Odd Vertex Equitable Even Labeling of Cycle Related Graphs

P. JEYANTHI¹ AND A. MAHESWARI²

¹ *Research Centre, Department of Mathematics,
Govindammal Aditanar College for Women,
Tiruchendur-628215, Tamilnadu, India.*

² *Department of Mathematics,
Kamaraj College of Engineering and Technology,
Virudhunagar, Tamil Nadu, India.*

jeyajeyanthi@rediffmail.com, bala_nithin@yahoo.co.in

ABSTRACT

Let G be a graph with p vertices and q edges and $A = \{1, 3, \dots, q\}$ if q is odd or $A = \{1, 3, \dots, q + 1\}$ if q is even. A graph G is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \rightarrow A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A , $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \dots, 2q$ where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even labeling is called an odd vertex equitable even graph. Here, we prove that the subdivision of double triangular snake ($S(D(T_n))$), subdivision of double quadrilateral snake ($S(D(Q_n))$), $DA(Q_m) \odot nK_1$ and $DA(T_m) \odot nK_1$ are odd vertex equitable even graphs.

RESUMEN

Sea G un grafo con p vértices y q aristas, y $A = \{1, 3, \dots, q\}$ si q es impar o $A = \{1, 3, \dots, q + 1\}$ si q es par. Se dice que un grafo G admite un etiquetado par equitativo de vértices impares si existe un etiquetado de vértices $f : V(G) \rightarrow A$ que induce un etiquetado de ejes f^* definido por $f^*(uv) = f(u) + f(v)$ para todos los ejes uv tales que para todo a y b en A , $|v_f(a) - v_f(b)| \leq 1$ y las etiquetas de ejes inducidas son $2, 4, \dots, 2q$ donde $v_f(a)$ es el número de vértices v con $f(v) = a$ para $a \in A$. Un grafo que admite un etiquetado par equitativo de vértices impares se dice grafo par equitativo de vértices impares. Aquí demostramos que la subdivisión de serpientes triangulares dobles ($S(D(T_n))$), la subdivisión de serpientes cuadriláteras dobles ($S(D(Q_n))$), $DA(Q_m) \odot nK_1$ y $DA(T_m) \odot nK_1$ son grafos pares equitativos de vértices impares.

Keywords and Phrases: Odd vertex equitable even labeling, odd vertex equitable even graph, double triangular snake, subdivision of double quadrilateral snake, double alternate triangular snake, double alternate quadrilateral snake, subdivision graph.

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1 Introduction:

All graphs considered here are simple, finite, connected and undirected. Let $G(V, E)$ be a graph with p vertices and q edges. We follow the basic notations and terminology of graph theory as in [2]. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions and a detailed survey of graph labeling can be found in [1]. The concept of vertex equitable labeling was due to Lourdusamy and Seenivasan [6]. Let G be a graph with p vertices and q edges and $A = \{0, 1, 2, \dots, \lceil \frac{q}{2} \rceil\}$. A graph G is said to be vertex equitable if there exists a vertex labeling $f : V(G) \rightarrow A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A , $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \dots, q$, where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. The vertex labeling f is known as vertex equitable labeling. A graph G is said to be a vertex equitable if it admits vertex equitable labeling. Motivated by the concept of vertex equitable labeling [6], Jeyanthi, Maheswari and Vijayalakshmi extend this concept and introduced a new labeling namely odd vertex equitable even (OVEE) labeling in [3]. A graph G with p vertices and q edges and $A = \{1, 3, \dots, q\}$ if q is odd or $A = \{1, 3, \dots, q + 1\}$ if q is even. A graph G is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \rightarrow A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A , $v_f(a) - v_f(b) \leq 1$ and the induced edge labels are $2, 4, \dots, 2q$ where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even (OVEE) labeling then G is called an odd vertex equitable even (OVEE) graph. In [3], [4] and [5] the same authors proved that nC_4 -snake, $CS(n_1, n_2, \dots, n_k, n_i \equiv 0 \pmod{4}, n_i \geq 4)$, be a generalized kC_n -snake, $\widehat{TQ}S_n$ and $\widetilde{TQ}S_n$ are odd vertex equitable even graphs. They also proved that the graphs path, $P_n \odot P_m$ ($n, m \geq 1$), $K_{1,n} \cup K_{1,n-2}$ ($n \geq 3$), $K_{2,n}$, T_p -tree, cycle C_n ($n \equiv 0$ or $1 \pmod{4}$), quadrilateral snake Q_n , ladder L_n , $L_n \odot K_1$, arbitrary super subdivision of any path P_n , $S(L_n)$, $L_m \widehat{OP}_n$, $L_n \odot \overline{K_m}$ and $\langle L_n \widehat{OK}_{1,m} \rangle$ are odd vertex equitable even graphs. Also they proved that the graphs $K_{1,n}$ is an odd vertex equitable even graph iff $n \leq 2$ and the graph $G = K_{1,n+k} \cup K_{1,n}$ is an odd vertex equitable even graph if and only if $k = 1, 2$ and cycle C_n is an odd vertex equitable even graph if and only if $n \equiv 0$ or $1 \pmod{4}$. Let G be a graph with p vertices and q edges and $p \leq \lceil \frac{q}{2} \rceil + 1$, then G is not an odd vertex equitable even graph. In addition they proved that if every edge of a graph G is an edge of a triangle, then G is not an odd vertex equitable even graph.

We use the following definitions in the subsequent section.

Definition 1.1. The double triangular snake $D(T_n)$ is a graph obtained from a path P_n with vertices $v_1, v_2, \dots, v_n$ by joining v_i and v_{i+1} to the new vertices w_i and u_i for $i = 1, 2, \dots, n - 1$.

Definition 1.2. The double quadrilateral snake $D(Q_n)$ is a graph obtained from a path P_n with vertices $u_1, u_2, \dots, u_n$ by joining u_i and u_{i+1} to the new vertices v_i, x_i and w_i, y_i respectively and then joining v_i, w_i and x_i, y_i for $i = 1, 2, \dots, n - 1$.

Definition 1.3. A double alternate triangular snake $DA(T_n)$ consists of two alternate triangular snakes that have a common path. That is, a double alternate triangular snake is obtained from

a path $u_1, u_2, \dots, u_n$ by joining u_i and u_{i+1} (alternatively) to the two new vertices v_i and w_i for $i = 1, 2, \dots, n - 1$.

Definition 1.4. A double alternate quadrilateral snake $DA(Q_n)$ consists of two alternate quadrilateral snakes that have a common path. That is, a double alternate quadrilateral snake is obtained from a path $u_1, u_2, \dots, u_n$ by joining u_i and u_{i+1} (alternatively) to the two new vertices v_i, x_i and w_i, y_i respectively and adding the edges $v_i w_i$ and $x_i y_i$ for $i = 1, 2, \dots, n - 1$.

Definition 1.5. Let G be a graph. The subdivision graph $S(G)$ is obtained from G by subdividing each edge of G with a vertex.

Definition 1.6. The corona $G_1 \odot G_2$ of the graphs G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 (with p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2 .

2 Main Results

In this section, we prove that $S(D(T_n))$, $S(D(Q_n))$, $DA(Q_m) \odot nK_1$ and $DA(T_m) \odot nK_1$ are odd vertex equitable even graphs.

Theorem 2.1. Let $G_1(p_1, q_1), G_2(p_2, q_2), \dots, G_m(p_m, q_m)$ be an odd vertex equitable even graphs with each q_i is even for $i = 1, 2, \dots, m - 1$, q_m is even or odd and let u_i, v_i be the vertices of G_i ($1 \leq i \leq m$) labeled by 1, q_i if q_i is odd or $q_i + 1$ if q_i is even. Then the graph G obtained by identifying v_1 with u_2 and v_2 with u_3 and v_3 with u_4 and so on until we identify v_{m-1} with u_m is also an odd vertex equitable even graph.

Proof. The graph G has $p_1 + p_2 + \dots + p_m - (m - 1)$ vertices and $\sum_{i=1}^m q_i$ edges and f_i be an odd vertex equitable even labeling of G_i ($1 \leq i \leq m$).

$$\text{Let } A = \left\{ \begin{array}{ll} 1, 3, 5, \dots, \sum_{i=1}^m q_i, & \text{if } \sum_{i=1}^m q_i \text{ is odd} \\ 1, 3, 5, \dots, \sum_{i=1}^m q_i + 1, & \text{if } \sum_{i=1}^m q_i \text{ is even} \end{array} \right\}.$$

Define a vertex labeling $f: V(G) \rightarrow A$ as follows: $f(x) = f_1(x)$ if $x \in V(G_1)$, $f(x) = f_i(x) + \sum_{k=1}^{i-1} q_k$ if $x \in V(G_i)$ for $2 \leq i \leq m$. The edge labels of the graph G_1 will remain fixed, the edge labels of the graph G_i ($2 \leq i \leq m$) are $2q_1 + 2, 2q_1 + 4, \dots, 2(q_1 + q_2); 2(q_1 + q_2) + 2, 2(q_1 + q_2) + 4, \dots, 2(q_1 + q_2 + q_3); \dots, 2 \sum_{i=1}^{m-1} q_i + 2, 2 \sum_{i=1}^{m-1} q_i + 4, \dots, 2 \sum_{i=1}^m q_i$. Hence the edge labels of G are distinct and is $\{2, 4, 6, \dots, 2 \sum_{i=1}^m q_i\}$. Also $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence G is an odd vertex equitable even graph. $\square$

Theorem 2.2. The graph $S(D(T_n))$ is an odd vertex equitable even graph.

Proof. Let $G_i = S(D(T_2))$ $1 \leq i \leq n - 1$ and u_i, v_i be the vertices with labels 1 and $q + 1$ respectively. By Theorem 2.1, $S(D(T_2))$ admits an odd vertex equitable even labeling. An odd vertex equitable even labeling of $G_i = S(D(T_2))$ is given in Figure 1.

Figure 1.

□

Theorem 2.3. *The graph $S(D(Q_n))$ is an odd vertex equitable even graph.*

Proof. Let $G_i = S(D(Q_2))$ $1 \leq i \leq n-1$ and u_i, v_i be the vertices with labels 1 and $q+1$ respectively. By Theorem 2.1, $S(D(Q_2))$ admits an odd vertex equitable even labeling. An odd vertex equitable even labeling of $G_i = S(D(Q_2))$ is given in Figure 2.

Figure 2.

□

Theorem 2.4. *The double quadrilateral graph $D(Q_{2n})$ is an odd vertex equitable even graph.*

Proof. Let $G_i = D(Q_4)$ $1 \leq i \leq n-1$ and u_i, v_i be the vertices with labels 1 and $q+1$ respectively. By Theorem 2.1, $D(Q_4)$ admits an odd vertex equitable even labeling. An odd vertex equitable even labeling of $G_i = D(Q_4)$ is given in Figure 3.

Figure 3.

□

Theorem 2.5. Let $G_1(p_1, q), G_2(p_2, q), \dots, G_m(p_m, q)$ be an odd vertex equitable even graphs with q odd and u_i, v_i be vertices of $G_i (1 \leq i \leq m)$ labeled by 1 and q . Then the graph G obtained by joining v_1 with u_2 and v_2 with u_3 and v_3 with u_4 and so on until joining v_{m-1} with u_m by an edge is also an odd vertex equitable even graph.

Proof. The graph G has $p_1 + p_2 + \dots + p_m$ vertices and $mq + (m - 1)$ edges.

Let f_i be the odd vertex equitable even labeling of $G_i (1 \leq i \leq m)$ and

let $A = \{1, 3, \dots, mq + (m - 1)\}$.

Define a vertex labeling $f : V(G) \rightarrow A$ as

$f(x) = f_i(x) + (i - 1)(q + 1)$ if $x \in G_i$ for $1 \leq i \leq m$.

The edge labels of G_i are increased by $2(i - 1)(q + 1)$ for $i = 1, 2, \dots, m$ under the new labeling f .

The bridge between the two graphs G_i, G_{i+1} will get the label $2i(q + 1)$, $1 \leq i \leq m - 1$.

Hence the edge labels of G are distinct and is $\{2, 4, \dots, 2(mq + m - 1)\}$.

Also $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.

Then the graph G is an odd vertex equitable even graph. □

Theorem 2.6. The graph $DA(T_2) \odot nK_1$ is an odd vertex equitable even graph for $n \geq 1$.

Proof. Let $G = DA(T_2) \odot nK_1$. Let $V(G) = \{u_1, u_2, u, w\} \cup \{u_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\} \cup \{v_i, w_i : 1 \leq i \leq n\}$ and

$E(G) = \{u_1u_2, u_1v, vu_2, u_1w, wu_2\} \cup \{u_iu_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\} \cup \{vv_i, ww_i : 1 \leq i \leq n\}$.

Here $|V(G)| = 4(n + 1)$ and $|E(G)| = 4n + 5$.

Let $A = \{1, 3, \dots, 4n + 5\}$.

Define a vertex labeling $f : V(G) \rightarrow A$ as follows:

For $1 \leq i \leq n$ $f(u_1) = 1$, $f(u_2) = 4n + 5$, $f(v) = 2n + 1$, $f(w) = 2n + 5$, $f(u_{1i}) = 2i - 1$, $f(u_{2i}) = 4n + 5 - 2(i - 1)$,

$$f(v_i) = \begin{cases} 3 & \text{if } i=1 \\ 2i + 3 & \text{if } 2 \leq i \leq n, \end{cases}$$

$$f(w_i) = \begin{cases} 2(n + i) + 1 & \text{if } 1 \leq i \leq n - 1 \\ 4n + 3 & \text{if } i=n. \end{cases}$$

It can be verified that the induced edge labels of $DA(T_2) \odot nK_1$ are $2, 4, \dots, 8n+10$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.

Hence f is an odd vertex equitable even labeling $DA(T_2) \odot nK_1$.

An odd vertex equitable even labeling of $DA(T_2) \odot 3K_1$ is shown in Figure 4.

Figure 4.

□

Theorem 2.7. *The graph $DA(Q_2) \odot nK_1$ is an odd vertex equitable even graph for $n \geq 1$.*

Proof. Let $G = DA(Q_2) \odot nK_1$. Let $V(G) = \{u_1, u_2, v, w, x, y\} \cup \{v_i, w_i, x_i, y_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\}$ and

$E(G) = \{u_1u_2, u_1v, vw, wu_2, u_1x, xy, yu_2\} \cup \{vv_i, ww_i, xx_i, yy_i : 1 \leq i \leq n\} \cup \{u_iu_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\}$.

Here $|V(G)| = 6(n+1)$ and $|E(G)| = 6n+7$.

Let $A = \{1, 3, \dots, 6n+7\}$.

Define a vertex labeling $f : V(G) \rightarrow A$ as follows:

For $1 \leq i \leq n$ $f(u_1) = 1$, $f(u_2) = 6n+7$, $f(u_{1i}) = 2i-1$, $f(u_{2i}) = 6n-2i+9$, $f(v) = 2n+1$, $f(w) = 2n+3$, $f(x) = 4n+5$, $f(y) = 4n+7$, $f(v_i) = 2i+1$, $f(w_i) = f(x_i) = 2n+2i+3$, $f(y_i) = 4n+2i+5$.

It can be verified that the induced edge labels of $DA(Q_2) \odot nK_1$ are $2, 4, \dots, 12n+14$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.

Hence f is an odd vertex equitable even labeling of $DA(Q_2) \odot nK_1$.

An odd vertex equitable even labeling of $DA(Q_2) \odot 4K_1$ is shown in Figure 5.

Figure 5.

□

Theorem 2.8. *The graph $DA(Q_m) \odot nK_1$ is an odd vertex equitable even graph for $m, n \geq 1$.*

Proof. By Theorem 2.7, $DA(Q_2) \odot nK_1$ is an odd vertex equitable even graph. Let $G_i = DA(Q_2) \odot nK_1$ for $1 \leq i \leq m-1$. Since each G_i has $6n+7$ edges, by Theorem 2.5, $DA(Q_m) \odot nK_1$ admits odd vertex equitable even labeling.

An odd vertex equitable even labeling of $DA(Q_4) \odot 4K_1$ is shown in Figure 6.

□

Figure 6.

Theorem 2.9. *The graph $DA(T_m) \odot nK_1$ is an odd vertex equitable even graph for $m, n \geq 1$.*

Proof. By Theorem 2.6, $DA(T_2) \odot nK_1$ is an odd vertex equitable even graph. Let $G_i = DA(T_2) \odot nK_1$ for $1 \leq i \leq m-1$. Since each G_i has $4n+5$ edges, by Theorem 2.5, $DA(T_m) \odot nK_1$ admits odd vertex equitable even labeling.

An odd vertex equitable even labeling of $DA(T_4) \odot 3K_1$ is shown in Figure 7.

Figure 7.

□

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