidence of f and g in X , then $(x, y) \in E(\tilde{G})$. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X .

Proof. Suppose that $C_{gf} \neq \emptyset$. We choose an $x_0 \in C_{gf}$ and keep it fixed. Since $f(X) \subseteq g(X)$, there exists a sequence (gx_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \dots$ and $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \dots$.

It is a well known fact that in a C^* -algebra $\mathbb{A}$, if $a, b \in \mathbb{A}_+$ and $a \preceq b$, then for any $x \in \mathbb{A}$ both x^*ax and x^*bx are positive elements and $x^*ax \preceq x^*bx$ [23].

For any $n \in \mathbb{N}$, we have by using condition (3.1) that

$$d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n) \preceq B^* d(gx_{n-1}, gx_n) B. \quad (3.2)$$

By repeated use of condition (3.2), we get

$$d(gx_n, gx_{n+1}) \preceq (B^*)^n d(gx_0, gx_1) B^n = (B^n)^* B_0 B^n, \quad (3.3)$$

for all $n \in \mathbb{N}$, where $B_0 = d(gx_0, gx_1) \in \mathbb{A}_+$.

For any $m, n \in \mathbb{N}$ with $m > n$, we have by using condition (3.3) that

$$\begin{aligned}
 d(gx_n, gx_m) &\preceq A[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\
 &\preceq Ad(gx_n, gx_{n+1}) + A^2d(gx_{n+1}, gx_{n+2}) + \cdots \\
 &\quad + A^{m-n-1}d(gx_{m-2}, gx_{m-1}) + A^{m-n-1}d(gx_{m-1}, gx_m) \\
 &\preceq A(B^*)^n B_0 B^n + A^2(B^*)^{n+1} B_0 B^{n+1} + A^3(B^*)^{n+2} B_0 B^{n+2} + \cdots \\
 &\quad + A^{m-n-1}(B^*)^{m-2} B_0 B^{m-2} + A^{m-n-1}(B^*)^{m-1} B_0 B^{m-1} \\
 &\preceq \sum_{k=1}^{m-n-1} A^k (B^*)^{n+k-1} B_0 B^{n+k-1} + A^{m-n} (B^*)^{m-1} B_0 B^{m-1} \\
 &= \sum_{k=1}^{m-n} A^k (B^*)^{n+k-1} B_0 B^{n+k-1} \\
 &\preceq \sum_{k=1}^{m-n} \|A^k (B^*)^{n+k-1} B_0 B^{n+k-1}\| I \\
 &\preceq \|B_0\| \sum_{k=1}^{m-n} \|A\|^k \|B\|^{2(n+k-1)} I \\
 &= \|B_0\| \|B\|^{2n} \|A\| \sum_{k=1}^{m-n} (\|A\| \|B\|^2)^{k-1} I \\
 &\preceq \frac{\|B_0\| \|B\|^{2n} \|A\|}{1 - \|A\| \|B\|^2} I, \text{ since } \|B\|^2 < \frac{1}{\|A\|} \\
 &\rightarrow \theta \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Therefore, (gx_n) is a Cauchy sequence with respect to $\mathbb{A}$. Since $g(X)$ is complete, there exists an $u \in g(X)$ such that $\lim_{n \rightarrow \infty} gx_n = u = gv$ for some $v \in X$.

As $x_0 \in C_{gf}$, it follows that $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geq 0$, and so by property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$.

Using condition (3.1), we have

$$\begin{aligned}
 d(fv, gv) &\preceq A[d(fv, fx_{n_i}) + d(fx_{n_i}, gv)] \\
 &\preceq AB^*d(gv, gx_{n_i})B + Ad(gx_{n_i+1}, gv) \\
 &\rightarrow \theta \text{ as } i \rightarrow \infty.
 \end{aligned}$$

This implies that $d(fv, gv) = \theta$ and hence $fv = gv = u$. Therefore, u is a point of coincidence of f and g .

The next is to show that the point of coincidence is unique. Assume that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property $(**)$, we have

$(u, u^*) \in E(\tilde{G})$. Then,

$$\begin{aligned} d(u, u^*) &= d(fv, fx) \\ &\preceq B^* d(gv, gx) B \\ &= B^* d(u, u^*) B, \end{aligned}$$

which implies that,

$$\begin{aligned} \|d(u, u^*)\| &\leq \|B^* d(u, u^*) B\| \\ &\leq \|B^*\| \|d(u, u^*)\| \|B\| \\ &= \|B\|^2 \|d(u, u^*)\|. \end{aligned}$$

Since $\|B\|^2 < \frac{1}{\|A\|} \leq 1$, it follows that $d(u, u^*) = \theta$ i.e., $u = u^*$. Therefore, f and g have a unique point of coincidence in X .

If f and g are weakly compatible, then by Proposition 2.13, f and g have a unique common fixed point in X . $\square$

The following corollary gives fixed point of Banach G -contraction in C^* -algebra valued b -metric spaces.

Corollary 3.3. *Let $(X, \mathbb{A}, d)$ be a complete C^* -algebra valued b -metric space endowed with a graph G and the mapping $f : X \rightarrow X$ be such that*

$$d(fx, fy) \preceq B^* d(x, y) B \quad (3.4)$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $B \in \mathbb{A}$ with $\|B\|^2 < \frac{1}{\|A\|}$. Suppose $(X, \mathbb{A}, d, G)$ has the following property:

(*) *If (x_n) is a sequence in X such that $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.*

Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the following property:

(***) *If x, y are fixed points of f in X , then $(x, y) \in E(\tilde{G})$.*

Proof. The proof can be obtained from Theorem 3.2 by considering $g = I$, the identity map on X . $\square$

Corollary 3.4. *Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space and the mappings $f, g : X \rightarrow X$ be such that (3.1) holds for all $x, y \in X$, where $B \in \mathbb{A}$ with $\|B\|^2 < \frac{1}{\|A\|}$. If $f(X) \subseteq g(X)$ and*

$g(X)$ is a complete subspace of X , then f and g have a unique point of coincidence in X . Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X .

Proof. The proof follows from Theorem 3.2 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. □

The following corollary is analogue of Banach Contraction Principle.

Corollary 3.5. *Let $(X, \mathbb{A}, d)$ be a complete C^* -algebra valued b -metric space and the mapping $f : X \rightarrow X$ be such that (3.4) holds for all $x, y \in X$, where $B \in \mathbb{A}$ with $\|B\|^2 < \frac{1}{\|\mathbb{A}\|}$. Then f has a unique fixed point u in X and $f^n x \rightarrow u$ for all $x \in X$.*

Proof. It follows from Theorem 3.2 by putting $G = G_0$ and $g = I$. □

Remark 3.6. *We observe that Banach contraction theorem in a complete metric space can be obtained from Corollary 3.5 by taking $\mathbb{A} = \mathbb{C}, \mathbb{A} = 1$. Thus, Theorem 3.2 is a generalization of Banach contraction theorem in metric spaces to C^* -algebra valued b -metric spaces.*

From Theorem 3.2, we obtain the following corollary concerning the fixed point of expansive mapping in C^* -algebra valued b -metric spaces.

Corollary 3.7. *Let $(X, \mathbb{A}, d)$ be a complete C^* -algebra valued b -metric space and let $g : X \rightarrow X$ be an onto mapping satisfying*

$$B^* d(gx, gy) B \succeq d(x, y)$$

for all $x, y \in X$, where $B \in \mathbb{A}$ with $\|B\|^2 < \frac{1}{\|\mathbb{A}\|}$. Then g has a unique fixed point in X .

Proof. The conclusion of the corollary follows from Theorem 3.2 by taking $G = G_0$ and $f = I$. □

Corollary 3.8. *Let $(X, \mathbb{A}, d)$ be a complete C^* -algebra valued b -metric space endowed with a partial ordering $\sqsubseteq$ and the mapping $f : X \rightarrow X$ be such that (3.4) holds for all $x, y \in X$ with $x \sqsubseteq y$ or, $y \sqsubseteq x$, where $B \in \mathbb{A}$ and $\|B\|^2 < \frac{1}{\|\mathbb{A}\|}$. Suppose $(X, \mathbb{A}, d, \sqsubseteq)$ has the following property:*

(†) *If (x_n) is a sequence in X such that $x_n \rightarrow x$ and x_n, x_{n+1} are comparable for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that x_{n_i}, x are comparable for all $i \geq 1$. If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for $m, n = 0, 1, 2, \dots$, then f has a fixed point in X . Moreover, f has a unique fixed point in X if the following property holds:*

(††) *If x, y are fixed points of f in X , then x, y are comparable.*

Proof. The proof can be obtained from Theorem 3.2 by taking $g = I$ and $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \sqsubseteq y \text{ or } y \sqsubseteq x\}$. □

Theorem 3.9. *Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space endowed with a graph G and the mappings $f, g : X \rightarrow X$ be such that*

$$d(fx, fy) \preceq B [d(fx, gy) + d(fy, gx)] \quad (3.5)$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, where $B \in \mathbb{A}'_+$ and $\|BA\| < \frac{1}{\|A\|+1}$. Suppose $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X with the property $()$. Then f and g have a point of coincidence in X if $C_{gf} \neq \emptyset$. Moreover, f and g have a unique point of coincidence in X if the graph G has the property $(**)$. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X .*

Proof. It follows from condition (3.5) that $B(d(fx, gy) + d(fy, gx))$ is a positive element.

Suppose that $C_{gf} \neq \emptyset$. We choose an $x_0 \in C_{gf}$ and keep it fixed. We can construct a sequence (gx_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \dots$. Evidently, $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \dots$.

For any $n \in \mathbb{N}$, we have by using condition (3.5) and Lemma 2.1(iii) that

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\preceq B[d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})] \\ &= B[d(fx_{n-1}, fx_{n-1}) + d(fx_n, fx_{n-2})] \\ &\preceq BA[d(fx_n, fx_{n-1}) + d(fx_{n-1}, fx_{n-2})] \\ &= BA d(gx_{n+1}, gx_n) + BA d(gx_n, gx_{n-1}) \end{aligned}$$

which implies that,

$$(I - BA)d(gx_n, gx_{n+1}) \preceq BA d(gx_n, gx_{n-1}). \quad (3.6)$$

Now, $A, B \in \mathbb{A}'_+$ implies that $BA \in \mathbb{A}'_+$. Since $\|BA\| < \frac{1}{2}$, by Lemma 2.1, it follows that $(I - BA)$ is invertible and $\|BA(I - BA)^{-1}\| = \|(I - BA)^{-1}BA\| < 1$. Moreover, by Lemma 2.1, $BA \preceq I$ i.e., $I - BA \succeq \theta$. Since $BA \in \mathbb{A}'_+$, we have $(I - BA) \in \mathbb{A}'_+$. Furthermore, $(I - BA)^{-1} \in \mathbb{A}'_+$. By using Lemma 2.1(iv), it follows from (3.6) that

$$d(gx_n, gx_{n+1}) \preceq (I - BA)^{-1}BA d(gx_n, gx_{n-1}) = td(gx_{n-1}, gx_n), \quad (3.7)$$

where $t = (I - BA)^{-1}BA \in \mathbb{A}'_+$.

By repeated use of condition (3.7), we get

$$d(gx_n, gx_{n+1}) \preceq t^n d(gx_0, gx_1) = t^n B_0, \quad (3.8)$$

for all $n \in \mathbb{N}$, where $B_0 = d(gx_0, gx_1) \in \mathbb{A}_+$.

We now prove that if $\|BA\| < \frac{1}{\|A\|+1}$, then $\|t\| < \frac{1}{\|A\|}$.
We have,

$$\begin{aligned}\|t\| &= \|(I - BA)^{-1}BA\| \\ &\leq \|(I - BA)^{-1}\| \|BA\| \\ &\leq \frac{1}{1 - \|BA\|} \|BA\| \\ &< \frac{1}{\|A\|}, \text{ since } \|BA\| < \frac{1}{\|A\|+1}.\end{aligned}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have by using condition (3.8) that

$$\begin{aligned}d(gx_n, gx_m) &\preceq A[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\ &\preceq Ad(gx_n, gx_{n+1}) + A^2d(gx_{n+1}, gx_{n+2}) + \dots \\ &\quad + A^{m-n-1}d(gx_{m-2}, gx_{m-1}) + A^{m-n-1}d(gx_{m-1}, gx_m) \\ &\preceq At^nB_0 + A^2t^{n+1}B_0 + A^3t^{n+2}B_0 + \dots \\ &\quad + A^{m-n-1}t^{m-2}B_0 + A^{m-n-1}t^{m-1}B_0 \\ &\preceq \sum_{k=1}^{m-n} A^k t^{n+k-1} B_0, \text{ since } A \succeq I \text{ and } A \in \mathbb{A}'_+ \\ &\preceq \sum_{k=1}^{m-n} \|A^k t^{n+k-1} B_0\| I \\ &\preceq \|B_0\| \|A\| \|t\|^n \sum_{k=1}^{m-n} (\|A\| \|t\|)^{k-1} I \\ &\preceq \|B_0\| \|A\| \|t\|^n \frac{1}{1 - \|A\| \|t\|} I \\ &\rightarrow \theta \text{ as } n \rightarrow \infty.\end{aligned}$$

Therefore, (gx_n) is a Cauchy sequence with respect to $\mathbb{A}$. As $g(X)$ is complete, there exists an $u \in g(X)$ such that $\lim_{n \rightarrow \infty} gx_n = u = gv$ for some $v \in X$. By property (*), there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$.

Using condition (3.5), we have

$$\begin{aligned}d(fv, gv) &\preceq A[d(fv, fx_{n_i}) + d(fx_{n_i}, gv)] \\ &\preceq AB[d(fv, gx_{n_i}) + d(fx_{n_i}, gv)] + Ad(gx_{n_i+1}, gv) \\ &\preceq ABA[d(fv, gv) + d(gv, gx_{n_i})] + ABd(gx_{n_i+1}, gv) + Ad(gx_{n_i+1}, gv)\end{aligned}$$

which implies that,

$$(I - BA^2)d(fv, gv) \preceq BA^2d(gv, gx_{n_i}) + ABd(gx_{n_i+1}, gv) + Ad(gx_{n_i+1}, gv).$$

Since $\|BA^2\| < \frac{\|A\|}{\|A\|+1} < 1$, we have $(I - BA^2)^{-1}$ exists. By using Lemma 2.1, it follows that

$$\begin{aligned} d(fv, gv) &\preceq (I - BA^2)^{-1} BA^2 d(gv, gx_{n_i}) + (I - BA^2)^{-1} ABd(gx_{n_i+1}, gv) \\ &\quad + (I - BA^2)^{-1} Ad(gx_{n_i+1}, gv) \\ &\rightarrow \theta \text{ as } i \rightarrow \infty. \end{aligned}$$

This implies that $d(fv, gv) = \theta$ i.e., $fv = gv = u$ and hence u is a point of coincidence of f and g .

Finally, to prove the uniqueness of point of coincidence, suppose that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property (**), we have $(u, u^*) \in E(\tilde{G})$. Then,

$$\begin{aligned} d(u, u^*) &= d(fv, fx) \\ &\preceq B[d(fv, gx) + d(fx, gv)] \\ &= B[d(u, u^*) + d(u, u^*)] \\ &\preceq AB[d(u, u^*) + d(u, u^*)] \end{aligned}$$

which implies that,

$$d(u, u^*) \preceq (I - AB)^{-1} AB d(u, u^*).$$

So, it must be the case that

$$\begin{aligned} \|d(u, u^*)\| &\leq \|(I - AB)^{-1} AB d(u, u^*)\| \\ &\leq \|(I - AB)^{-1} AB\| \|d(u, u^*)\|. \end{aligned}$$

Since $\|(I - AB)^{-1} AB\| < 1$, we have $\|d(u, u^*)\| = 0$ i.e., $u = u^*$. Therefore, f and g have a unique point of coincidence in X .

If f and g are weakly compatible, then by Proposition 2.13, f and g have a unique common fixed point in X . $\square$

Corollary 3.10. *Let $(X, \mathbb{A}, d)$ be a complete C^* -algebra valued b -metric space endowed with a graph G and the mapping $f : X \rightarrow X$ be such that*

$$d(fx, fy) \preceq B[d(fx, y) + d(fy, x)] \quad (3.9)$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $B \in \mathbb{A}'_+$ and $\|BA\| < \frac{1}{\|A\|+1}$. Suppose $(X, \mathbb{A}, d, G)$ has the property $()'$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the property $(***)'$.*

Proof. The proof can be obtained from Theorem 3.9 by putting $g = I$. $\square$

Corollary 3.11. *Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space and the mappings $f, g : X \rightarrow X$ be such that (3.5) holds for all $x, y \in X$, where $B \in \mathbb{A}'_+$ and $\|BA\| < \frac{1}{\|A\|+1}$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then f and g have a unique point of coincidence in X . Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X .*

Proof. The proof can be obtained from Theorem 3.9 by taking $G = G_0$. □

Corollary 3.12. *Let $(X, \mathbb{A}, d)$ be a complete C^* -algebra valued b -metric space and the mapping $f : X \rightarrow X$ be such that (3.9) holds for all $x, y \in X$, where $B \in \mathbb{A}'_+$ with $\|BA\| < \frac{1}{\|A\|+1}$. Then f has a unique fixed point in X .*

Proof. The proof follows from Theorem 3.9 by taking $G = G_0$ and $g = I$. □

Remark 3.13. *We observe that Brian Fisher's theorem in a complete metric space can be obtained from Corollary 3.12 by taking $\mathbb{A} = \mathbb{C}, A = 1$. Thus, Theorem 3.9 is a generalization of Brian Fisher's theorem in metric spaces to C^* -algebra valued b -metric spaces.*

Corollary 3.14. *Let $(X, \mathbb{A}, d)$ be a complete C^* -algebra valued b -metric space endowed with a partial ordering $\sqsubseteq$ and the mapping $f : X \rightarrow X$ be such that (3.9) holds for all $x, y \in X$ with $x \sqsubseteq y$ or, $y \sqsubseteq x$, where $B \in \mathbb{A}'_+$ with $\|BA\| < \frac{1}{\|A\|+1}$. Suppose $(X, \mathbb{A}, d, \sqsubseteq)$ has the property $(\dagger)$. If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for $m, n = 0, 1, 2, \dots$, then f has a fixed point in X . Moreover, f has a unique fixed point in X if the property $(\dagger\dagger)$ holds.*

Proof. The proof can be obtained from Theorem 3.9 by taking $G = G_2$ and $g = I$. □

Theorem 3.15. *Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space endowed with a graph G and the mappings $f, g : X \rightarrow X$ be such that*

$$d(fx, fy) \preceq B [d(fx, gx) + d(fy, gy)] \quad (3.10)$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, where $B \in \mathbb{A}'_+$ and $\|B\| < \frac{1}{\|A\|+1}$. Suppose $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X with the property $()$. Then f and g have a point of coincidence in X if $C_{gf} \neq \emptyset$. Moreover, f and g have a unique point of coincidence in X if the graph G has the property $(**)$. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X .*

Proof. We observe that $B(d(fx, gx) + d(fy, gy))$ is a positive element.

Suppose that $C_{gf} \neq \emptyset$. We choose an $x_0 \in C_{gf}$ and keep it fixed. We can construct a sequence (gx_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \dots$. Evidently, $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \dots$.

For any $n \in \mathbb{N}$, we have by using condition (3.10) that

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\preceq B[d(fx_{n-1}, gx_{n-1}) + d(fx_n, gx_n)] \\ &= B d(gx_n, gx_{n-1}) + B d(gx_n, gx_{n+1}) \end{aligned}$$

which implies that,

$$(I - B)d(gx_n, gx_{n+1}) \preceq B d(gx_n, gx_{n-1}). \quad (3.11)$$

Since $B \in \mathbb{A}'_+$ and $\|B\| < \frac{1}{2}$, by Lemma 2.1, it follows that $B \preceq I$ and $(I - B)$ is invertible with $\|B(I - B)^{-1}\| = \|(I - B)^{-1}B\| < 1$. Furthermore, $(I - B), (I - B)^{-1} \in \mathbb{A}'_+$ and so, $(I - B)^{-1}B \in \mathbb{A}'_+$. Again, by using Lemma 2.1(iv), it follows from condition (3.11) that

$$d(gx_n, gx_{n+1}) \preceq (I - B)^{-1}B d(gx_n, gx_{n-1}) = t d(gx_{n-1}, gx_n), \quad (3.12)$$

where $t = (I - B)^{-1}B \in \mathbb{A}'_+$.

By repeated use of condition (3.12), we get

$$d(gx_n, gx_{n+1}) \preceq t^n d(gx_0, gx_1) = t^n B_0, \quad (3.13)$$

for all $n \in \mathbb{N}$, where $B_0 = d(gx_0, gx_1) \in \mathbb{A}_+$.

We now prove that if $\|B\| < \frac{1}{\|A\|+1}$, then $\|t\| < \frac{1}{\|A\|}$.
We have,

$$\begin{aligned} \|t\| &= \|(I - B)^{-1}B\| \\ &\leq \|(I - B)^{-1}\| \|B\| \\ &\leq \frac{1}{1 - \|B\|} \|B\| \\ &< \frac{1}{\|A\|}, \text{ since } \|B\| < \frac{1}{\|A\|+1}. \end{aligned}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have by using condition (3.13) that

$$\begin{aligned}
 d(gx_n, gx_m) &\preceq A[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\
 &\preceq Ad(gx_n, gx_{n+1}) + A^2d(gx_{n+1}, gx_{n+2}) + \dots \\
 &\quad + A^{m-n-1}d(gx_{m-2}, gx_{m-1}) + A^{m-n-1}d(gx_{m-1}, gx_m) \\
 &\preceq At^nB_0 + A^2t^{n+1}B_0 + A^3t^{n+2}B_0 + \dots \\
 &\quad + A^{m-n-1}t^{m-2}B_0 + A^{m-n-1}t^{m-1}B_0 \\
 &\preceq \sum_{k=1}^{m-n} A^k t^{n+k-1} B_0, \text{ since } A \succeq I \text{ and } A \in \mathbb{A}'_+ \\
 &\preceq \sum_{k=1}^{m-n} \|A^k t^{n+k-1} B_0\| I \\
 &\preceq \|B_0\| \|A\| \|t\|^n \sum_{k=1}^{m-n} (\|A\| \|t\|)^{k-1} I \\
 &\preceq \|B_0\| \|A\| \|t\|^n \frac{1}{1 - \|A\| \|t\|} I \\
 &\rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Therefore, (gx_n) is a Cauchy sequence with respect to $\mathbb{A}$. By completeness of $g(X)$, there exists an $u \in g(X)$ such that $\lim_{n \rightarrow \infty} gx_n = u = gv$ for some $v \in X$. By property (*), there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$.

Using condition (3.10), we have

$$\begin{aligned}
 d(fv, gv) &\preceq A[d(fv, fx_{n_i}) + d(fx_{n_i}, gv)] \\
 &\preceq AB[d(fv, gv) + d(fx_{n_i}, gx_{n_i})] + Ad(gx_{n_i+1}, gv)
 \end{aligned}$$

which implies that,

$$(I - AB)d(fv, gv) \preceq ABd(gx_{n_i+1}, gx_{n_i}) + Ad(gx_{n_i+1}, gv).$$

Since $\|AB\| < \frac{\|A\|}{\|A\|+1} < 1$, we have $(I - AB)^{-1}$ exists and $(I - AB) \in \mathbb{A}'_+$. By using Lemma 2.1, it follows that

$$d(fv, gv) \preceq (I - AB)^{-1}ABd(gx_{n_i+1}, gx_{n_i}) + (I - AB)^{-1}Ad(gx_{n_i+1}, gv).$$

Then,

$$\begin{aligned}
 \|d(fv, gv)\| &\leq \|(I - AB)^{-1}AB\| \|d(gx_{n_i+1}, gx_{n_i})\| \\
 &\quad + \|(I - AB)^{-1}A\| \|d(gx_{n_i+1}, gv)\| \\
 &\leq \|(I - AB)^{-1}AB\| \|t\|^{n_i} \|B_0\| \\
 &\quad + \|(I - AB)^{-1}A\| \|d(gx_{n_i+1}, gv)\| \\
 &\rightarrow 0 \text{ as } i \rightarrow \infty.
 \end{aligned}$$

This implies that $d(fv, gv) = \theta$ i.e., $fv = gv = u$ and hence u is a point of coincidence of f and g .

Finally, to prove the uniqueness of point of coincidence, suppose that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property (**), we have $(u, u^*) \in E(\tilde{G})$. Then,

$$\begin{aligned} d(u, u^*) &= d(fv, fx) \\ &\preceq B[d(fv, gv) + d(fx, gx)] \\ &= \theta \end{aligned}$$

which implies that, $d(u, u^*) = \theta$ i.e., $u = u^*$. Therefore, f and g have a unique point of coincidence in X .

If f and g are weakly compatible, then by Proposition 2.13, f and g have a unique common fixed point in X . $\square$

Corollary 3.16. *Let $(X, \mathbb{A}, d)$ be a complete C^* -algebra valued b -metric space endowed with a graph G and the mapping $f : X \rightarrow X$ be such that*

$$d(fx, fy) \preceq B [d(fx, x) + d(fy, y)] \quad (3.14)$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $B \in \mathbb{A}'_+$ and $\|B\| < \frac{1}{\|\mathbb{A}\|+1}$. Suppose $(X, \mathbb{A}, d, G)$ has the property $()$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the property $(**)$.*

Proof. The proof can be obtained from Theorem 3.15 by putting $g = I$. $\square$

Corollary 3.17. *Let $(X, \mathbb{A}, d)$ be a C^* -algebra valued b -metric space and the mappings $f, g : X \rightarrow X$ be such that (3.10) holds for all $x, y \in X$, where $B \in \mathbb{A}'_+$ and $\|B\| < \frac{1}{\|\mathbb{A}\|+1}$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then f and g have a unique point of coincidence in X . Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X .*

Proof. The proof can be obtained from Theorem 3.15 by taking $G = G_0$. $\square$

Corollary 3.18. *Let $(X, \mathbb{A}, d)$ be a complete C^* -algebra valued b -metric space and the mapping $f : X \rightarrow X$ be such that (3.14) holds for all $x, y \in X$, where $B \in \mathbb{A}'_+$ with $\|B\| < \frac{1}{\|\mathbb{A}\|+1}$. Then f has a unique fixed point in X .*

Proof. The proof follows from Theorem 3.15 by taking $G = G_0$ and $g = I$. $\square$

Remark 3.19. We observe that Kannan's fixed point theorem in a complete metric space can be obtained from Corollary 3.18 by taking $\mathbb{A} = \mathbb{C}$, $A = 1$. Thus, Theorem 3.15 is a generalization of Kannan's fixed point theorem in metric spaces to C^* -algebra valued $\mathbf{b}$ -metric spaces.

Corollary 3.20. Let $(X, \mathbb{A}, d)$ be a complete C^* -algebra valued $\mathbf{b}$ -metric space endowed with a partial ordering $\sqsubseteq$ and the mapping $f : X \rightarrow X$ be such that (3.14) holds for all $x, y \in X$ with $x \sqsubseteq y$ or, $y \sqsubseteq x$, where $B \in \mathbb{A}'_+$ with $\|B\| < \frac{1}{\|\mathbb{A}\|+1}$. Suppose $(X, \mathbb{A}, d, \sqsubseteq)$ has the property $(\dagger)$. If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for $m, n = 0, 1, 2, \dots$, then f has a fixed point in X . Moreover, f has a unique fixed point in X if the property $(\dagger\dagger)$ holds.

Proof. The proof can be obtained from Theorem 3.15 by taking $G = G_2$ and $g = I$. $\square$

We furnish some examples in favour of our results.

Example 3.21. Let $X = \mathbb{R}$ and $B(H)$ be the set of all bounded linear operators on a Hilbert space H . Define $d : X \times X \rightarrow B(H)$ by $d(x, y) = |x - y|^3 I$ for all $x, y \in X$, where I is the identity operator on H . Then $(X, B(H), d)$ is a complete C^* -algebra valued $\mathbf{b}$ -metric space with the coefficient $A = 4I$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(\frac{1}{n}, 0) : n = 1, 2, 3, \dots\}$.

Let $f, g : X \rightarrow X$ be defined by

$$\begin{aligned} fx &= \frac{x}{5}, \text{ if } x \neq \frac{4}{5} \\ &= 1, \text{ if } x = \frac{4}{5} \end{aligned}$$

and $gx = 2x$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$.

If $x = 0, y = \frac{1}{2n}, n = 1, 2, 3, \dots$, then $gx = 0, gy = \frac{1}{n}$ and so $(gx, gy) \in E(\tilde{G})$.

For $x = 0, y = \frac{1}{2n}$, we have

$$\begin{aligned} d(fx, fy) &= d\left(0, \frac{1}{10n}\right) \\ &= \frac{1}{10^3 \cdot n^3} I \\ &\prec \frac{1}{25n^3} I \\ &= \frac{1}{25} d(gx, gy) \\ &= B^* d(gx, gy) B, \end{aligned}$$

where $B = \frac{1}{5} I \in B(H)$.

Therefore,

$$d(fx, fy) \preceq B^* d(gx, gy) B$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, where $B \in B(H)$ and $\|B\|^2 < \frac{1}{\|A\|}$. We can verify that $0 \in C_{gf}$. In fact, $gx_n = fx_{n-1}$, $n = 1, 2, 3, \dots$ gives that $gx_1 = f0 = 0 \Rightarrow x_1 = 0$ and so $gx_2 = fx_1 = 0 \Rightarrow x_2 = 0$. Proceeding in this way, we get $gx_n = 0$ for $n = 0, 1, 2, \dots$ and hence $(gx_n, gx_m) = (0, 0) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \dots$.

Also, any sequence (gx_n) with the property $(gx_n, gx_{n+1}) \in E(\tilde{G})$ must be either a constant sequence or a sequence of the following form

$$\begin{aligned} gx_n &= 0, \text{ if } n \text{ is odd} \\ &= \frac{1}{n}, \text{ if } n \text{ is even} \end{aligned}$$

where the words 'odd' and 'even' are interchangeable. Consequently it follows that property (*) holds. Furthermore, f and g are weakly compatible. Thus, we have all the conditions of Theorem 3.2 and 0 is the unique common fixed point of f and g in X .

Remark 3.22. It is worth mentioning that weak compatibility condition in Theorem 3.2 cannot be relaxed. In Example 3.21, if we take $gx = 2x - 9$ for all $x \in X$ instead of $gx = 2x$, then $5 \in C_{gf}$ and $f(5) = g(5) = 1$ but $g(f(5)) \neq f(g(5))$ i.e., f and g are not weakly compatible. However, all other conditions of Theorem 3.2 are satisfied. We observe that 1 is the unique point of coincidence of f and g without being any common fixed point.

Remark 3.23. In Example 3.21, f is a C^* -algebra valued G -contraction but it is not a C^* -algebra valued contraction. In fact, for $x = \frac{4}{5}$, $y = 0$, we have

$$\begin{aligned} d(fx, fy) &= d(1, 0) \\ &= I \\ &= \frac{125}{64} \cdot \frac{64}{125} I \\ &= \frac{125}{64} d(x, y) \\ &\succ B^* d(x, y) B, \end{aligned}$$

for any $B \in B(H)$ with $\|B\|^2 < \frac{1}{\|A\|}$. This implies that f is not a C^* -algebra valued contraction.

The following example shows that property (*) is necessary in Theorem 3.2.

Example 3.24. Let $X = [0, \infty)$ and $B(H)$ be the set of all bounded linear operators on a Hilbert space H . Define $d : X \times X \rightarrow B(H)$ by $d(x, y) = |x - y|^3 I$ for all $x, y \in X$, where I is the identity operator on H . Then $(X, B(H), d)$ is a complete C^* -algebra valued b -metric space with the coefficient

$A = 4I$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(x, y) : (x, y) \in (0, 1] \times (0, 1], x \geq y\}$.

Let $f, g : X \rightarrow X$ be defined by

$$\begin{aligned} fx &= \frac{x}{6}, \text{ if } x \neq 0 \\ &= 1, \text{ if } x = 0 \end{aligned}$$

and $gx = \frac{x}{2}$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$.

For $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, we have

$$\begin{aligned} d(fx, fy) &= \frac{1}{27} d(gx, gy) \\ &\preceq \frac{1}{9} d(gx, gy) \\ &= B^* d(gx, gy) B, \end{aligned}$$

where $B = \frac{1}{3}I \in B(H)$ with $\|B\|^2 < \frac{1}{\|A\|}$.

We see that f and g have no point of coincidence in X . We now verify that the property $(*)$ does not hold. In fact, (gx_n) is a sequence in X with $gx_n \rightarrow 0$ and $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \in \mathbb{N}$ where $x_n = \frac{2}{n}$. But there exists no subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, 0) \in E(\tilde{G})$.

Example 3.25. Let $X = \mathbb{R}$ and $B(H)$ be the set of all bounded linear operators on a Hilbert space H . Choose a positive operator $T \in B(H)$. Define $d : X \times X \rightarrow B(H)$ by $d(x, y) = |x - y|^5 T$ for all $x, y \in X$. Then $(X, B(H), d)$ is a complete C^* -algebra valued b -metric space with the coefficient $A = 16I$. Let $f, g : X \rightarrow X$ be defined by

$$\begin{aligned} fx &= 2, \text{ if } x \neq 5 \\ &= 3, \text{ if } x = 5 \end{aligned}$$

and $gx = 3x - 4$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$.

Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(2, 3), (3, 5)\}$. If $x = 2, y = \frac{7}{3}$, then $gx = 2, gy = 3$ and so $(gx, gy) \in E(\tilde{G})$.

Again, if $x = \frac{7}{3}, y = 3$, then $gx = 3, gy = 5$ and so $(gx, gy) \in E(\tilde{G})$.

It is easy to verify that condition (3.5) of Theorem 3.9 holds for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$. Furthermore, $2 \in C_{gf}$ i.e., $C_{gf} \neq \emptyset$, f and g are weakly compatible, and $(X, B(H), d, G)$ has the property $(*)$. Thus, all the conditions of Theorem 3.9 are satisfied and 2 is the unique common fixed point of f and g in X .

Remark 3.26. It is observed that in Example 3.25, f is not a Fisher G -contraction. In fact,

for $x = 3, y = 5$, we have

$$\begin{aligned}
 B[d(fx, y) + d(fy, x)] &= B[d(2, 5) + d(3, 3)] \\
 &= 243BT \\
 &= \frac{243}{16}BAT \\
 &= \frac{243}{16 \times 17}17BAT \\
 &\prec T \\
 &= d(fx, fy),
 \end{aligned}$$

for any $B \in B(H)'_+$ with $\|BA\| < \frac{1}{\|A\|+1}$. This implies that f is not a Fisher G -contraction.

The following example supports our Theorem 3.15.

Example 3.27. Let $X = [0, \infty)$ and $B(H)$ be the set of all bounded linear operators on a Hilbert space H . Choose a positive operator $T \in B(H)$. Define $d : X \times X \rightarrow B(H)$ by $d(x, y) = |x - y|^2 T$ for all $x, y \in X$. Then $(X, B(H), d)$ is a complete C^* -algebra valued b -metric space with the coefficient $A = 2I$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(4^t x, 4^t(x+1)) : x \in X \text{ with } x \geq 2, t = 0, 1, 2, \dots\}$.

Let $f, g : X \rightarrow X$ be defined by $fx = 4x$ and $gx = 16x$ for all $x \in X$. Clearly, $f(X) = g(X) = X$.

If $x = 4^{t-2}z$, $y = 4^{t-2}(z+1)$, then $gx = 4^t z$, $gy = 4^t(z+1)$ and so $(gx, gy) \in E(\tilde{G})$ for all $z \geq 2$.

For $x = 4^{t-2}z$, $y = 4^{t-2}(z+1)$, $z \geq 2$ with $B = \frac{1}{117}I$, we have

$$\begin{aligned}
 d(fx, fy) &= d(4^{t-1}z, 4^{t-1}(z+1)) \\
 &= 4^{2t-2}T \\
 &\preceq \frac{1}{117}4^{2t-2}(18z^2 + 18z + 9)T \\
 &= \frac{1}{117} [d(4^{t-1}z, 4^t z) + d(4^{t-1}(z+1), 4^t(z+1))] \\
 &= B[d(fx, gx) + d(fy, gy)].
 \end{aligned}$$

Thus, condition (3.10) is satisfied for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$. It is easy to verify that $0 \in C_{gf}$. Also, any sequence (gx_n) with $gx_n \rightarrow x$ and $(gx_n, gx_{n+1}) \in E(\tilde{G})$ must be a constant sequence and hence property $(*)$ holds. Furthermore, f and g are weakly compatible. Thus, we have all the conditions of Theorem 3.15 and 0 is the unique common fixed point of f and g in X .

Remark 3.28. It is easy to observe that in Example 3.27, f is a C^* -algebra valued G -Kannan operator with $B = \frac{16}{117}I$. But f is not a C^* -algebra valued Kannan operator because, if $x = 4$, $y = 0$,

then for any arbitrary $B \in B(H)'_+$ with $\|B\| < \frac{1}{\|A\|+1} = \frac{1}{3}$ (which implies $3B \prec I$), we have

$$\begin{aligned} B[d(fx, x) + d(fy, y)] &= B[d(f4, 4) + d(f0, 0)] \\ &= 144BT \\ &= \frac{144}{3 \times 256}(3B)(256T) \\ &\prec 256T \\ &= d(fx, fy). \end{aligned}$$

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