Postulation of general unions of lines and +lines in positive characteristic

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ABSTRACT

A +line is a scheme \( R \subset \mathbb{P}^r \) with a line as its reduction \( L = R_{\text{red}} \) which is the union of \( L \) and a tangent vector \( v \not\subseteq L \) with \( v_{\text{red}} \in L \). Here we prove in arbitrary characteristic that for \( r \geq 4 \) a general union of lines and +lines has maximal rank. We use the case \( r = 3 \) proved by myself in a previous paper and adapt the characteristic zero proof of the case \( r > 3 \) given in the same paper.

RESUMEN

Una +línea es un esquema \( R \subset \mathbb{P}^r \) con una línea como su reducción \( L = R_{\text{red}} \) que es la unión de \( L \) y un vector tangente \( v \not\subseteq L \), con \( v_{\text{red}} \in L \). Acá demostramos que para \( r \geq 4 \) una unión general de líneas y +líneas tiene rango máximo en característica arbitraria. Usamos el caso \( r = 3 \) demostrado por el autor en un artículo anterior y adaptamos la demostración en característica cero del caso \( r > 3 \) dado en el mismo artículo anterior.

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1 Introduction

The aim of this note is to extend to the positive characteristic case a results in [1]. This extension is sufficient to extend [2, 3] to the positive characteristic case.

A scheme $X \subset \mathbb{P}^r$ is said to have maximal rank if $h^0(I_X(t)) \cdot h^1(I_X(t)) = 0$ for all $t \in \mathbb{N}$. Fix a line $L \subset \mathbb{P}^r$, $r \geq 2$, and $P \in L$. A tangent vector of $\mathbb{P}^r$ with $P$ as its support is a zero-dimensional scheme $Z \subset \mathbb{P}^r$ such that $\deg(Z) = 2$ and $Z_{\text{red}} = \{P\}$. The tangent vector $Z$ is uniquely determined by $P$ and the line $\langle Z \rangle$ spanned by $Z$. Conversely, for each line $D \subset \mathbb{P}^r$ with $P \in D$ there is a unique tangent vector $\nu$ with $\nu_{\text{red}} = P$ and $\langle \nu \rangle = D$. A +line $M \subset \mathbb{P}^r$ supported by $L$ and with nilradical supported by $P$ is the union $\nu \cup L$ of $L$ and a tangent vector $\nu$ with $P$ as its support and spanning a line $\langle \nu \rangle \neq L$. The set of all +lines of $\mathbb{P}^r$ supported by $L$ and with a nilradical at $P$ is an irreducible variety of dimension $r-1$ (the complement of $L$ in the $(r-1)$-dimensional projective space of all lines of $\mathbb{P}^r$ containing $P$). Hence the set of all +lines of $\mathbb{P}^r$ supported by $L$ is parametrized by an irreducible variety of dimension $r$. For any +line $R$ and every integer $k > 0$ we have $h^0(O_R(k)) = k+2$ and $h^1(O_R(k)) = 0$.

For any integers $r \geq 3$, $t \geq 0$, $c \geq 0$ with $(t, c) \neq (0, 0)$ let $L(r, t, c)$ be the set of all schemes $X \subset \mathbb{P}^r$ which are the disjoint union of $t$ lines and $c$ +lines. Every element of $L(r, t, c)$ has the map $k \mapsto (k+1)t + (k+2)c$ as its Hilbert function.

Consider the following statement.

**Theorem 1.1.** For all integers $r \geq 3$, $a \geq 0$ and $b \geq 0$, $(a, b) \neq (0, 0)$, a general union $X \subset \mathbb{P}^r$ of $a$ lines and $b$ +lines has maximal rank.

This statement was proved in [1] when either $r = 3$ or $r \geq 4$ and the algebraically closed base field has characteristic zero. The aim of this note is to prove Theorem 1.1 in positive characteristic (using the case $r = 3$ proved in [1]). Hence we may assume $r \geq 4$. We also use numerical lemmas and elementary remarks contained in [1]. We only need to change all parts which quote [4, Lemma 1.4] or [6], the only characteristic zero tool used in [1]. We recall that the case $c = 0$ is due to R. Hartshorne and A. Hirschowitz ([7]).

2 Proof of Theorem 1.1

For all integers $r \geq 3$ and $k \geq 0$ let $H_{r, k}$ denote the following statement:

**Assertion** $H_{r, k}$, $r \geq 3$, $k \geq 0$: Fix $(t, c) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ and take a general $X \in L(r, t, c)$. If $(k+1)t + (k+2)c \geq \binom{t+k}{k}$, then $h^0(I_X(k)) = 0$. If $(k+1)t + (k+2)c \leq \binom{t+k}{k}$, then $h^1(I_X(k)) = 0$.

For all integers $r \geq 3$ and $k \geq 0$ define the integers $m_{r, k}$ and $n_{r, k}$ by the relations

$$ (k+1)m_{r, k} + n_{r, k} = \binom{r+k}{r}, \quad 0 \leq n_{r, k} \leq k \quad (2.1) $$
From (2.1) for the pairs $(r, k)$ and $(r, k - 1)$ we get
\[ m_{r,k-1} + (k+1)(m_{r,k} - m_{r,k-1}) + n_{r,k} - n_{r,k-1} = \frac{(r+k-1)}{r-1} \] (2.2)
for all $k > 0$.

For all integers $r \geq 3$ and $k \geq 0$ set $u_{r,k} := \lceil \left(\frac{r+k}{r}\right) / (k+2) \rceil$ and $v_{r,k} := (k+2)u_{r,k} - \left(\frac{r+k}{r}\right)$.
We have
\[ (k+2)(u_{r,k} - v_{r,k}) + (k+1)v_{r,k} = \frac{r+k}{r}, \quad 0 \leq v_{r,k} \leq k+1 \] (2.3)

As in [1] we need the following assumption $B_{r,k}$:

**Assumption** $B_{r,k}$, $r \geq 4$, $k > 0$. Fix a hyperplane $H \subset \mathbb{P}^r$. There is $X \in L(r, m_{r,k} - n_{r,k}, n_{r,k})$ such that the support of the nilradical sheaf of $X$ is contained in $H$ and $h^0(I_X(k)) = 0$.

For all $X \in L(r, m_{r,k} - n_{r,k}, n_{r,k})$ we have $h^0(O_X(k)) = \left(\frac{r+k}{r}\right)$ and $h^1(I_X(k)) = h^0(I_X(k))$.

**Lemma 2.1.** We have $m_{r,k} - m_{r,k-1} \geq n_{r,k-1} + n_{r,k}$ if $r \geq 4$ and $k \geq 2$.

**Proof.** Assume $m_{r,k} - m_{r,k-1} \leq n_{r,k-1} + n_{r,k} - 1$. From (2.1) we get
\[ m_{r,k-1} + kn_{r,k-1} + (k+2)n_{r,k} - k - 1 \geq \frac{(r+k-1)}{r-1} \]

Since $n_{r,k-1} \leq k-1$ and $n_{r,k} \leq k$, we get $m_{r,k-1} \geq \frac{(r+k-1)}{r-1} - 2k^2 + 1$. Since $km_{r,k-1} \leq \frac{(r+k-1)}{r-1}$ and $k\left(\frac{r+k-1}{r-1}\right) - \left(\frac{r+k-1}{r}\right) = (r-1)\left(\frac{r+k-1}{r}\right)$, we get
\[ 2k^3 - k \geq (r-1)\left(\frac{r+k-1}{r}\right) \] (2.4)

This inequality is false if $r = 4$ and $k \geq 2$, because it is equivalent to the inequality $k(2k^2 - 1) \geq (k+3)(k+2)(k+1)k/8$. Since the right hand side of (2.4) is an increasing function of $r$, we conclude for all $r \geq 5$ and $k \geq 2$.

**Lemma 2.2.** Fix an integer $r \geq 4$ and assume that Theorem [1] is true in $\mathbb{P}^{r-1}$. Then $B_{r,k}$ is true for all $k > 0$.

**Proof.** Since the case $k = 1$ is true ([1], Remark 3), we may assume $k \geq 2$ and use induction on $k$. By Lemma 2.1 we have $m_{r,k} - m_{r,k-1} \geq n_{r,k-1} + n_{r,k}$. Fix a solution $X \in L(r, m_{r,k-1} - n_{r,k-1}, n_{r,k-1})$ of $B_{r,k-1}$, say $X = A \cup B$ with $A \in L(r, m_{r,k-1} - n_{r,k-1}, 0)$, $B \in L(r, 0, n_{r,k-1})$ and the tangent vectors of $B$ have support $S \subset H$. By semicontinuity we may assume that no irreducible component of $X_{red}$ is contained in $H$, that no tangent vector associated to the nilradical of $B$ is contained in $H$ and that $S$ is a general subset of $H$ with cardinality $n_{r,k-1}$. Let $C_1 \subset H$ be a general union of $m_{r,k} - m_{r,k-1} - n_{r,k} - n_{r,k-1}$ lines. Let $C_2 \subset H$ be a general union of $n_{r,k-1}$ lines, each of them containing a different point of $S$. Let $E \subset H$ be a general union of $n_{r,k} +$ lines. Since $S$ is
general, \( C_1 \cup C_2 \cup E \) is a general element of \( L(\tau - 1, m_{r,k} - m_{r,k-1} - n_{r,k}, r_{n,k}) \). Since Theorem [1] is true in \( \mathbb{P}^{r-1} \), by (2.2) we get \( h^1(H, \mathcal{I}_{C_1 \cup C_2 \cup E}(k)) = 0 \) and \( h^0(H, \mathcal{I}_{C_1 \cup C_2 \cup E}(k)) = m_{r,k-1} - n_{r,k-1} \). Deforming \( A \) with \( B \cup C_1 \cup C_2 \cup E \) fixed, we may assume \( A \cap (B \cup C_1 \cup C_2 \cup E) = \emptyset \) and that \( h^1(H, \mathcal{I}_{C_1 \cup C_2 \cup E(U \cap H)}(k)) = 0, i = 0, 1 \). Since \( A \cap (B \cup C_1 \cup C_2 \cup E) = \emptyset \), \( Y := A \cup B \cup C_1 \cup C_2 \cup E \) is a disjoint union of \( n_{r,k} \) + lines with support in \( H \) (even contained in \( H \)), \( m_{r,k} - 2n_{r,k-1} - n_{r,k} \) lines and \( n_{r,k-1} \) sundials in the sense of [3]. Hence \( Y \) is a flat limit of a family of elements \( L(r, m_{r,k-1} - n_{r,k-1}, n_{r,k-1}) \) whose nilpotent sheaf is contained in \( H \) ([7], [9]). By the semicontinuity theorem to prove \( B_{r,k} \) it is sufficient to prove that \( h^0(\mathcal{I}_Y(k)) = 0 \). Since no tangent vector of \( B \) is contained in \( H \), then \( \text{Res}_H(Y) = X \) and \( Y \cap H = C_1 \cup C_2 \cup E \cap (A \cap H) \). Since \( h^0(\mathcal{I}_X(k-1)) = 0 \) and \( h^0(H, \mathcal{I}_{C_1 \cup C_2 \cup E(U \cap H)}(k)) = 0 \), a residual exact sequence gives \( h^0(\mathcal{I}_Y(k)) = 0 \).

\[ \text{Lemma 2.3.} \quad \text{Assume } \tau \geq 4 \text{ and that Theorem [1] is true in } H = \mathbb{P}^{r-1}. \text{ Fix an integer } k \geq 2 \text{ and assume that } H_{r,k-1} \text{ is true. Fix integers } a \geq 0, b \geq 0, e \geq 0 \text{ such that } e \leq 2[(k + 2)/2], (k + 2)a + (k + 1)b + 4[(k + 2)/2] \leq (r+1-k-1). \text{ Let } X \subset H \text{ be a general union of } \alpha \text{ +lines, } b \text{ lines and } e \text{ tangent vectors. Then } h^1(H, \mathcal{I}_X(k)) = 0. \]

\[ \text{Proof.} \quad \text{It is sufficient to do the case } e = [(k + 2)/2]. \text{ Let } A \subset H \text{ be a general union of } \alpha \text{ lines and } b \text{ 2-lines.} \]

First assume that \( k \) is even. Let \( L_1, L_2 \subset H \) be general lines. Fix a general \( S_1 \subset L_1 \) with \( z(S_1) = k/2 \) and a general \( P_1 \subset L_1, i = 1, 2 \). Let \( L_i \subset H \) be a general tangent vector of \( H \) with \( P_i \) as its support; in particular we assume \( v_i \subset L_i \). Let \( E_i \subset L_i \) be the union of the \( k/2 \) tangent vectors of \( L_i \) with \( (E_i)_{\text{red}} = S_i \). Set \( Y := L_1 \cup V_1 \cup L_2 \cup V_2 \). Let \( R_i \) the +lines with \( L_i \) as their supports and with \( v_i \) as the tangent vectors associated to their nilpotent sheaf. We have \( h^0(\mathcal{O}_{AE_1 \cup E_2 \cup V_1 \cup V_2}(k)) = h^0(\mathcal{O}_{AUR_1 \cup R_2}(k)) \), \( h^1(\mathcal{O}_{AE_1 \cup E_2 \cup V_1 \cup V_2}(k)) = h^1(\mathcal{O}_{AUR_1 \cup R_2}(k)) \) and \( h^0(\mathcal{I}_{AE_1 \cup E_2 \cup V_1 \cup V_2}(k)) = h^0(\mathcal{I}_{AUR_1 \cup R_2}(k)) \). Therefore we have \( h^1(\mathcal{I}_{AE_1 \cup E_2 \cup V_1 \cup V_2}(k)) = h^1(\mathcal{I}_{AUR_1 \cup R_2}(k)) \). Since \( (k + 2)a + (k + 1)b + 2(k + 2) \leq (r+1-k-1) \) and Theorem [1] is true in \( \mathbb{P}^{r-1} \), we have \( h^1(\mathcal{I}_{AUR_1 \cup R_2}(k)) = 0 \). Hence \( h^1(\mathcal{I}_{AE_1 \cup E_2 \cup V_1 \cup V_2}(k)) = 0 \). The semicontinuity theorem gives \( h^1(H, \mathcal{I}_X(k)) = 0 \).

Now assume that \( k \) is odd. Let \( F_1 \subset L_1 \) be any disjoint union of \( (k + 1)/2 \) tangent vectors. We have \( h^0(\mathcal{O}_{AUF_1 \cup F_2}(k)) = h^0(\mathcal{O}_{AUL_1 \cup L_2}(k)) \), \( h^1(\mathcal{O}_{AUF_1 \cup F_2}(k)) = h^1(\mathcal{O}_{AUL_1 \cup L_2}(k)) \) and \( h^2(\mathcal{I}_{AUF_1 \cup F_2}(k)) = h^2(\mathcal{I}_{AUL_1 \cup L_2}(k)) \). Therefore we obtain \( h^1(\mathcal{I}_{AUF_1 \cup F_2}(k)) = h^1(\mathcal{I}_{AUL_1 \cup L_2}(k)) \). Since \( (k + 2)a + (k + 1)b + 2(k + 1) \leq (r-k+1) \) and Theorem [1] is true in \( \mathbb{P}^{r-1} \), we have \( h^1(\mathcal{I}_{AUL_1 \cup L_2}(k)) = 0 \). Therefore \( h^1(\mathcal{I}_{AUF_1 \cup F_2}(k)) = 0 \). The semicontinuity theorem gives \( h^1(H, \mathcal{I}_X(k)) = 0 \).

\[ \text{Proof of Theorem [1]} \quad \text{By [1] we may assume } \tau \geq 4. \text{ By induction on } \tau \text{ we may also assume that Theorem [1] is true in } \mathbb{P}^{r-1}. \text{ By [1] Remark 3} \] it is sufficient to prove \( H_{r,k} \) for all integers \( k \geq 1 \). Hence we may assume \( k \geq 2 \) and that \( H_{r,k-1} \) is true. By [1] Remark 4 it is sufficient to prove \( H_{r,k} \) for the pairs \( (t,c) \) such that either \( t = 0 \) and \( r-k-1 \leq c(k+2) \leq (r-k) \) or \( t(k+1) + (k+2)c = (r+k) \) and \( c > 0 \); in the former case either
\(v_{r,k} = 0\) and \(c = u_{r,k}\) or \(v_{r,k} > 0\) and \(c = u_{r,k} - 1\); in the latter case we have \(t + c \geq u_{r,k}\). If \(c < n_{r,k-1}\), then we use step (b) of the proof of Theorem 1 in [1], because we gave a characteristic free proof of \(B_{r,k}\) (Lemma 2.2). The case \(c \geq n_{r,k-1}\) and \(t \geq m_{r,k-1} - n_{r,k-1}\) was proved as step (a1) without using the characteristic zero assumption. Hence we may assume \(c \geq n_{r,k-1}\) and \(t < m_{r,k-1} - n_{r,k-1}\), i.e. the case of step (a2) of the proof in [1].

(i) Assume \(t = 0\) and hence either \(v_{r,k} = 0\) and \(c = u_{r,k}\) or \(v_{r,k} > 0\) and \(c = u_{r,k} - 1\). Fix a general \(U \in L(r,0,v_{r,k-1},u_{r,k-1} - v_{r,k-1})\), say \(U = A \cup B\) with \(A\) the union of the \(v_{r,k-1}\) lines. By \(H_{r,k-1}\) we have \(h^i(\mathcal{I}_U(k-1)) = 0\), \(i = 0,1\). It is easy to check using (2.3) that \(u_{r,k} > u_{r,k-1}\). Hence \(c \geq u_{r,k-1}\).

Let \(E \subset H\) be a general union of \(c - u_{r,k-1}\) + lines. We may assume \(E \cap (H \cap U) = \emptyset\.
Let \(G \subset H\) be a general union of \(v_{r,k-1}\) tangent vectors of \(H\) with the only restriction that \(G_{red} = A \cap H\). For general \(A\) (and hence a general \(A \cap H\)) the scheme \(E \cup G\) is a general union inside \(H\) of \(u_{r,k} - u_{k-1}\) + lines and \(v_{r,k-1}\) tangent vectors. We have \(v_{r,k-1} \leq k\). Using (2.3) for the integer \(k-1\) is easy to check that if \(v_{r,k-1} > 0\), then \(u_{r,k-1} - v_{r,k-1} \geq 2(k+2) - 2v_{r,k-1}\).

Hence Lemma 2.3 gives \(h^1(H; \mathcal{I}_{\text{EG}}(k)) = 0\). Since \(B \cap H\) is a general union of lines.

(ii) Assume \(t > 0\), \(c > 0\), \((k+1)+(k+2)c = (\binom{t+k}{r})\) and \(t < m_{r,k-1} - n_{r,k-1}\). First assume \(t \leq 2[(k+2)/2]\). In this case we may use the proof given in [1] (step a2)) quoting Lemma 2.8 instead of [1] Lemma 1.4 for the postulation of the \(t\) tangent vectors, because \(m_{r,k-1} - t \geq 2k+2\) in this case. Therefore we may assume \(t \geq k+1\). Since \(t < m_{r,k-1} - n_{r,k-1}\), we have \(k \geq 3\) and \(kt < (\binom{t+k-1}{r-1})\). Set \(d := \lfloor (t+k-1)/k \rfloor (k+1)\) and \(z := (k+1)(k+1)\).

Fix a general \(W \in L(r,t,d)\). Since \(H_{r,k}\) holds for \(x = k-1, k-2\), we have \(h^0(\mathcal{I}_W(k-2)) = 0\) and \(h^1(\mathcal{I}_W(k)) = 0\) and \(h^0(\mathcal{I}_W(k)) = 0\). Since \(S\) is general in \(H\) and \(z(S) = z\), we get \(h^1(\mathcal{I}_{W_0}(k-1)) = 0\), \(i = 0,1\). Since \(kt+(k+1)t+z = (\binom{t+k-1}{r-1})\) and \((k+1)+(k+2)c = (\binom{t+k}{r})\), we get

\[
\begin{aligned}
t + d + (k+2)(c-d) + (k+1)z &= \binom{t+k-1}{r-1}-(c-d) + (k+1)z\\ &= \binom{t+k-1}{r-1}-(c-d) + (k+1)z \\
&= \binom{t+k-1}{r-1}-(k+1)z,
\end{aligned}
\]

Claim 1: We have \(c \geq d + z\).

Proof of Claim 1: We have \(c \leq d + z - 1\). From (2.5) we get \(t + d + (k+1)z - (k+1)z \geq (\binom{t+k-1}{r-1})\) and hence \(k(t + d) + (k+1)zk - (k+1)k \geq k(\binom{t+k-1}{r-1})\). Since \(kt + (k+1)d + z = (\binom{t+k}{r})\) and \(z \leq k\), we get \((k+1)k^2 - k(k+1) \geq k(\binom{t+k-1}{r-1} - (\binom{t+k}{r})\), i.e. \(k^2 - 2k \geq (r-1)(\binom{t+k-1}{r})\). Call \(\phi(t,k)\) the difference between the right hand side and the left hand side of this inequality. We have \(\phi(t,k) = (r-1)(\binom{t+k-1}{r}) - k^2 + 2k\), which is positive if \(r \geq 4\) and \(k \geq 2\).

Let \(M \subset H\) be a general union of \(c - d + z\) + lines of \(H\). Let \(N \subset H\) be \(z\) general lines of \(H\), each of them containing a different point of \(Z\). Since \(S\) is general, \(M \cup N\) has the Hilbert function of a general element of \(L(r-1,z,c-d-z)\) and hence it has maximal rank. By (2.5) we have \(h^1(H; \mathcal{I}_{M \cup N}(k)) = 0\) and \(h^0(\mathcal{I}_{M \cup N}(k)) = t + d\). Let \(Z \subset F^r\) be a general union of \(z\) + lines of \(F^r\) with \(N\) as their support. We have \(G \cap H = N\) and \(\text{Res}_H(Z) = S\). Since \(W \cup M \cup Z \in L(r,t,c)\), it is sufficient to prove that \(h^1(\mathcal{I}_{\text{WU} \cup M \cup Z}(k)) \geq 0\), \(i = 0,1\). Since \(\text{Res}_H(W \cup M \cup Z) = W \cup S\), we have \(h^1(\mathcal{I}_{\text{Res}_H(W \cup M \cup Z)}(k-1)) = 0\). Since \(W \cap H\) is a general union of \(d + c\) points of \(H\) and \((W \cup M \cup Z) = (W \cap H) \cup M \cup N\) as schemes, (2.5) gives \(h^1(H; \mathcal{I}_{\text{Res}_H(W \cup M \cup Z)}(k)) = 0\). Apply the

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Castelnuovo’s lemma.

References


