Odd Vertex Equitable Even Labeling of Cycle Related Graphs

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ABSTRACT

Let $G$ be a graph with $p$ vertices and $q$ edges and $A = \{1, 3, \ldots, q\}$ if $q$ is odd or $A = \{1, 3, \ldots, q + 1\}$ if $q$ is even. A graph $G$ is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling $f^* \text{ defined by } f^*(uv) = f(u) + f(v)$ for all edges $uv$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \ldots, 2q$ where $v_f(a)$ be the number of vertices $v$ with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even labeling is called an odd vertex equitable even graph. Here, we prove that the subdivision of double triangular snake $(S(D(T_n)))$, subdivision of double quadrilateral snake $(S(D(Q_n)))$, $DA(Q_m) \odot nK_1$ and $DA(T_m) \odot nK_1$ are odd vertex equitable even graphs.
RESUMEN

Sea $G$ un grafo con $p$ vértices y $q$ aristas, y $A = \{1, 3, \ldots, q\}$ si $q$ es impar o $A = \{1, 3, \ldots, q + 1\}$ si $q$ es par. Se dice que un grafo $G$ admite un etiquetado par equitativo de vértices impares si existe un etiquetado de vértices $f : V(G) \to A$ que induce un etiquetado de ejes $f^* \mathrm{def} = f(u) + f(v)$ para todos los ejes $uv$ tales que para todo $a$ y $b$ en $A$, $|v_f(a) - v_f(b)| \leq 1$ y las etiquetas de ejes inducidas son $2, 4, \ldots, 2q$ donde $v_f(a)$ es el número de vértices $v$ con $f(v) = a$ para $a \in A$. Un grafo que admite un etiquetado par equitativo de vértices impares se dice grafo par equitativo de vértices impares. Aquí demostramos que la subdivisión de serpientes triangulares dobles ($S(D(T_n)))$, la subdivisión de serpientes cuadriláteras dobles ($S(D(Q_n)))$, $DA(Q_m) \odot nK_1$ y $DA(T_m) \odot nK_1$ son grafos pares equitativos de vértices impares.

**Keywords and Phrases:** Odd vertex equitable even labeling, odd vertex equitable even graph, double triangular snake, subdivision of double quadrilateral snake, double alternate triangular snake, double alternate quadrilateral snake, subdivision graph.

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1 Introduction:

All graphs considered here are simple, finite, connected and undirected. Let \( G(V, E) \) be a graph with \( p \) vertices and \( q \) edges. We follow the basic notations and terminology of graph theory as in [2]. The vertex set and the edge set of a graph are denoted by \( V(G) \) and \( E(G) \) respectively. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions and a detailed survey of graph labeling can be found in [1]. The concept of vertex equitable labeling was due to Lourdusamy and Seenivasan [6]. Let \( G \) be a graph with \( p \) vertices and \( q \) edges and \( A = \{0, 1, 2, \ldots, \left\lfloor \frac{q}{2} \right\rfloor \} \). A graph \( G \) is said to be vertex equitable if there exists a vertex labeling \( f: V(G) \rightarrow A \) that induces an edge labeling \( f^* \) defined by \( f^*(uv) = f(u) + f(v) \) for all edges \( uv \) such that for all \( a \) and \( b \) in \( A \), \( |v_1(a) - v_1(b)| \leq 1 \) and the induced edge labels are \( 1, 2, 3, \ldots, q \), where \( v_1(a) \) be the number of vertices \( v \) with \( f(v) = a \) for \( a \in A \). The vertex labeling \( f \) is known as vertex equitable labeling. A graph \( G \) is said to be a vertex equitable if it admits vertex equitable labeling. Motivated by the concept of vertex equitable labeling [6], Jeyanthi, Maheswari and Vijayalakshmi extend this concept and introduced a new labeling namely odd vertex equitable even (OVEE) labeling in [3]. A graph \( G \) with \( p \) vertices and \( q \) edges and \( A = \{1, 3, \ldots, q\} \) if \( q \) is odd or \( A = \{1, 3, \ldots, q + 1\} \) if \( q \) is even. A graph \( G \) is said to admit an odd vertex equitable even labeling if there exists a vertex labeling \( f: V(G) \rightarrow A \) that induces an edge labeling \( f^* \) defined by \( f^*(uv) = f(u) + f(v) \) for all edges \( uv \) such that for all \( a \) and \( b \) in \( A \), \( v_1(a) - v_1(b) \leq 1 \) and the induced edge labels are \( 2, 4, \ldots, 2q \) where \( v_1(a) \) be the number of vertices \( v \) with \( f(v) = a \) for \( a \in A \). A graph that admits an odd vertex equitable even (OVEE) labeling then \( G \) is called an odd vertex equitable even (OVEE) graph. In [3], [4] and [5] the same authors proved that \( nC_4 \)-snake, \( CS(n_1, n_2, \ldots, n_k, n_i \equiv 0(\text{mod}4), n_i \geq 4 \) is a generalized \( kC_n \)-snake, \( T\tilde{O}QS_n \) and \( T\tilde{O}QS_n \) are odd vertex equitable even graphs. They also proved that the graphs path, \( P_n \circ P_m(n, m \geq 1) \), \( K_{1,n} \cup K_{1,n-2}(n \geq 3) \), \( K_{2,n} \) and \( T_p \)-tree, cycle \( C_n (n \equiv 0 \text{ or } 1 (\text{mod}4)) \), quadrilateral snake \( Q_n \), ladder \( L_n \), \( L_n \circ K_1 \), arbitrary super subdivision of any path \( P_n \), \( S(L_n) \), \( L_n \tilde{O}P_m \), \( L_n \circ K_m \), and \( \bigcup_{n \equiv 0}^{1}(\text{mod}4) \) are odd vertex equitable even graphs. Also they proved that the graphs \( K_{1,n} \) is an odd vertex equitable even graph if \( n \leq 2 \) and the graph \( G = K_{1,n+k} \cup K_{1,n} \) is an odd vertex equitable even graph if and only if \( k = 1 \) and cycle \( C_n \) is an odd vertex equitable even graph if and only if \( n \equiv 0 \text{ or } 1(\text{mod}4) \). Let \( G \) be a graph with \( p \) vertices and \( q \) edges and \( p \leq \left\lfloor \frac{q}{2} \right\rfloor + 1 \), then \( G \) is not an odd vertex equitable even graph. In addition they proved that if every edge of a graph \( G \) is an edge of a triangle, then \( G \) is not an odd vertex equitable even graph.

We use the following definitions in the subsequent section.

**Definition 1.1.** The double triangular snake \( D(T_n) \) is a graph obtained from a path \( P_n \) with vertices \( v_1, v_2, \ldots, v_n \) by joining \( v_i \) and \( v_{i+1} \) to the new vertices \( w_i \) and \( u_i \) for \( i = 1, 2, \ldots, n-1 \).

**Definition 1.2.** The double quadrilateral snake \( D(Q_n) \) is a graph obtained from a path \( P_n \) with vertices \( u_1, u_2, \ldots, u_n \) by joining \( u_i \) and \( u_{i+1} \) to the new vertices \( v_i, x_i \) and \( w_i, y_i \) respectively and then joining \( v_i, w_i \) and \( x_i, y_i \) for \( i = 1, 2, \ldots, n-1 \).

**Definition 1.3.** A double alternate triangular snake \( DA(T_n) \) consists of two alternate triangular snakes that have a common path. That is, a double alternate triangular snake is obtained from
a path \( u_1, u_2, \ldots, u_n \) by joining \( u_i \) and \( u_{i+1} \) (alternatively) to the two new vertices \( v_i \) and \( w_i \) for \( i = 1, 2, \ldots, n-1 \).

**Definition 1.4.** A double alternate quadrilateral snake \( DA(Q_n) \) consists of two alternate quadrilateral snakes that have a common path. That is, a double alternate quadrilateral snake is obtained from a path \( u_1, u_2, \ldots, u_n \) by joining \( u_i \) and \( u_{i+1} \) (alternatively) to the two new vertices \( v_i \), \( x_i \) and \( w_i \), \( y_i \) respectively and adding the edges \( v_i w_i \) and \( x_i y_i \) for \( i = 1, 2, \ldots, n-1 \).

**Definition 1.5.** Let \( G \) be a graph. The graph \( S(G) \) is obtained from \( G \) by subdividing each edge of \( G \) with a vertex.

**Definition 1.6.** The corona \( G_1 \odot G_2 \) of the graphs \( G_1 \) and \( G_2 \) is defined as the graph obtained by taking one copy of \( G_1 \) (with \( p \) vertices) and \( p \) copies of \( G_2 \) and then joining the \( i \)th vertex of \( G_1 \) to every vertex of the \( i \)th copy of \( G_2 \).

## 2 Main Results

In this section, we prove that \( S(D(T_n)), S(D(Q_n)), DA(Q_m) \odot nK_1 \) and \( DA(T_m) \odot nK_1 \) are odd vertex equitable even graphs.

**Theorem 2.1.** Let \( G_1(p_1, q_1), G_2(p_2, q_2), \ldots, G_m(p_m, q_m) \) be an odd vertex equitable even graphs with each \( q_i \) is even for \( i = 1, 2, \ldots, m-1 \), \( q_m \) is even or odd and let \( u_i, v_i \) be the vertices of \( G_i(1 \leq i \leq m) \) labeled by 1, \( q_i \) if \( q_i \) is odd or \( q_i + 1 \) if \( q_i \) is even. Then the graph \( G \) obtained by identifying \( v_1 \) with \( u_2 \) and \( v_2 \) with \( u_3 \) and \( v_3 \) with \( u_4 \) and so on until we identify \( v_{m-1} \) with \( u_m \) is also an odd vertex equitable even graph.

**Proof.** The graph \( G \) has \( p_1 + p_2 + \ldots + p_m - (m-1) \) vertices and \( \sum_{i=1}^{m} q_i \) edges and \( f_i \) be an odd vertex equitable even labeling of \( G_i(1 \leq i \leq m) \).

Let \( A = \left\{ 1, 3, 5, \ldots, \sum_{i=1}^{m} q_i, \right\} \) if \( \sum_{i=1}^{m} q_i \) is odd

\( A = \left\{ 1, 3, 5, \ldots, \sum_{i=1}^{m} q_i + 1, \right\} \) if \( \sum_{i=1}^{m} q_i \) is even

Define a vertex labeling \( f : V(G) \rightarrow A \) as follows: \( f(x) = f_i(x) \) if \( x \in V(G_i) \), \( f(x) = f_i(x) + \sum_{k=1}^{i-1} q_k \) if \( x \in V(G_i) \) for \( 2 \leq i \leq m \). The edge labels of the graph \( G_1 \) will remain fixed, the edge labels of the graph \( G_i(2 \leq i \leq m) \) are \( 2q_1 + 2, 2q_1 + 4, \ldots, 2(q_1 + q_2); 2(q_1 + q_2) + 2, 2(q_1 + q_2) + 4, \ldots, 2(q_1 + q_2 + q_3); \ldots, 2 \sum_{i=1}^{m-1} q_i + 2, 2 \sum_{i=1}^{m-1} q_i + 4, \ldots, 2 \sum_{i=1}^{m-1} q_i \). Hence the edge labels of \( G \) are distinct and is \( \{ 2, 4, 6, \ldots, 2 \sum_{i=1}^{m} q_i \} \). Also \( |v_f(a) - v_f(b)| \leq 1 \) for all \( a, b \in A \). Hence \( G \) is an odd vertex equitable even graph. \( \square \)

**Theorem 2.2.** The graph \( S(D(T_n)) \) is an odd vertex equitable even graph.

**Proof.** Let \( G_i = S(D(T_2)) \) \( 1 \leq i \leq n-1 \) and \( u_i, v_i \) be the vertices with labels 1 and \( q + 1 \) respectively. By Theorem 2.1, \( S(D(T_2)) \) admits an odd vertex equitable even labeling. An odd vertex equitable even labeling of \( G_i = S(D(T_2)) \) is given in Figure 1.
Theorem 2.3. The graph $S(D(Q_n))$ is an odd vertex equitable even graph.

Proof. Let $G_i = S(D(Q_2))$ for $1 \leq i \leq n-1$ and $u_i, v_i$ be the vertices with labels 1 and $q+1$ respectively. By Theorem 2.1, $S(D(Q_2))$ admits an odd vertex equitable even labeling. An odd vertex equitable even labeling of $G_i = S(D(Q_2))$ is given in Figure 2.

Theorem 2.4. The double quadrilateral graph $D(Q_{2n})$ is an odd vertex equitable even graph.

Proof. Let $G_i = D(Q_4)$ for $1 \leq i \leq n-1$ and $u_i, v_i$ be the vertices with labels 1 and $q+1$ respectively. By Theorem 2.1, $D(Q_4)$ admits an odd vertex equitable even labeling. An odd vertex equitable even labeling of $G_i = D(Q_4)$ is given in Figure 3.
Theorem 2.5. Let $G_1(p_1, q), G_2(p_2, q), \ldots, G_m(p_m, q)$ be an odd vertex equitable even graphs with $q$ odd and $u_i, v_i$ be vertices of $G_i(1 \leq i \leq m)$ labeled by $1$ and $q$. Then the graph $G$ obtained by joining $v_1$ with $u_2$ and $v_2$ with $u_3$ and $v_3$ with $u_4$ and so on until joining $v_{m-1}$ with $u_m$ by an edge is also an odd vertex equitable even graph.

Proof. The graph $G$ has $p_1 + p_2 + \ldots + p_m$ vertices and $mq + (m - 1)$ edges. Let $f_i$ be the odd vertex equitable even labeling of $G_i(1 \leq i \leq m)$ and let $A = \{1, 3, \ldots, mq + (m - 1)\}$.

Define a vertex labeling $f : V(G) \rightarrow A$ as
\[ f(x) = f_i(x) + (i - 1)(q + 1) \] if $x \in G_i$ for $1 \leq i \leq m$.

The edge labels of $G_i$ are increased by $2(i - 1)(q + 1)$ for $i = 1, 2, \ldots, m$ under the new labeling $f$. The bridge between the two graphs $G_i, G_{i+1}$ will get the label $2i(q + 1), 1 \leq i \leq m - 1$.

Hence the edge labels of $G$ are distinct and is $\{2, 4, \ldots, 2(mq + m - 1)\}$.

Also $|\nu_i(a) - \nu_i(b)| \leq 1$ for all $a, b \in A$.

Then the graph $G$ is an odd vertex equitable even graph. \[ \square \]

Theorem 2.6. The graph $DA(T_2) \odot nK_1$ is an odd vertex equitable even graph for $n \geq 1$.

Proof. Let $G = DA(T_2) \odot nK_1$. Let $V(G) = \{u_{i1}, u_{i2}, u_i, w\} \cup \{u_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\} \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(G) = \{u_{i1}u_{i2}, u_{i1}v, vu_{i1}, u_{i1}w, wu_{i1}\} \cup \{u_{ij}u_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\} \cup \{vv_i, wv_i : 1 \leq i \leq n\}$.

Here $|V(G)| = 4(n + 1)$ and $|E(G)| = 4n + 5$.

Let $A = \{1, 3, \ldots, 4n + 5\}$.

Define a vertex labeling $f : V(G) \rightarrow A$ as follows:

For $1 \leq i \leq n$, $f(u_{i1}) = 1$, $f(u_{i2}) = 4n + 5$, $f(v) = 2n + 1$, $f(w) = 2n + 5$, $f(u_{11}) = 2i - 1$, $f(u_{21}) = 4n + 5 - 2(i - 1)$,

\[ f(v_i) = \begin{cases} 
3 & \text{if } i = 1 \\
2i + 3 & \text{if } 2 \leq i \leq n,
\end{cases} \]

\[ f(w_i) = \begin{cases} 
2(n + i) + 1 & \text{if } 1 \leq i \leq n - 1 \\
4n + 3 & \text{if } i = n.
\end{cases} \]
It can be verified that the induced edge labels of $DA(T_2) \odot nK_1$ are $2, 4, ..., 8n+10$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.

Hence $f$ is an odd vertex equitable even labeling $DA(T_2) \odot nK_1$.

An odd vertex equitable even labeling of $DA(T_2) \odot 3K_1$ is shown in Figure 4.

![Graph Image](image-url)

Figure 4.

**Theorem 2.7.** The graph $DA(Q_2) \odot nK_1$ is an odd vertex equitable even graph for $n \geq 1$.

**Proof.** Let $G = DA(Q_2) \odot nK_1$. Let $V(G) = \{u_1, u_2, v, w, x, y\} \cup \{v_i, w_i, x_i, y_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\}$ and $E(G) = \{u_1u_2, u_1v, vw, wu_2, u_1x, xy, yu_2\} \cup \{vw_i, ww_i, xx_i, yy_i : 1 \leq i \leq n\} \cup \{u_{ij}u_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\}$.

Here $|V(G)| = 6(n+1)$ and $|E(G)| = 6n + 7$.

Let $A = \{1, 3, ..., 6n + 7\}$.

Define a vertex labeling $f : V(G) \to A$ as follows:

For $1 \leq i \leq n$, $f(u_1) = 1$, $f(u_2) = 6n + 7$, $f(u_{11}) = 2i - 1$, $f(u_{2i}) = 6n - 2i + 9$, $f(v) = 2n + 1$, $f(w) = 2n + 3$, $f(x) = 4n + 5$, $f(y) = 4n + 7$, $f(v_1) = 2i + 1$, $f(w_i) = f(x_i) = 2n + 2i + 3$, $f(y_i) = 4n + 2i + 5$.

It can be verified that the induced edge labels of $DA(Q_2) \odot nK_1$ are $2, 4, ..., 12n + 14$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.

Hence $f$ is an odd vertex equitable even labeling of $DA(Q_2) \odot nK_1$. 
An odd vertex equitable even labeling of $\text{DA}(Q_2) \odot 4K_1$ is shown in Figure 5.

**Theorem 2.8.** The graph $\text{DA}(Q_m) \odot nK_1$ is an odd vertex equitable even graph for $m, n \geq 1$.

**Proof.** By Theorem 2.7, $\text{DA}(Q_2) \odot nK_1$ is an odd vertex equitable even graph. Let $G_i = \text{DA}(Q_2) \odot nK_1$ for $1 \leq i \leq m - 1$. Since each $G_i$ has $6n+7$ edges, by Theorem 2.5, $\text{DA}(Q_m) \odot nK_1$ admits odd vertex equitable even labeling.

An odd vertex equitable even labeling of $\text{DA}(Q_4) \odot 4K_1$ is shown in Figure 6.
**Theorem 2.9.** The graph $DA(T_m) \odot nK_1$ is an odd vertex equitable even graph for $m, n \geq 1$.

**Proof.** By Theorem 2.6, $DA(T_2) \odot nK_1$ is an odd vertex equitable even graph. Let $G_i = DA(T_2) \odot nK_1$ for $1 \leq i \leq m - 1$. Since each $G_i$ has $4n+5$ edges, by Theorem 2.5, $DA(T_m) \odot nK_1$ admits odd vertex equitable even labeling.

An odd vertex equitable even labeling of $DA(T_4) \odot 3K_1$ is shown in Figure 7.

![Figure 7](image_url)

References


