Some remarks on the non-real roots of polynomials

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ABSTRACT

Let $f \in \mathbb{R}(t)[x]$ be given by $f(t,x) = x^n + t \cdot g(x)$ and $\beta_1 < \cdots < \beta_m$ the distinct real roots of the discriminant $\Delta_{f,t}(x)$ of $f(t,x)$ with respect to $x$. Let $\gamma$ be the number of real roots of $g(x) = \sum_{k=0}^{s} t_{s-k} x^{s-k}$. For any $\xi > |\beta_m|$, if $n-s$ is odd then the number of real roots of $f(\xi,x)$ is $\gamma + 1$, and if $n-s$ is even then the number of real roots of $f(\xi,x)$ is $\gamma, \gamma + 2$ if $t_s > 0$ or $t_s < 0$ respectively. A special case of the above result is constructing a family of degree $n \geq 3$ irreducible polynomials over $\mathbb{Q}$ with many non-real roots and automorphism group $S_n$.

RESUMEN

Sea $f \in \mathbb{R}(t)[x]$ dada por $f(t,x) = x^n + t \cdot g(x)$ y $\beta_1 < \cdots < \beta_m$ las diferentes raíces reales del discriminante $\Delta_{f,t}(x)$ de $f(t,x)$ con respecto a $x$. Sea $\gamma$ el número de raíces reales de $g(x) = \sum_{k=0}^{s} t_{s-k} x^{s-k}$. Para todo $\xi > |\beta_m|$, si $n-s$ es impar entonces el número de raíces reales de $f(\xi,x)$ es $\gamma + 1$, y si $n-s$ es par entonces el número de raíces reales de $f(\xi,x)$ es $\gamma, \gamma + 2$ si $t_s > 0$ o $t_s < 0$, respectivamente. Un caso especial del resultado anterior es construyendo una familia de polinomios irreducibles sobre $\mathbb{Q}$ de grado $n \geq 3$ con muchas raíces no-reales y grupo de automorfismos $S_n$.

Keywords and Phrases: Polynomials, non-real roots, discriminant, Bezoutian, Galois groups.

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1 Introduction

Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree $n \geq 2$ and Gal $(f)$ its Galois group over $\mathbb{Q}$. Let us assume that over $\mathbb{R}$, $f(x)$ is factored as

$$f(x) = a \prod_{j=1}^{r}(x - \alpha_j) \prod_{i=1}^{s}(x^2 + a_i x + b_i),$$

where $a_i^2 < 4b_i$, for all $i = 1, \ldots, s$. The pair $(r, s)$ is called the signature of $f(x)$. Obviously $\text{deg } f = 2s + r$. If $s = 0$ then $f(x)$ is called totally real and if $r = 0$ it is called totally complex. Equivalently the above terminology can be defined for binary forms $f(x, z)$. By a reordering of the roots we may assume that if $f(x)$ has $2s$ non-real roots then

$$\alpha := (1, 2)(3, 4) \cdots (2s - 1, 2s) \in \text{Gal}(f).$$

In [4] it is proved that if $\text{deg } f = p$, for a prime $p$, and $s$ satisfies

$$s (s \log s + 2 \log s + 3) \leq p$$

then $\text{Gal}(f) = A_p, S_p$. Moreover, a list of all possible groups for various values of $r$ is given for $p \leq 29$; see [4, Thm. 2]. There are some follow up papers to [4].

In [1] the author proves that if $p \geq 4s + 1$, then the Galois group is either $S_p$ or $A_p$. This improves the bound given in [4]. The author also studies when polynomials with non-real roots are solvable by radicals, which are consequences of Table 2 and Theorem 2 in [4]. In [13] the author uses Bezoutians of a polynomial and its derivative to construct polynomials with real coefficients where the number of real roots can be counted explicitly. Thereby, irreducible polynomials in $\mathbb{Q}[x]$ of prime degree $p$ are constructed for which the Galois group is either $S_p$ or $A_p$.

In this paper we study a family of polynomials with non-real roots whose degree is not necessarily prime. Given a polynomial $g(x) = \sum_{i=0}^{\gamma} t_i x^i$ and with $\gamma$ number of non-real roots we construct a polynomial $f(t, x) = x^n + t g(x)$ which has $\gamma, \gamma + 1, \gamma + 2$ non-real roots for certain values of $t \in \mathbb{R}$; see Theorem 3.2. The values of $t \in \mathbb{R}$ are given in terms of the Bezoutian matrix of polynomials or equivalently the discriminant of $f(t, x)$ with respect to $x$. This is the focus of Section 3 in the paper.

While most of the efforts have been focusing on the case of irreducible polynomials over $\mathbb{Q}$ which have real roots, the case of polynomials with no real roots is equally interesting. How should an irreducible polynomial over $\mathbb{Q}$ with all non-real roots must look like? What can be said about the Galois group of such totally complex polynomials? In [5] is developed a reduction theory for such polynomials via the hyperbolic center of mass. A special case of Theorem 3.2 provides a class of totally complex polynomials.

**Notation** For any polynomial $f(x)$ we denote by $\Delta(f, x)$ its discriminant with respect to $x$. If $f$ is a univariate polynomial then $\Delta_f$ is used and the leading coefficient is denoted by $\text{led}(f)$. Throughout this paper the ground field is a field of characteristic zero.
2 Preliminaries

Let $f_1(x), f_2(x)$ be polynomials over a field $F$ of characteristic zero and, let $n$ be an integer which is greater than or equal to $\max\{\deg f_1, \deg f_2\}$. Then, we put
\[
B_n(f_1, f_2) := \frac{f_1(x)f_2(y) - f_1(y)f_2(x)}{x - y} = \sum_{i,j=1}^{n} \alpha_{ij} x^{n-i} y^{n-j} \in F[x,y],
\]
\[
M_n(f_1, f_2) := (\alpha_{ij})_{1 \leq i,j \leq n}.
\]
The matrix $M_n(f_1, f_2)$ is called the Bezoutian of $f_1$ and $f_2$. Clearly, $B_n(f_1, f_1) = 0$ and hence $M_n(f_1, f_1)$ is the zero matrix. The following properties hold true; see [6, Theorem 8.25] for details.

**Proposition 1.** The following are true:

1. $M_n(f_1, f_2)$ is an $n \times n$ symmetric matrix over $F$.
2. $B_n(f_1, f_2)$ is linear in $f_1$ and $f_2$, separately.
3. $B_n(f_1, f_2) = -B_n(f_2, f_1)$.

When $f_2 = f_1'$, the formal derivative of $f_1$ (with respect to the indeterminate $x$), we often write $B_n(f_1) := B_n(f_1, f_1')$. From now on, for any degree $n \geq 2$ polynomial $f(x) \in \mathbb{R}[x]$ we will denote by $M_n(f) := M_n(f, f')$ as above. The matrix $M_n(f)$ is called the **Bezoutian matrix** of $f$.

**Remark 2.1.** It is often the case that the matrix $M'_n(f_1, f_2) = (\alpha'_{ij})_{1 \leq i,j \leq n}$ defined by the generating function
\[
B'_n(f_1, f_2) := \frac{f_1(x)f_2(y) - f_1(y)f_2(x)}{x - y} = \sum_{i,j=1}^{n} \alpha'_{ij} x^{i-1} y^{j-1} \in F[x,y]
\]
is called the Bezoutian of $f_1$ and $f_2$. But no difference can be seen between these two definitions as far as we consider the corresponding quadratic forms
\[
\sum_{i,j=1}^{n} \alpha_{ij} x_i x_j \quad \text{and} \quad \sum_{i,j=1}^{n} \alpha'_{ij} x_i x_j.
\]

In fact, these two quadratic forms are equivalent over the prime field $\mathbb{Q} \subset F$ since we have $M'_n(f_1, f_2) = J_n M_n(f_1, f_2)|_n$, where
\[
J_n = \begin{bmatrix}
0 & & & 1 \\
& \ddots & & \\
& & \ddots & 1 \\
1 & & & 0
\end{bmatrix}
\]
is an $n \times n$ anti-identity matrix. This implies that above two quadratic forms are equivalent over $\mathbb{Q}$ or more precisely, over the ring of rational integers $\mathbb{Z}$. 
Let $f(x) \in \mathbb{R}[x]$ be a degree $n \geq 2$ polynomial which is given by

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

Then over $\mathbb{R}$ this polynomial is factored as

$$f(x) = a \prod_{j=1}^{r} (x - \alpha_j) \prod_{i=1}^{s} (x^2 + a_i x + b_i)$$

for some $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ and $a_i, b_i, a \in \mathbb{R}$, where $a_i^2 < 4b_i$, for all $i = 1, \cdots, s$.

Throughout this paper, for a univariate polynomial $f$, its discriminant will be denoted by $\Delta_f$. For any two polynomials $f_1(x), f_2(x)$ the resultant with respect to $x$ will be denoted by $\text{Res}(f_1, f_2, x)$. We notice the following elementary fact, its proof is elementary and we skip the details.

**Remark 2.2.** For any polynomial $f(x)$, the determinant of the Bezoutian is the same as the discriminant up to a multiplication by a constant. More precisely,

$$\Delta_f = \frac{1}{\text{led}(f)} \det M_n(f),$$

where $\text{led}(f)$ is the leading coefficient of $f(x)$.

If $f(x) \in \mathbb{Q}[x]$ is irreducible and its degree is a prime number, say $\deg f = p$, then there is enough known for the Galois group of polynomials with some non-real roots; see [4], [1], [13] for details. If the number of non-real roots is "small" enough with respect to the prime degree $\deg f = p$ of the polynomial, then the Galois group is $A_p$ or $S_p$. Furthermore, using the classification of finite simple groups one can provide a complete list of possible Galois groups for every polynomial of prime degree $p$ which has non-real roots; see [4] for details.

On the other extreme are the polynomials which have all roots non-real. We called them above, totally complex polynomials. We have the following:

**Lemma 2.1.** The followings are equivalent:

i) $f(x) \in \mathbb{R}[x]$ is totally complex

ii) $f(x)$ can be written as

$$f(x) = a \prod_{i=1}^{n} f_i$$

where $f_i = x^2 + a_i x + b_i$, for $i = 1, \ldots, n$ and $a_i, b_i, a \in \mathbb{R}$, where $a_i^2 < 4b_i$, for all $i = 1, \ldots, n$.

Moreover, the determinant of the Bezoutian $M_n(f)$ is given by

$$\Delta_f = \frac{1}{\text{led}(f)^2 \cdot \det M_n(f)} = \prod_{i=1}^{n} \Delta_{f_i} \cdot \prod_{i,j;i \neq j} (\text{Res}(f_i, f_j, x))^2$$
where \(\text{led}(f)\) is the leading coefficient of \(f(x)\).

ii) the index of inertia of Bezoutian \(M(f)\) is 0

iii) if \(\Delta_f \neq 0\) then the equivalence class of \(M(f)\) in the Witt ring \(W(\mathbb{R})\) is 0.

Proof. The equivalence between i), ii), and iii) can be found in [6].

It is not clear when such polynomials are irreducible over \(\mathbb{Q}\). If that’s the case, what is the Galois group \(\text{Gal}(f)\)? Clearly the group generated by the involution \((1,2)(3,4)\cdots(2n-1,2n)\) is embedded in \(\text{Gal}(f)\). Is \(\text{Gal}(f)\) larger in general?

3 On the number of real roots of polynomials

For any degree \(n \geq 2\) polynomial \(f(x) \in \mathbb{R}[x]\) and any symmetric matrix \(M := M_n(f)\) with real entries, let \(N_f\) be the number of distinct real roots of \(f\) and \(\sigma(M)\) be the index of inertia of \(M\), respectively. The next result plays a fundamental role throughout this section ([6, Theorem 9.2]).

Proposition 2. For any real polynomial \(f \in \mathbb{R}[x]\), the number \(N_f\) of its distinct real roots is the index of inertia of the Bezoutian matrix \(M_n(f)\). In other words,

\[
N_f = \sigma(M_n(f)).
\]

Let us cite one more result which says that the roots of a polynomial depend continuously on its coefficients ([11, Theorem 1.4], [16, Theorem 1.3.1]).

Proposition 3. Let be given a polynomial

\[
f(x) = \sum_{l=0}^{n} a_l x^l \in \mathbb{C}[x],
\]

with distinct roots \(\alpha_1, \ldots, \alpha_k\) of multiplicities \(m_1, \ldots, m_k\) respectively. Then, for any given a positive

\[
\varepsilon < \min_{1 \leq i < j \leq k} \left\{ \frac{|\alpha_i - \alpha_j|}{2} \right\},
\]

there exists a real number \(\delta > 0\) such that any monic polynomial \(g(x) = \sum_{l=0}^{n} b_l x^l \in \mathbb{C}[x]\) whose coefficients satisfy

\[
|b_l - a_l| < \delta,
\]

for \(l = 0, \cdots, n - 1\), has exactly \(m_j\) roots in the disk

\[
D(\alpha_j; \varepsilon) = \{z \in \mathbb{C} \mid |z - \alpha_j| < \varepsilon\} \ (j = 1, \cdots, k).
\]
Let $n$, $s$ be positive integers such that $n > s$ and let
\[
g(t_0, \cdots, t_s; x) = \sum_{k=0}^{s} t_{s-k}x^{s-k},
\]
(3.1)
be polynomials in $x$ over $E_1 = \mathbb{R}(t_0, \cdots, t_s)$, $E_2 = \mathbb{R}(t_0, \cdots, t_s, t)$, respectively. Here, $E_1$ (resp., $E_2$) is a rational function field with $s + 1$ (resp., $(s + 2)$) variables $t_0, \cdots, t_s$ (resp., $(t_0, \cdots, t_s, t)$).

To ease notation, let us put
\[
g(x) = g(t_0, \cdots, t_s; x), \quad f(t; x) = f^{(n)}(t_0, \cdots, t_s, t; x)
\]
and for any real vector $v = (v_0, \cdots, v_s) \in \mathbb{R}^{s+1}$, we put
\[
g_v(x) = g(v_0, \cdots, v_s; x), \quad f_v(t; x) = f^{(n)}(v_0, \cdots, v_s, t; x).
\]
(3.2)
By using Proposition\footnote{2} we can prove the next theorem (\cite{13} Main Theorem 1.3).

**Theorem 3.1.** Let $r = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be a vector such that $N_{g_r} = s$. Let us consider
\[
f_r(t; x) = f^{(n)}(r_0, \cdots, r_s, t; x)
\]
as a polynomial over $\mathbb{R}(t)$ in $x$ and put
\[
P_r(t) = \det M_{\alpha}(f_r(t; x)) = \det M_{\alpha}(f_r(t; x), f'_r(t; x)),
\]
where $f'_r(t; x)$ is a derivative of $f_r(t; x)$ with respect to $x$. Then, for any real number $\xi > \alpha_r = \max\{\alpha \in \mathbb{R} \mid P_r(\alpha) = 0\}$, we have
\[
N_{f_r(\xi; x)} = \begin{cases} 
  s + 1 & \text{if } n - s \text{ : odd} \\
  s & \text{if } n - s \text{ : even, } r_s > 0 \\
  s + 2 & \text{if } n - s \text{ : even, } r_s < 0.
\end{cases}
\]

By this theorem and a theorem of Oz Ben-Shimol\footnote{1} Theorem 2.6, we can obtain an algorithm to construct prime degree $p$ polynomials with given number of real roots, and whose Galois groups are isomorphic to the symmetric group $S_p$ or the alternating group $A_p$ (\cite{13} Corollary 1.6).

In this section, we extend this theorem as follows;

**Theorem 3.2.** Let $r = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be a vector such that $g_r(x)$ is a degree $s$ separable polynomial satisfying $N_{g_r(x)} = \gamma$ $(0 \leq \gamma \leq s)$. Let us consider
\[
f_r(t; x) = f^{(n)}(r_0, \cdots, r_s, t; x)
\]
as a polynomial over $\mathbb{R}(t)$ in $x$ and put
\[
P_r(t) = \det M_{\alpha}(f_r(t; x)) = \det M_{\alpha}(f_r(t; x), f'_r(t; x)),
\]
where $f'_r(t; x)$ is a derivative of $f_r(t; x)$ with respect to $x$. Then, for any real number $\xi > \alpha_r = \max\{\alpha \in \mathbb{R} \mid P_r(\alpha) = 0\}$, we have
\[
N_{f_r(\xi; x)} = \begin{cases} 
  \gamma + 1 & \text{if } n - s \text{ : odd} \\
  \gamma & \text{if } n - s \text{ : even, } r_s > 0 \\
  \gamma + 2 & \text{if } n - s \text{ : even, } r_s < 0.
\end{cases}
\]
(3.3)
The above theorem can be restated as follows:

**Corollary 1.** Let $f \in \mathbb{R}(t)[x]$ be given by

$$f(t, x) = x^n + t \cdot \sum_{k=0}^{s} t_{s-k} x^{s-k}$$

and $\beta_1 < \cdots < \beta_m$ the distinct real roots of the degree $s$ polynomial

$$P(t) := \frac{1}{t^{n-1}} \Delta_{(f, x)}(t).$$

For any $\xi > |\beta_m|$, the number of real roots of $f(\xi, x)$ is

$$N_{f(\xi, x)} = \begin{cases} \gamma + 1 & \text{if } n - s : \text{odd}, \ t_s > 0 \\ \gamma & \text{if } n - s : \text{even}, \ t_s > 0 \\ \gamma + 2 & \text{if } n - s : \text{even}, \ t_s < 0. \end{cases}$$

where $\gamma$ is the number of real roots of $g(x) = \frac{f(x)-x^n}{t} \in \mathbb{R}[x]$.

The rest of the section is concerned with proving Thm. 3.2.

### 3.1 The Bezoutian of $f(t; x)$

First, let us put

$$A(t_0, \cdots, t_s, t) = (a_{ij}(t_0, \cdots, t_s, t))_{1 \leq i, j \leq n} = M_n(f(t; x)) \in \text{Sym}_n(E_2),$$

$$B(t_0, \cdots, t_s) = (b_{ij}(t_0, \cdots, t_s))_{1 \leq i, j \leq s} = M_s(g(x)) \in \text{Sym}_s(E_1).$$

For ease of notation, we also write

$$A(t_0, \cdots, t_s, t) = A(t) = (a_{ij}(t))_{1 \leq i, j \leq n}, \ B(t_0, \cdots, t_s) = B = (b_{ij})_{1 \leq i, j \leq s}$$

and we put $B(t) = (b_{ij}(t))_{1 \leq i, j \leq s} = t^2B$. Then, by Proposition, we have

$$A(t) = M_n(x^n + tg(x), nx^{n-1} + tg'(x))$$

$$= nM_n(x^n, x^{n-1}) - ntM_n(x^{n-1}, g(x)) + tM_n(x^n, g'(x)) + t^2M_n(g(x), g'(x))$$

$$= nM_n(x^n, x^{n-1}) - nt \sum_{k=0}^{s} t_{s-k} M_n(x^{n-1}, x^{s-k})$$

$$+ t \sum_{k=0}^{s-1} (s-k) t_{s-k} M_n(x^n, x^{s-k-1}) + t^2M_n(g(x), g'(x)).$$

**Lemma 3.1.** Let $\lambda, \mu, \nu$ be integers such that $\lambda \geq \mu > \nu \geq 0$. Then $M_\lambda(x^\mu, x^\nu) = (m_{ij})_{1 \leq i, j \leq \lambda}$, where

$$m_{ij} = \begin{cases} 1 & i + j = 2\lambda - (\mu + \nu) + 1 \ (\lambda - \mu + 1 \leq i, j \leq \lambda - \nu), \\ 0 & \text{otherwise.} \end{cases}$$
Proof. By definition, we have
\[
B_\lambda(x^\mu, x^\nu) = \frac{x^\mu y^\nu - x^\nu y^\mu}{x - y}
= \sum_{k=1}^{\mu - \nu} x^{\mu - k} y^{\nu + k - 1} = \sum_{k=1}^{\mu - \nu} x^{\lambda - (\lambda - \mu + k)} y^{\lambda - (\lambda - \nu + k + 1)},
\]
which implies
\[
m_{ij} = \begin{cases} 
1 & (i, j) = (\lambda - \mu + k, \lambda - \nu - k + 1) \ (1 \leq k \leq \mu - \nu) \\
0 & \text{otherwise}
\end{cases}
= \begin{cases} 
1 & i + j = 2\lambda - (\mu + \nu) + 1 \ (\lambda - \mu + 1 \leq i, j \leq \lambda - \nu), \\
0 & \text{otherwise}.
\end{cases}
\]
This completes the proof. \(\square\)

Here, let us divide \(A(t)\) into two parts \(A(t)\) and \(\tilde{A}(t)\), where
\[
\tilde{A}(t) = (\tilde{a}_{ij}(t))_{1 \leq i, j \leq n} = nM_n(x^n, x^{n-1}) - nt \sum_{k=0}^{s} t_{s-k} M_n(x^{n-1}, x^{s-k})
+ t \sum_{k=0}^{s-1} (s-k) t_{s-k} M_n(x^n, x^{s-k-1}),
\]
and put \(l_k = n - s + k + 2 \ (= 2n - (n + s - k - 1) + 1)\). Then, by lemma 3.1, we have
\[
\tilde{a}_{1l}(t) = n
\]
\[
\tilde{a}_{1l', 1l''}(t) = \tilde{a}_{l' - 1, l''}(t) = (s-k) t_{s-k} t \ (0 \leq k \leq s - 1).
\]
Moreover, when \(i + j = l_k\), we have
\[
\tilde{a}_{ij}(t) = -nt t_{s-k} + t(s-k) t_{s-k} = -(l_k - 2) t_{s-k} t \ (2 \leq i, j \leq l_k - 2, \ 0 \leq k \leq s). \quad (3.4)
\]

Remark 3.3. Note that, if \(s = n - 1\), we have
\[
-nt \sum_{k=0}^{s} t_{s-k} M_n(x^{n-1}, x^{s-k}) = -nt \sum_{k=1}^{s} t_{s-k} M_n(x^{n-1}, x^{s-k}),
\]
Thus, when \(i + j = l_k\), equation (3.4) should be modified by
\[
\tilde{a}_{ij}(t) = -nt t_{s-k} + t(s-k) t_{s-k} = -(l_k - 2) t_{s-k} t \ (2 \leq i, j \leq l_k - 2, \ 1 \leq k \leq s).
\]
We avoid this minor defect by considering that there is no entries satisfying \(2 \leq i, j \leq l_0 - 2\) when \(s = n - 1\) since \(l_0 - 2 = n - s = 1\).
Proposition 4. Put \( l_k = n - s + k + 2 \). Then

\[
\hat{a}_{ij}(t) = \begin{cases} 
  n & (i,j) = (1,1) \\
  (s-k)t_{s-k}^t & (i,j) = (1, l_k - 1) \text{ or } (l_k - 1, 1) \quad (0 \leq k \leq s - 1) \\
  -(l_k - 2)t_{s-k}^t & i + j = l_k, \; 2 \leq i,j \leq l_k - 2, \quad (0 \leq k \leq s) \\
  0 & \text{otherwise.}
\end{cases}
\]

\[
\hat{a}_{ij}(t) = \begin{cases} 
  b_{t - (n-s), j - (n-s)}t^2 & n - s + 1 \leq i,j \leq n \\
  0 & \text{otherwise.}
\end{cases}
\]

Proof. The statement for \( \hat{a}_{ij}(t) \) has just been proved. For \( \hat{a}_{ij}(t) \), it is enough to see that we can denote

\[
M_s(g(x)) = \sum_{\ell=0}^{s} \sum_{m=1}^{s} m_{t}t_{m}M_s(x^{\ell}, x^{m-1}),
\]

\[
M_n(g(x)) = \sum_{\ell=0}^{s} \sum_{m=1}^{s} m_{t}t_{m}M_n(x^{\ell}, x^{m-1}),
\]

that is, we can obtain \( M_n(g(x)) \) from \( M_s(g(x)) \) by just replacing \( s \) with \( n \) for all \( M_s(x^{\ell}, x^{m}) \), which, by Lemma 3.1 means that \( s \times s \) matrix \( M_s(g(x)) \) occupies the part \( \{b_{ij} | n - s + 1 \leq i,j \leq n\} \) of the matrix \( M_n(g(x)) = (b_{ij})_{1 \leq i,j \leq n} \).

By Proposition 4 we can express the matrix \( A(t) \) as follows;

\[
A(t) = \begin{bmatrix}
  n & 0 & \cdots & 0 & st_1 & (s-1)t_{s-1} & \cdots & t_{t-1} \\
  0 & -|n-s|t_1 & \cdots & -|n-s+1|t_{s-1} & \cdots & -|n-1|t_1 & -nt_1 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & 0 \\
  0 & -|n-s|t_1 & \cdots & 0 & 0 & \vdots & \vdots \\
  st_1 & -(n-s+1)t_{s-1} & \cdots & \vdots & \vdots & \ddots & 0 \\
  \vdots & \vdots & \cdots & \vdots & \ddots & \ddots & 0 \\
  (s-1)t_{s-1} & \cdots & \vdots & \vdots & \vdots & \ddots & 0 \\
  t_{t-1} & -nt_{t-1} & 0 & 0 & 0 & \cdots & \vdots \\
\end{bmatrix}
\]

Here, \( C(t) = (c_{ij}(t))_{1 \leq i,j \leq s} = C(t_0, \cdots, t_s, t) = (c_{ij}(t_0, \cdots, t_s, t))_{1 \leq i,j \leq s} \) is an \( s \times s \) symmetric matrix whose entries are of the form

\[
c_{ij}(t_0, \cdots, t_s, t) = b_{ij}t^2 + \lambda_{ij}t
\]

\[
= b_{ij}(t_0, \cdots, t_s)t^2 + \lambda_{ij}(t_0, \cdots, t_s)t \quad (\lambda_{ij} = \lambda_{ij}(t_0, \cdots, t_s) \in E_1).
\]

Next, let \( A(t)_1 = (a_{ij}(t)_1)_{1 \leq i,j \leq n} = A(t_0, \cdots, t_s, t)_1 = (a_{ij}(t_0, \cdots, t_s, t))_{1 \leq i,j \leq n} \) be the \( n \times n \) symmetric matrix obtained from \( A(t) \) by multiplying the first row and the first column by \( 1/\sqrt{n} \) and then sweeping out the entries of the first row and the first column by the \((1,1)\) entry 1. Here, let \( Q_m(k,c) = (q_{ij})_{1 \leq i,j \leq m} \) and \( R_m(k,t,c) = (r_{ij})_{1 \leq i,j \leq m} \) be \( m \times m \) elementary matrices such that
where \( q_{kk} = c \) and \( r_{kl} = c \). Moreover, for any \( m \times m \) matrices \( M_1, M_2, \ldots, M_l \), put \( \prod_{k=1}^{l} M_k = M_1 M_2 \cdots M_l \). Then, we have \( A(t) = t S(t) A(t) S(t) \), where

\[
S(t) = Q_n(1; 1/\sqrt{n}) \prod_{k=0}^{s-1} R_n(1, l_k - 1; -a_{1,l_k-1}(t)/\sqrt{n}).
\]

The matrix \( A(t) \) can be expressed as follows;

\[
A(t) = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & -(n-s) t_1 t & -(n-s+1) t_{s-1} t & \ldots & -(n-1) t_1 t - nt_0 t \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & -(n-s) t_s t & \cdots & 0 & -n t_0 t & \cdots & 0 \\
0 & -(n-s+1) t_{s-1} t & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & -(n-1) t_1 t & \ddots & 0 & -n t_0 t & 0 & 0 \\
0 & -n t_0 t & \cdots & 0 & 0 & \cdots & 0 \\
\end{bmatrix} + C(t)_1
\]

(3.6)

Here, \( C(t)_1 = (c_{ij}(t) \mathcal{1}_{i,j \leq s} = C(t_0, \ldots, t_s, t)_1 = (c_{ij}(t_0, \ldots, t_s, t) \mathcal{1}_{i,j \leq s} is an \( s \times s \) symmetric matrix whose entries are of the form

\[
c_{ij}(t_0, \ldots, t_s, t) = b_{ij}(t_0, \ldots, t_s) t^2 + \lambda_{ij}(t_0, \ldots, t_s) t \quad (b_{ij}(t_0, \ldots, t_s) \in E_1),
\]

where

\[
b_{ij}(t_0, \ldots, t_s) = b_{ij}(t_0, \ldots, t_s) - \frac{(s-i+1)(s-j+1)}{n} t_{s-i+1} t_{s-j+1} \quad (3.7)
\]

for any \( i, j \ (1 \leq i, j \leq s) \). We put \( b_{ij}(t_0, \ldots, t_s) = \tilde{b}_{ij} \) and \( B = (b_{ij})_{1 \leq i,j \leq s} \).

**3.2 Some results for the Bezoutian of \( f_r(t; x) \)**

Let \( r = (r_0, \ldots, r_s) \in \mathbb{R}^{s+1} \) be a vector as in Theorem 3.2. We put

\[
A_r(t) = (a_{ij}^{(r)}(t))_{1 \leq i,j \leq n} = A(r_0, \ldots, r_s, t) \in \text{Sym}_n(\mathbb{R}(t)),
\]

\[
B_r = (b_{ij}^{(r)})_{1 \leq i,j \leq s} = B(r_0, \ldots, r_s) \in \text{Sym}_s(\mathbb{R})
\]
and $B_r(t) = t^2B_r$. Let us also put $A_r(t)_1 = A(r_0, \cdots, r_s, t)_1$. By equation (3.6), the matrix $A_r(t)_1$ can be expressed as follows:

$$A_r(t)_1 = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & -(n-s)r_s \\
\vdots & \vdots & \ddots & \vdots \\
0 & -(n-s)r_s & & 0 \\
0 & -(n-s-1)r_s & & \ddots \\
\vdots & \vdots & \ddots & \ddots \\
0 & -(n-1)r_1 & & \cdots & 0 \\
0 & -nr_0 & & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
-(n-s+1)r_{s-1} & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-(n-1)r_1 & & \cdots & 0 \\
r_{s-1} & & \cdots & 0 \\
\end{bmatrix}.
$$

Here, $C_r(t)_1 = (c_{ij}^{(r)}(t)_1)_{1 \leq i, j \leq s} = C(r_0, \cdots, r_s, t)_1$ and

$$c_{ij}^{(r)}(t)_1 = b_{ij}(r_0, \cdots, r_s)t^2 + \lambda_{ij}(r_0, \cdots, r_s)t \ (b_{ij}(r_0, \cdots, r_s), \lambda_{ij}(r_0, \cdots, r_s) \in \mathbb{R}).$$

Note that, by equation (3.7), we have

$$b_{ij}(r_0, \cdots, r_s) = b_{ij}^{(r)} = \frac{(s-i+1)(s-j+1)}{n}r_{s-i+1}r_{s-j+1} \ (1 \leq i, j \leq s).$$

To ease notation, we put $\bar{b}_{ij}(r_0, \cdots, r_s) = \bar{b}_{ij}^{(r)}$ and $\bar{B}_r = (\bar{b}_{ij}^{(r)})_{1 \leq i, j \leq s}$.

In particular, since

$$M_s(g_r) = M_s\left( r_sx^s, \sum_{k=0}^{s-1} (s-k)r_s x^{s-k-1} \right) + M_s\left( \sum_{k=1}^s r_{s-k} x^{s-k-1}, g_r' \right) = \sum_{k=0}^{s-1} (s-k)r_s r_{s-k} M_s(x^s, x^{s-k-1}) + M_s\left( \sum_{k=1}^s r_{s-k} x^{s-k-1}, g_r' \right),$$

we have

$$b_{k,k+1}^{(r)} = b_{k+1,k}^{(r)} = (s-k)r_s r_{s-k} \ (0 \leq k \leq s - 1) \quad (3.8)$$

by Lemma 3.1 and hence

$$\bar{b}_{ij}^{(r)} = (s-j+1)r_s r_{s-j+1} - \frac{s(s-j+1)}{n}r_s r_{s-j+1} \quad (3.9)$$

$$= (s-j+1)\left(1 - \frac{s}{n}\right)r_s r_{s-j+1} \ (1 \leq j \leq s).$$

**Lemma 3.2.** Put $\bar{B}_r(t) = t^2\bar{B}_r$. Then, $B_r(\xi)$ and $\bar{B}_r(\xi)$ are equivalent over $\mathbb{R}$ for any real number $\xi$, and we have $\sigma(B_r(\xi)) = N_q$, for any non-zero real number $\xi$.

**Proof.** Let us denote by $B_r^* = (b_{ij}^{(r)*})_{1 \leq i, j \leq s}$ ($\bar{B}_r^* = (\bar{b}_{ij}^{(r)*})_{1 \leq i, j \leq s}$) the matrix obtained from $B_r$ ($\bar{B}_r$) by multiplying the first row and the first column by $1/\sqrt{\bar{b}_{11}^{(r)}} \left( 1/\pm \sqrt{b_{11}^{(r)}} \right)$ (the sign
before $\sqrt{b_{11}^r} \left( \sqrt{b_{11}^r} \right)$ are the same as the sign of $r_s$; see the definition of $d$ (\( \hat{d} \)) below and then sweeping out the entries of the first row and the first column by the $(1, 1)$ entry $1$. Since $b_{11} = sr_s^2 (> 0)$ and $b_{11} = s(1 - s/n)r_s^2 (> 0)$ by (3.8) and (3.9), we have

$$B_s^* = t^TB_s^T, \quad B_s^* = t^TB_s^T,$$

where

$$T = Q_s(1; 1/d) \prod_{k=2}^{s} R_s(1, k; -b_{1k}^{(r)}) / d \ (d = \sqrt{s} \cdot r_s),$$

$$\bar{T} = Q_s(1; 1/d) \prod_{k=2}^{s} R_s(1, k; -b_{1k}^{(r)}) / \bar{d} \ (\bar{d} = \sqrt{s(1 - s/n)} \cdot r_s).$$

Note that in [13, Lemma 3.3], we have proved $b_{ij}^{(r,s)} = b_{ij}^{(r,r)}$ (1 ≤ i, j ≤ s) and hence $t^2B_s^* = t^2\bar{B}_s^*$, which, by (3.10), implies that symmetric matrices $B_r(\xi)$ and $\bar{B}_r(\xi)$ are equivalent over $\mathbb{R}$ for any real number $\xi$. Then, since $N_{g_r} = \sigma(B_r) = \sigma(B_r(\xi))$ for any $\xi \in \mathbb{R} \setminus \{0\}$, the latter half of the statement have also been proved.

\[\square\]

3.3 Nonvanishingness of some coefficients

In this subsection, we prove the next lemma.

Lemma 3.3. \hspace{1cm} Let

$$\Phi(x) = \Phi(t_0, \cdots, t_s; x) = \sum_{k=0}^{s} h_{s-k}(t_0, \cdots, t_s)x^{s-k} \in E_1[x]$$

be the characteristic polynomial of $\bar{B}$. Then, $h_{s-k}(t_0, \cdots, t_s)$ is a non-zero polynomial in $E_1$ for any $k$ (1 ≤ k ≤ s).

Proof. Lemma [5.3] is clear for $s = 1$, since we have

$$B = M_1(t_1x + t_0) = \begin{bmatrix} t_1^2 \end{bmatrix}$$

and hence, by equation (3.7),

$$\bar{B} = \begin{bmatrix} t_1^2 - \frac{1}{n}t_1^2 \end{bmatrix} = \begin{bmatrix} \frac{r_s}{n}t_1^2 \end{bmatrix}.$$ 

Next, suppose $s \geq 2$. Then, by equation (3.7) and the definition of the Bezoutian, we have $h_{s-k}(t_0, \cdots, t_s) \in \mathbb{R}[t_0, \cdots, t_s]$ for any $k$ (1 ≤ k ≤ s). Thus, we have only to prove that $h_{s-k}(t_0, \cdots, t_s) \neq 0$ for any $k$ (1 ≤ k ≤ s), which is clear from the next Lemma 3.4. \[\square\]

Lemma 3.4. Suppose $s \geq 2$ and put $u_0 = u_s = 1$, $u_1 = t_1$ and $u_k = 0$ (2 ≤ k ≤ s - 1). Then, $h_{s-k}(u_0, \cdots, u_s)$ is a non-constant polynomial in $\mathbb{R}(t_1)$ for any $k$ (1 ≤ k ≤ s), i.e., $h_{s-k}(u_0, \cdots, u_s) \in \mathbb{R}(t_1) \setminus \mathbb{R} (1 \leq k \leq s)$. 

\[\square\]
To prove lemma 3.3 let us put \( u = (u_0, \ldots, u_s) \) and

\[
\begin{align*}
g_u(x) &= g(u_0, \ldots, u_s; x) = x^s + t_1x + 1 \in \mathbb{R}(t_1)[x], \\
f_u(t; x) &= x^n + t g_u(x) \in \mathbb{R}(t_1, t)[x] \ (n > s), \\
A_u(t) &= (a_{ij}^{(u)}(t))_{1 \leq i, j \leq n} = A(u_0, \ldots, u_s, t) \in \text{Sym}_n(\mathbb{R}(t_1, t)), \\
B_u &= (b_{ij}^{(u)})_{1 \leq i, j \leq s} = B(t_0, \ldots, u_s) \in \text{Sym}_s(\mathbb{R}(t_1)), \quad B_u(t) = t^2 B_u.
\end{align*}
\]

Then, by equation (3.5), we have

\[
\begin{align*}
u
\end{align*}
\]

where \( C_u(t) = (c_{ij}^{(u)}(t))_{1 \leq i, j \leq s} = C(u_0, \ldots, u_s, t) \) and

\[
c_{ij}^{(u)}(t) = b_{ij}(u_0, \ldots, u_s) t^2 + \lambda_{ij}(u_0, \ldots, u_s) t \quad (\lambda_{ij}(u_0, \ldots, u_s) \in \mathbb{R}(t_1)).
\]

Moreover, by equation (3.9), we also have

\[
\begin{align*}
A_u(t)_1 = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & -(n-s)t \\
\vdots & \vdots & \ddots & \vdots \\
0 & -(n-s)t & \cdots & 0 \\
0 & 0 & \cdots & -(n-1)t_1t \\
0 & \vdots & \ddots & \vdots \\
0 & -(n-1)t_1t & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{bmatrix}
\end{align*}
\]

Here, \( C_u(t)_1 = (c_{ij}^{(u)}(t)_1)_{1 \leq i, j \leq s} = C(u_0, \ldots, u_s, t)_1 \) and

\[
c_{ij}^{(u)}(t)_1 = \tilde{b}_{ij}(u_0, \ldots, u_s) t^2 + \lambda_{ij}(u_0, \ldots, u_s) t \quad (\tilde{b}_{ij}(u_0, \ldots, u_s) \in \mathbb{R}).
\]

Note that, by equation (3.7), we have

\[
\begin{align*}
\tilde{b}_{ij}^{(u)} &= \begin{cases}
\frac{b_{ij}^{(u)} - (s^2/n)}{s}, & (i, j) = (1, 1) \\
\frac{b_{ij}^{(u)} - (s/n)t_1}{s}, & (i, j) = (1, s) \text{ or } (s, 1) \\
\frac{b_{ij}^{(u)} - (1/n)t_1^2}{s}, & (i, j) = (s, s) \\
b_{ij}^{(u)}, & \text{otherwise}.
\end{cases}
\end{align*}
\]
Let us put $\bar{B}_u = (\bar{b}_{ij}^{(u)})_{1 \leq i, j \leq s}$ and $\bar{B}_u(t) = t^2 \bar{B}_u$. Then, since
\[
M_s(g_u) = M_s(x^s + t_1 x + 1, sx^{s-1} + t_1)
\]
\[
= sM_s(x^s, x^{s-1}) + t_1 M_s(x^s, 1) - st_1 M_s(x^{s-1}, x) - sM_s(x^{s-1}, 1)
\]
\[
+ t_1^2 M_s(x, 1) + t_1 M_s(1, 1),
\]
we have
(a) if $s = 2$,
\[
B_u = \begin{bmatrix}
2 & t_1 \\
t_1 & t_1^2 - 2
\end{bmatrix},
\]
(b) if $s \geq 3$,
\[
b_{ij}^{(u)} = \begin{cases}
    s & (i, j) = (1, 1) \\
    t_1 & (i, j) = (1, s) \text{ or } (s, 1) \\
    (1-s)t_1 & i + j = s + 1, 2 \leq i, j \leq s - 1 \\
    -s & i + j = s + 2 \\
    t_1^2 & (i, j) = (s, s), \\
    0 & \text{otherwise},
\end{cases}
\]
which, by equation (3.12), implies
(a') if $s = 2$,
\[
\bar{B}_u = \begin{bmatrix}
2(n-2)/n & (n-2)t_1/n \\
(n-2)t_1/n & (n-1)t_1^2/n - 2
\end{bmatrix},
\]
(b') if $s \geq 3$,
\[
\bar{b}_{ij}^{(u)} = \begin{cases}
    s(n-s)/n & (i, j) = (1, 1) \\
    (n-s)t_1/n & (i, j) = (1, s) \text{ or } (s, 1) \\
    (1-s)t_1 & i + j = s + 1, 2 \leq i, j \leq s - 1 \\
    -s & i + j = s + 2 \\
    (n-1)t_1^2/n & (i, j) = (s, s), \\
    0 & \text{otherwise}.
\end{cases}
\]
Therefore, if $s \geq 3$, the matrix $\bar{B}_u = (\bar{b}_{ij}^{(u)})_{1 \leq i, j \leq s}$ has the expression of the form
\[
\begin{bmatrix}
s(n-s)/n & 0 & 0 & 0 & \cdots & 0 & (n-s)t_1/n \\
0 & 0 & 0 & 0 & \cdots & 0 & (1-s)t_1 & -s \\
0 & 0 & \cdots & (1-s)t_1 & -s & 0 \\
0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & \cdots & (1-s)t_1 & \ddots & \ddots & 0 \\
0 & (1-s)t_1 & -s & \ddots & \ddots & \ddots & 0 \\
(n-s)t_1/n & -s & 0 & \cdots & 0 & (n-1)t_1^2/n
\end{bmatrix}.
\]
Here, let us denote by

$$
\Phi_u(x) = \sum_{k=0}^{s} h_{s-k}^{(u)} x^{s-k} = \Phi(u_0, \ldots, u_s; x) \left( = \sum_{k=0}^{s} h_{s-k}(u_0, \ldots, u_s)x^{s-k} \right)
$$

the characteristic polynomial of $\overline{B}_u$. Note that since we have $h_{s-k}^{(u)} \in \mathbb{R}[t_1]$ by the proof of Lemma 3.3, we have only to prove $h_{s-k}^{(u)}$ is non-constant for any $k$ ($1 \leq k \leq s$).

By the above expression of $\overline{B}_u$, we have

(a'') if $s = 2$,

$$
\Phi_u(x) = x^2 - \frac{(n - 1)t_1^2 - 4}{n} x + \frac{(n - 2)t_1^2 - 4n + 8}{n},
$$

(b'') if $s \geq 3,

$$
\Phi_u(x) = \begin{cases}
    \begin{vmatrix}
        x - (n - s)/n & \ldots & x - (s - 1)/n & s \\
        \ldots & \ldots & \ldots & \ldots \\
        x & \ldots & x & \ldots \\
        \ldots & \ldots & \ldots & \ldots \\
        \ldots & \ldots & \ldots & \ldots \\
        \ldots & \ldots & \ldots & \ldots \\
        \end{vmatrix} & (s \text{ is odd}), \\
    \begin{vmatrix}
        x - (n - s)/n & \ldots & x - (s - 1)/n & s \\
        \ldots & \ldots & \ldots & \ldots \\
        x & \ldots & x & \ldots \\
        \ldots & \ldots & \ldots & \ldots \\
        \ldots & \ldots & \ldots & \ldots \\
        \ldots & \ldots & \ldots & \ldots \\
        \end{vmatrix} & (s \text{ is even}).
\end{cases}
$$

Example 3.1. (1) Put $s = 7$ and $n = 10$. Then, we have

$$
g_u(x) = x^7 + t_1 x + 1, \quad f_u(t; x) = x^{10} + t(x^7 + t_1 x + 1),
$$
Proof of Lemma 3.4. To prove Lemma 3.4, it is enough to prove \( \deg h_{s-k}^{(u)} \geq 1 \) for any \( k \) (\( 1 \leq k \leq s \)). This is clear for \( s = 2 \) by \((a^n)\) and we suppose \( s \geq 3 \) hereafter. To prove \( \deg h_{s-k}^{(u)} \geq 1 \) (\( 1 \leq k \leq s \)),
let us compute the leading term of \( h^{(u)}_{s-k} (\in \mathbb{R}[t_1]) \). Then, since \( h^{(u)}_{s-k} \) is the coefficient of the term \( h^{(u)}_{s-k} x^{s-k} \) of the characteristic polynomial \( \Phi_u(x) \), we need to maximize the degree in \( t_1 \) when we take 's − k' \( x \) and the remaining \( k \) elements from \( \mathbb{R}[t_1] \).

(a) Suppose \( s \) is odd. Let us divide the case into three other sub-cases.

(a1) Suppose \( k \) is odd and \( 1 \leq k \leq s - 2 \).

In this case, the degree of the leading term of \( h^{(u)}_{s-k} \) is \( k + 1 \). In fact, it is obtained by taking

(a11) \(-(n-1)t_1^2/n \) from the \((s,s)\) entry \( x - (n-1)t_1^2/n \),

(a12) 'k − 1' \((s−1)t_1\) from entries of the form \((i, s + 1 − i)\) \((2 \leq i \leq s − 1)\).

First, suppose we take the \((s,s)\) entry \( x - (n-1)t_1^2/n \) from the \(s\)-th row. Then we must take the \((1,1)\) entry from the first row. Next, let us proceed to the \((s-1)\)-th row. If we take the \((s-1,s-1)\) entry \( x \) from the \((s-1)\)-th row, then we must also take \( x \) from the second row, while if we take \((s-1)t_1\) from the \((s-1)\)-th row, then we must also take \((s-1)t_1\) from the second row. The situation is the same for the \((s-2)\)-th row, the \((s-3)\)-th row ... and so on, which implies that \((s-1)t_1\) must occur in pair.

Hence, the leading term of \( h^{(u)}_{s-k} \) is

\[-\frac{n-1}{n}t_1^2 \cdot \left(\frac{(s-3)/2}{(k-1)/2}\right)^{((-1)} \cdot (s-1)^2t_1^2\right)^{(k-1)/2} \quad \left(\begin{array}{c} n \\ 0 \end{array}\right) = 1 (n \geq 0)\]

and the degree of this term is \( k + 1 \) \((\geq 2)\).

(a2) Suppose \( k \) is odd and \( k = s \).

If \( k = s \), \( h^{(u)}_{s-k} = h^{(u)}_0 \) is the constant term of \( \Phi_u(x) \). In this case, the degree of the leading term of \( h^{(u)}_0 \) is \( s \). In fact, it is obtained by taking

(a21) \(-(n-1)t_1^2/n \) from the \((s,s)\) entry \( x - (n-1)t_1^2/n \),

(a22) If \( s \geq 5 \) \((\Leftrightarrow (s,k) \neq (3,3))\), \('(s-3)/2\)' pairs of \((s-1)t_1\) from entries of the form \((i, s + 1 − i)\) \((2 \leq i \leq (s-1)/2, (s+3)/2 \leq i \leq s - 1)\),

(a23) \((s-1)t_1\) from the \((s+1)/2\), \((s+1)/2\) entry \( x + (s-1)t_1 \),

(a24) \(-s(n-s)/n \) from the \((1,1)\) entry \( x - s(n-s)/n \)

or by taking

(a25) all anti-diagonal entries.
Therefore, the leading term of $h_{3}^{[u]}$ is
\[
-\frac{n-1}{n}t_{1}^{2} \cdot (s-3)/2 \cdot (s-1)t_{1} \cdot \left( -\frac{s(n-s)}{n} \right)
\]
\[
+ (-1) \cdot \left( -\frac{n-s}{n}t_{1} \right)^{2} \cdot (s-3)/2 \cdot (s-1)t_{1}
\]
\[
= \left( \frac{n-s}{n} \right) \cdot (s-3)/2 \cdot (s-1)t_{1}^{2}
\]
\[
= (-1)^{(s-3)/2} \frac{(n-s)(s-1)s-1}{n}t_{1}^{3}
\]
for any $s (s \geq 3)$ and the degree of this term is $s$.

(a3) Suppose $k$ is even.

In this case, we have $2 \leq k \leq s-1$ and the degree of the leading term of $h_{s-k}^{[u]}$ is $k+1$. In fact, it is obtained by taking

\[(a31)\] $-\frac{n-1}{n}t_{1}^{2}/n$ from the $(s,s)$ entry $x - (n-1)t_{1}^{2}/n$.

(a32) If $s \geq 5$ (if $(s,k) \neq (3,2)$), $\{(k-2)/2\}$ pairs of $(s-1)t_{1}$ from entries of the form $(i,s+1-i)$ ($2 \leq i \leq (s-1)/2$, $(s+3)/2 \leq i \leq s-1$),

(a33) $(s-1)t_{1}$ from the $((s+1)/2,(s+1)/2)$ entry $x + (s-1)t_{1}$.

Therefore, the leading term of $h_{s-k}^{[u]}$ is
\[
-\frac{n-1}{n}t_{1}^{2} \cdot \left( \frac{(s-3)/2}{(k-2)/2} \right) \cdot (-1) \cdot (s-1)t_{1}^{2} \cdot (s-1)t_{1}
\]
for any $s (s \geq 3)$ and the degree of this term is $k+1 (\geq 3)$.

(b) Suppose $s$ is even ($s \geq 4$). We also divide this case into three other sub-cases.

(b1) Suppose $k$ is odd.

In this case, we have $1 \leq k \leq s-1$ and the degree of the leading term of $h_{s-k}^{[u]}$ is $k+1$. In fact, it is obtained by taking

\[(b11)\] $-\frac{n-1}{n}t_{1}^{2}/n$ from the $(s,s)$ entry $x - (n-1)t_{1}^{2}/n$.

(b12) $\{(k-1)/2\}$ pairs of $(s-1)t_{1}$ from entries of the form $(i,s+1-i)$ ($2 \leq i \leq s-1$).

Therefore, the leading term of $h_{s-k}^{[u]}$ is
\[
-\frac{n-1}{n}t_{1}^{2} \cdot \left( \frac{(s-2)/2}{(k-1)/2} \right) \cdot (-1) \cdot (s-1)t_{1}^{2} \cdot (k-1)/2
\]
and the degree of this term is $k+1 (\geq 2)$.

(b2) Suppose $k$ is even and $2 \leq k \leq s-2$.

In this case, the degree of the leading term of $h_{s-k}^{[u]}$ is $k$. In fact, it is obtained by taking
(b21) \(-(n-1)t_1^2/n\) from the \((s,s)\) entry \(x-(n-1)t_1^2/n\),

(b22) ‘\((k-2)/2\)’ pairs of \((s-1)t_1\) from entries of the form \((i,s+1-i)\) \((2 \leq i \leq s-1)\),

(b23) \(-s(n-s)/n\) from the \((1,1)\) entry \(x-s(n-s)/n\)

or by taking

(b24) \(-(n-1)t_1^2/n\) from the \((s,s)\) entry \(x-(n-1)t_1^2/n\),

(b25) If \(s \geq 6 \Leftrightarrow (s,k) \neq (4,2)\), ‘\((k-2)/2\)’ pairs of \((s-1)t_1\) from entries of the form \((i,s+1-i)\) \((2 \leq i \leq (s-2)/2, (s+4)/2 \leq i \leq s-1)\),

(b26) \(s\) from the \(((s+2)/2, (s+2)/2)\) entry \(x+s\)

or by taking

(b27) ‘\(k/2\)’ pairs of \((s-1)t_1\) from entries of the form \((i,s+1-i)\) \((2 \leq i \leq s-1)\)

or by taking

(b28) One pair of \(-(n-s)t_1/n\) from the \((1,1)\) and the \((s,1)\) entry.

(b29) ‘\((k-2)/2\)’ pairs of \((s-1)t_1\) from entries of the form \((i,s+1-i)\) \((2 \leq i \leq s-1)\).

Here, note that if we take the \((s,1)\) entry \(-(n-s)t_1/n\) from the \(s\)-th row, we must also take the \((1,s)\) entry \(-(n-s)t_1/n\) from the first row.

Therefore, the leading term of \(h_{s-k}^{(n)}\) is

\[
-\frac{n-1}{n}t_1^2 \cdot \left(\frac{(s-2)/2}{(k-2)/2}\right) \left(\frac{(n-s)}{n}\right) \left(\frac{(s-1)}{n}\right) \left(\frac{(s-1)^2 t_1^2}{n^2}\right) \left(\frac{(s-1)^2 t_1^2}{n^2}\right) \left(\frac{(s-1)^2 t_1^2}{n^2}\right)
\]

for any \(s\) \((s \geq 4)\). Then, since

\[
\left(\frac{(s-2)/2}{(k-2)/2}\right) = \frac{s-k}{s-2} \left(\frac{(s-2)/2}{(k-2)/2}\right), \quad \left(\frac{(s-2)/2}{k/2}\right) = \frac{s-k}{k} \left(\frac{(s-2)/2}{(k-2)/2}\right),
\]

we have
\[
\frac{s(n-s)(n-1)}{n^2} \left(\frac{(s-2)/2}{(k-2)/2}\right) - \frac{s(n-1)}{n} \left(\frac{(s-4)/2}{(k-2)/2}\right)
\]
\[
= \left(\frac{s(s-2)(n-1)}{n^2} - \frac{s(s-k)(n-1)}{n(s-2)}\right) - \frac{(s-1)^2}{(k-2)/2} \left(\frac{(s-2)/2}{(k-2)/2}\right)
\]
\[
= s \left(\frac{(k+s^2-4s+2)-s^3+4s^2-5s+2}{nk(s-2)}\right) - \frac{k(k+s^2-4s+2)}{k+s^2-4s+2-s^3+4s^2-5s+2}
\]

Hence, if the above value becomes zero, we have
\[
(k(k+s^2-4s+2)-s^3+4s^2-5s+2)n - k(k+s^2-4s+2) = 0,
\]
which implies
\[
k(k+s^2-4s+2) = 0, \quad -s^3+4s^2-5s+2 = 0 \quad (3.14)
\]
or
\[
n = \frac{k(k+s^2-4s+2)}{k+s^2-4s+2-s^3+4s^2-5s+2}. \quad (3.15)
\]
Here, (3.14) is impossible since \(-s^3+4s^2-5s+2 = -(s-1)^2(s-2)\) and \(s \geq 4\). Also, (3.15) is impossible since, for any \(s \geq 4\) and \(2 \leq k \leq s-2\), we have
\[
k(k+s^2-4s+2) \geq 2(2+s^2-4s+2) \geq 2(s-2)^2 > 0
\]
and
\[
k(k+s^2-4s+2)-s^3+4s^2-5s+2
\]
\[
\leq (s-2)((s-2)+s^2-4s+2)-s^3+4s^2-5s+2
\]
\[
= -s^2+s+2
\]
\[
= -(s+1)(s-2) < 0,
\]
which implies \(n < 0\), a contradiction. Thus, the above value (3.13) is non-zero and the degree of the leading term of \(h_{s-k}^{(u)}\) is \(k\).

(b3) Suppose \(k\) is even and \(k = s\).
If \(k = s\), \(h_{s-k}^{(u)} = h_{0}^{(u)}\) is the constant term of \(\Phi_u(x)\). In this case, the degree of the leading term of \(h_{0}^{(u)}\) is \(s\). In fact, it is obtained by taking

(b31) \(-(n-1)t_1^2/n\) from the \((s,s)\) entry \(x-(n-1)t_1^2/n\),

(b32) \((s-2)/2\) pairs of \((s-1)t_1\) from entries of the form \((i,s+1-i)\) \(2 \leq i \leq s-1\),

(b33) \(-s(n-s)/n\) from the \((1,1)\) entry \(x-s(n-s)/n\)
or by taking
(b34) all anti-diagonal entries.

Therefore, the leading term of $h_0^{(u)}$ is

\[
-\frac{n-1}{n} t_1^2 \cdot \left( (-1) \cdot (s-1)^2 t_1^2 \right) \cdot \left( -\frac{s(n-s)}{n} \right) + (-1) \cdot \left( -\frac{n-s}{n} t_1 \right) \cdot \left( (-1) \cdot (s-1)^2 t_1^2 \right) \cdot \left( \frac{s(n-s)}{n} \right)
\]

\[
= (-1)^{(s-2)/2} \frac{n-s}{n} (s-1)^{s-1} t_1^s
\]

and the degree of this term is $s$ ($s \geq 4$).

**Lemma 3.5.** Let $v = (v_0, \cdots, v_s) \in \mathbb{R}^{s+1}$ be a real vector and $n (> s)$ be an integer. Put

\[
P_v(t) = \text{det} M_n(f_v(t';x)) = \text{det} M_n(f^{(n)}(v_0, \cdots, v_s, t; x))
\]

and $\alpha_v = \max(\alpha \in \mathbb{R} | P_v(\alpha) = 0)$. If there exists a real number $\rho_0 (> \alpha_v)$ such that $N_{f_v(\xi,t,x)} = \gamma_0$ for any $\xi > \rho_0$, we have $N_{f_v(\xi,t,x)} = \gamma_0$ for any $\xi > \alpha_v$.

**Proof.** Put $A_v(t) = M_n(f_v(t';x))$. Then, by Proposition 2, we have $\gamma_0 = \sigma(A_v(\xi))$ for any $\xi > \rho_0$. Let us also put

\[
R = \{ \rho \in \mathbb{R} | \rho > \alpha_v, \sigma(A_v(\xi)) = \gamma_0 \text{ for any } \xi > \rho \}.
\]

Since $R$ is a nonempty set ($\rho_0 \in R$) having a lower bound $\alpha_v$, $R$ has the infimum $\rho_v$: $\rho_v = \inf R$. Then, it is enough to prove $\rho_v = \alpha_v$. Here, suppose to the contrary that $\rho_v > \alpha_v$ and we denote by

\[
\Omega_v(t;x) = \sum_{k=0}^{n} \omega_k(t)x^k \in \mathbb{R}(t)[x]
\]

the characteristic polynomial of $A_v(t)$. Note that $\omega_k(t) \in \mathbb{R}[t]$ ($0 \leq k \leq n$) and for any $\xi > \alpha_v$, $\Omega_v(\xi;x)$ has $n$ non-zero real roots (counted with multiplicity) since $A_v(\xi)$ is symmetric and $\text{det} A_v(\xi) \neq 0$. Then, by Proposition 3 there exists a positive real number $\delta$ such that $\rho_v - \delta > \alpha_v$ and for any $\xi \in [\rho_v - \delta, \rho_v + \delta]$, $\Omega_v(\xi;x)$ has the same number of positive and hence negative real roots with $\Omega_v(\rho_v;x)$. On the other hand, since $\rho_v = \inf R$, there exist real numbers $\xi_+$ ($\rho_v < \xi_+ < \rho_v + \delta$) and $\xi_-$ ($\rho_v - \delta < \xi_- < \rho_v$) such that $\sigma(A_v(\xi_+)) \neq \sigma(A_v(\xi_-))$, which implies $\Omega_v(\xi_+;x)$ and $\Omega_v(\xi_-;x)$ have different number of positive and hence negative real roots. This is a contradiction and we have $\rho_v = \alpha_v$. \(
\)

### 3.4 Proof of Theorem 3.2

Let $r = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be the vector as in Theorem 3.2 and put

\[
n_0 = \begin{cases} (n-s+1)/2, & n-s-1 : \text{even} \\ (n-s+2)/2, & n-s-1 : \text{odd} \end{cases}
\]
When $n-s \geq 2$, we inductively define the matrix $A_r(t)_k = (a_{ij}(r)(t)_k)_{1 \leq i,j \leq n}$ $(2 \leq k \leq n-s)$ as the matrix obtained from $A_r(t)_{k-1}$ by sweeping out the entries of the $k$-th row ($k$-th column) by the $(k,l_0-k)$ entry $-(n-s)r_s t$ ($l_0-k,k)$ entry $-(n-s)r_s t$). That is, we define $A_r(t)_k = tS_r(t)_kA_r(t)_k-1S_r(t)_k$, where

$$S_r(t)_k = \begin{cases} \prod_{m=1}^{n} R_n \left( l_0-k,m; -a_{km}(r)(t)_{k-1} \right) & (2 \leq k \leq n_0) \\ \prod_{m=1}^{n} R_n \left( l_0-k,m; -a_{km}(r)(t)_{k-1} \right) & (n_0 < k \leq n-s). \end{cases}$$

Then, if $n-s \geq 1$, we can express the matrix $A_r(t)_{n-s}$ as follows:

$$A_r(t)_{n-s} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & -(n-s)r_s t \\ \vdots & \vdots & \ddots & 0 \\ 0 & -(n-s)r_s t & 0 & 0 \end{bmatrix} O \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} C_r(t)_{n-s}.$$

Note that $a_{km}(r)(t)_{k-1}$ and $a_{kk}(r)(t)_{k-1}$ appearing in $S_r(t)_k$ are degree 1 monomials in $t$ and hence the numbers $-a_{km}(r)(t)_{k-1}/-(n-s)r_s t$, $-a_{kk}(r)(t)_{k-1}/-2(n-s)r_s t$ appearing in $S_r(t)_k$ are just real numbers. Therefore, the entries of the $s \times s$ symmetric matrix $C_r(t)_{n-s} = (c_{ij}(r)(t)_{n-s})_{1 \leq i,j \leq s}$ $(n-s \geq 1)$ are of the form

$$c_{ij}(r)(t)_{n-s} = b_{ij}(r)t^2 + \tilde{\lambda}_{ij}(r)t \quad (\tilde{\lambda}_{ij}(r) \in \mathbb{R}). \quad (3.16)$$

Moreover, since the matrix

$$D_r(t)_{n-s} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & -(n-s)r_s t \\ \vdots & \vdots & \ddots & 0 \\ 0 & -(n-s)r_s t & 0 & 0 \end{bmatrix}$$
In this case, we have

\[ \text{(i) The case} \]

is equivalent to the matrix

\[
\begin{pmatrix}
1 & 0 & -(n-s)r_s t \\
0 & -(n-s)r_s t & 0 \\
-(n-s)r_s t & 0 & -(n-s)r_s t \\
& \ddots & \ddots & \ddots
\end{pmatrix}
\]

over \( \mathbb{R} \), we have

\[
\sigma(D_r(\xi)) = \sigma(\bar{D}_r(\xi)) = \begin{cases} 
1 & n-s \text{ : odd} \\
0 & n-s \text{ : even, } r_s > 0 \\
2 & n-s \text{ : even, } r_s < 0
\end{cases} \quad (3.17)
\]

for any real number \( \xi > \alpha_r \geq 0 \). Here, note that since \( P_r(0) = 0 \), we have \( \alpha_r \geq 0 \).

Next, let \( \Phi_r(t;x), \Psi_r(t;x) \) be characteristic polynomials of \( \bar{B}_r(t), C_r(t)_{n-s} \), respectively. Then, by equations (3.11) and (3.12), we have

\[
\Phi_r(t;x) = x^s + h_{s-1}^{(r)} t^2 x^{s-1} + \cdots + h_1^{(r)} t^{2s-2} x + h_0^{(r)} t^{2s}
\]

where \( h_{k-s}^{(r)} = h_{s-k}(r_0, \ldots, r_s) \in \mathbb{R} \) \((1 \leq k \leq s)\),

\[
\Psi_r(t;x) = x^s + \left( h_{s-1}^{(r)} t^2 + \psi_{s-1}(t) \right) x^{s-1} + \cdots \\
+ \left( h_1^{(r)} t^{2s-2} + \psi_1(t) \right) x + \left( h_0^{(r)} t^{2s} + \psi_0(t) \right)
\]

(\( \psi_0(t), \ldots, \psi_{s-1}(t) \in \mathbb{R}[t], \deg \psi_{s-k}(t) < 2k \) \((1 \leq k \leq s)\)).

Here, let us divide the proof into next two cases.

(i) The case \( h_0^{(r)} h_1^{(r)} \cdots h_{s-1}^{(r)} \neq 0 \).

In this case, we have

\[
\Psi_r(t;x) = x^s + h_{s-1}^{(r)} t^2 \left( 1 + \frac{\psi_{s-1}(t)}{h_{s-1}^{(r)} t^2} \right) x^{s-1} + \cdots \\
+ h_1^{(r)} t^{2s-2} \left( 1 + \frac{\psi_1(t)}{h_1^{(r)} t^{2s-2}} \right) x + h_0^{(r)} t^{2s} \left( 1 + \frac{\psi_0(t)}{h_0^{(r)} t^{2s}} \right)
\]
and \(1 + \psi_{s-k}(t)/h_{s-k}^{(r)}t^{2k} \to 1\) (\(t \to \infty\)) for any \(k\) (\(1 \leq k \leq s\)). Moreover, since \(h_0^{(r)}h_1^{(r)} \cdots h_{s-1}^{(r)} \neq 0\), we have \(h_0^{(r)} \neq 0\), which implies that for any non-zero real number \(\xi\), \(\Phi_r(\xi; x)\) have \(s\) non-zero real roots (counted with multiplicity). Thus, there exists a real number \(\rho_0 (> \alpha_r)\) such that for any real number \(\xi > \rho_0\), \(\Psi_r(\xi; x)\) have the same number of positive (hence also negative) real roots with \(\Phi_r(\xi; x)\) by Proposition 3.4, which implies \(\sigma(C_r(\xi)) = \sigma(B_r(\xi))\) and hence \(\sigma(C_r(\xi)) = N_{g_r} = \gamma (\xi > \rho_0)\) by Lemma 3.2. Then, by the equation (3.17), we have

\[
\sigma(A_r(\xi)) = \begin{cases} 
\gamma + 1 & n - s: \text{odd} \\
\gamma & n - s: \text{even}, r_s > 0 \\
\gamma + 2 & n - s: \text{even}, r_s < 0 
\end{cases}
\]

for any \(\xi > \rho_0\), which implies

\[
N_{f_r(\xi; x)} = \sigma(A_r(\xi)) = \begin{cases} 
\gamma + 1 & n - s: \text{odd} \\
\gamma & n - s: \text{even}, r_s > 0 \\
\gamma + 2 & n - s: \text{even}, r_s < 0 
\end{cases}
\]

for any \(\xi > \rho_0\) since \(A_r(\xi)\) and \(A_r(\xi)_{n-s}\) are equivalent over \(\mathbb{R}\). Hence, by Lemma 3.5, we have

\[
N_{f_r(\xi; x)} = \begin{cases} 
\gamma + 1 & n - s: \text{odd} \\
\gamma & n - s: \text{even}, r_s > 0 \\
\gamma + 2 & n - s: \text{even}, r_s < 0 
\end{cases}
\]

for any \(\xi > \alpha_r\).

(ii) General case.

Let \(\varepsilon_0\) be a positive real number and for any vector \(v \in \mathbb{R}^{s+1}\), set

\[\alpha'_v = \max(|\alpha| : \alpha \in \mathbb{C}, P_v(\alpha) = 0)\].

Clearly, we have \(\alpha'_v \geq \alpha_v\) for any \(v \in \mathbb{R}^{s+1}\). Here, let us put \(\rho'_0 = \alpha'_v + \varepsilon_0\). Then, by Lemma 3.5, it is enough to prove the next claim.

**Claim 1.** For any real number \(\xi > \rho'_0\), we have

\[
N_{f_r(\xi; x)} = \begin{cases} 
\gamma + 1 & n - s: \text{odd} \\
\gamma & n - s: \text{even}, r_s > 0 \\
\gamma + 2 & n - s: \text{even}, r_s < 0 
\end{cases}
\]

**Proof.** By the assumption that \(g_r(x)\) is a separable polynomial of degree \(s\) and the fact that the non-real roots must occur in pair with its complex conjugate, there exists a real number \(\delta_0\) such that for any vector \(v = (v_0, \cdots, v_s) \in \mathbb{R}^{s+1}\) satisfying \(|r - v|_0 = \max_{0 \leq k \leq s}(|r_k - v_k|) < \delta_0\), \(g_v(x)\) is also a degree \(s\) separable polynomial satisfying \(N_{g_v} = N_{g_r} = \gamma\) by Proposition 3.

(S1) If a vector \(v \in \mathbb{R}^{s+1}\) satisfies \(|r - v|_0 < \delta_0\), then \(g_v(x)\) is also a degree \(s\) separable polynomial satisfying \(N_{g_v} = N_{g_r} = \gamma\).
Next, we put
\[ P(t) = \sum_{k \geq 0} x_k(t_0, \ldots, t_s) t^k = \det A(t) \ (A(t) = A(t_0, \ldots, t_s, t)) \]
and let us consider \( P(t) \) as a polynomial over \( E_1 = \mathbb{R}(t_0, \ldots, t_s) \) in \( t \). Then, since \( x_k(t_0, \ldots, t_s) \in \mathbb{R}[t_0, \ldots, t_s] \) for any \( k \geq 0 \), there exists a real number \( \delta_1 > 0 \) such that for any vector \( v \in \mathbb{R}^{s+1} \) satisfying \( |r - v|_0 < \delta_1 \), we have \( |\alpha'_r - \alpha'_w| < \varepsilon_0 \) by Proposition 3

\( \text{(S2)} \) If a vector \( v \in \mathbb{R}^{s+1} \) satisfies \( |r - v|_0 < \delta_1 \), we have \( |\alpha'_r - \alpha'_w| < \varepsilon_0 \).

Here, let \( \xi \) be any real number such that \( \xi > \rho'_0 = \alpha'_r + \varepsilon_0 \) and let
\[ \Omega(t_0, \ldots, t_s, \xi; x) = \sum_{k=0}^n y_k(t_0, \ldots, t_s)x^k \in E_1[x] \]
be the characteristic polynomial of the Bezoutian
\[ A(t_0, \ldots, t_s, \xi; x) = M_n(f^{(n)}(t_0, \ldots, t_s, \xi; x), f^{(n)}(t_0, \ldots, t_s, \xi; x)'). \]
Here, \( f^{(n)}(t_0, \ldots, t_s, \xi; x)' \) is the derivative of
\[ f^{(n)}(t_0, \ldots, t_s, \xi; x) = \sum_{k=0}^n z_k(t_0, \ldots, t_s)x^k \in E_1[x] \]
with respect to \( x \). Then, since \( z_k(t_0, \ldots, t_s) \in \mathbb{R}[t_0, \ldots, t_s] \) \( (0 \leq k \leq n) \), we also have \( y_k(t_0, \ldots, t_s) \in \mathbb{R}[t_0, \ldots, t_s] \) \( (0 \leq k \leq n) \). Moreover, since \( \xi > \rho'_0 > \alpha_r \), we have \( \det A_r(\xi) = \det A(\tau_0, \ldots, \tau_s, \xi) \neq 0 \).

By these arguments, we can also deduce that there exists a positive real number \( \delta_2 \) such that for any vector \( v \in \mathbb{R}^{s+1} \) satisfying \( |r - v|_0 < \delta_2 \), the characteristic polynomial \( \Omega_r(\xi; x) \) have the same number of positive and hence negative real roots with \( \Omega_r(\xi; x) \) (counted with multiplicity), which implies \( N_{E_r}(\xi, x) = \sigma(A_r(\xi)) = \sigma(A_r(\xi)) = N_{E_r}(\xi, x) \).

\( \text{(S3)} \) If a vector \( v \in \mathbb{R}^{s+1} \) satisfies \( |r - v|_0 < \delta_2 \), we have \( N_{E_r}(\xi, x) = N_{E_r}(\xi, x) \).

Put \( \delta = \min(\delta_0, \delta_1, \delta_2) > 0 \). Then, there exists a vector \( w = (w_0, \ldots, w_s) \in \mathbb{R}^{s+1} \) such that
\[ (a) \ |r - w|_0 < \delta, \ (b) \ h^{(w)}_0 \cdot h^{(w)}_1 \cdot \ldots \cdot h^{(w)}_{s-1} \neq 0. \]
Here, we put \( h^{(w)}_{s-k} = h_{s-k}(w_0, \ldots, w_s) \) for any \( k \) \( (1 \leq k \leq s) \). In fact, since \( h_{s-k}(t_0, \ldots, t_s) \) is a non-zero polynomial for any \( k \) \( (1 \leq k \leq s) \) by Lemma 3, the product \( \prod_{k=1}^s h_{s-k}(t_0, \ldots, t_s) \) is also non-zero, which implies that there exists a vector \( w \in \mathbb{R}^{s+1} \) satisfying (a) and (b).

Let \( w \in \mathbb{R}^{s+1} \) be the vector as above. Then, since \( |r - w|_0 < \delta \leq \delta_0 \), \( g_w(x) \) is a degree \( s \) separable polynomial satisfying \( N_{g_w} = \gamma \) by (S1) and also, by (S2), we have \( \alpha_w \leq \alpha'_w < \alpha'_r + \varepsilon_0 = \rho'_0 < \xi \). Thus, by (b) and the case (i), we have
\[ N_{E_r}(\xi, x) = \begin{cases} \gamma + 1 & n - s : \text{odd} \\ \gamma & n - s : \text{even, } r_s > 0 \\ \gamma + 2 & n - s : \text{even, } r_s < 0, \end{cases} \]
which, by (S3), implies

$$N_{f_r(\xi,x)} = \begin{cases} 
\gamma + 1 & n - s \text{ : odd} \\
\gamma & n - s \text{ : even, } r_s > 0 \\
\gamma + 2 & n - s \text{ : even, } r_s < 0.
\end{cases}$$

Since $\xi$ is any real number such that $\xi > \rho'_0$, this completes the proof of Claim and hence the proof of Theorem 3.2. □

Proposition 5. Let $g(x) = \sum_{i=0}^s a_i x^i$ be a polynomial in $\mathbb{R}[x]$ such that $\Delta_g \neq 0$ and

$$f(t,x) = x^n + t \cdot g(x) \quad (3.18)$$

If $g(x)$ is totally complex, $(n - s)$ is even, and $a_s > 0$ then $f(\beta,x)$ is totally complex for all $\beta > \max\{\alpha | \Delta_{f(\beta,x)}(\alpha) = 0\}$.

Proof. We have to show that $f(\beta,x)$ has no real roots. Since $g(x)$ is totally complex we have that $\gamma = 0$. $N_{f(\beta,x)} = \gamma$ as $\beta > \max\{\alpha | \Delta_{f(\beta,x)}(\alpha) = 0\}$ and $a_s > 0$, so $N_{f(\beta,x)} = \gamma = 0$. Hence, $f(\beta,x)$ is totally complex. □

Let $K := \mathbb{Q}(t,a_0,\ldots,a_s)$ be the field of transcendental degree $s + 1$ and $g(x) = \sum_{i=0}^s a_i x^i$. Then we have the following.

Corollary 2. Let $K := \mathbb{Q}(t,a_0,\ldots,a_s)$ be the field of transcendental degree $s + 1$, $g(x) = \sum_{i=0}^s a_i x^i$ and

$$f(t,x) = x^n + t \cdot g(x)$$

For any value of $(\lambda_0,\ldots,\lambda_s) \in \mathbb{Z}^{s+1}$, if $g(\lambda_0,\ldots,\lambda_s,x) \in \mathbb{Z}[x]$ is irreducible and satisfies the conditions of the Eisenstein criteria, then $f(x)$ is irreducible, over $\mathbb{Q}$.

We also note:

Remark 3.4. It can be verified computationally by Maple that if $n \leq 9$ and $1 \leq s < n$ then the Galois group $\text{Gal}_K(f,x)$ is isomorphic to $S_n$.

Remark 3.5. Polynomials in Eq. (3.18) for $s = 1$ and $t = 1$ has been treated by Y. Zarhin in [18] while studying Mori trinomials. It is shown there that the Galois group of $f(x)$ over $\mathbb{Q}$ is isomorphic to $S_n$; see [18, Cor. 3.5] for details.

In general, if we let $K := \mathbb{Q}(t,a_0,\ldots,a_s)$ be the field of transcendental degree $s + 1$, for $1 \leq s < n$, then we expect that $\text{Gal}_K(f) \cong S_n$ for all $n \geq 1$. If true, this would generalize Zarhin’s result to a more general class of polynomials.

References


[13] Shuichi Otake, *Counting the number of distinct real roots of certain polynomials by Bezoutian and the Galois groups over the rational number field*, Linear Multilinear Algebra 61 (2013), no. 4, 429–441. MR3005628


