Generalized trace pseudo-spectrum of matrix pencils

AYMEN AMMAR, AREF JERIBI AND KAMEL MAHFOUDHI

Department of Mathematics
Faculty of Sciences of Sfax, University of Sfax
Route de soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisia
ammar_aymen84@yahoo.fr, aref.jeribi@fss.rnu.tn, kamelmahfoudhi72@yahoo.com

ABSTRACT

The objective of the study was to investigate a new notion of generalized trace pseudo-spectrum for matrix pencils. In particular, we prove many new interesting properties of the generalized trace pseudo-spectrum. In addition, we show an analogue of the spectral mapping theorem for the generalized trace pseudo-spectrum in the matrix algebra.

RESUMEN

El objetivo de este estudio es investigar una nueva noción de pseudo-espectro traza generalizado para pinceles de matrices. En particular, demostramos varias propiedades nuevas e interesantes del pseudo-espectro traza generalizado. Adicionalmente, mostramos un análogo del teorema espectral de aplicaciones para el pseudo-espectro traza generalizado en el álgebra de matrices.

Keywords and Phrases: pseudo-spectrum, condition pseudo-spectrum, trace pseudo-spectrum.

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1 Introduction

Let $M_n(C)$ ($M_n(R)$) denote the algebra of all $n \times n$ complex (real) matrices, $I$ denotes the $n \times n$ identity matrix and the conjugate transpose of $U$ is denoted by $U^*$. We denote by $\text{Tr}$, (resp. $\text{Det}$) the trace (resp. determinant) map on $M_n(C)$. In the present paper, we study the problem of finding the eigenvalues of the generalized eigenvalue problem

$$UX = \lambda VX.$$ 

Next, let $\lambda \in C$ and

$$s_n(\lambda V - U) \leq \ldots \leq s_2(\lambda V - U) \leq s_1(\lambda V - U)$$

be the singular values of the matrix pencils $\lambda V - U$ where $s_1(\lambda V - U)$ is the smallest and $s_n(\lambda V - U)$ is largest singular values of the matrix pencil. Let $U, V \in M_n(C)$, then the set of all eigenvalues of the matrix pencils of the form $\lambda V - U$ is denoted by $\sigma(U, V)$ and is defined as

$$\sigma(U, V) = \left\{ \lambda \in C : \lambda V - U \text{ is not invertible} \right\},$$

and its spectral radius by

$$r(U, V) = \sup \left\{ |\lambda| : \lambda \in \sigma(U, V) \right\}.$$

For an $n \times n$ complex matrices $U$ and $V$ and a non-negative real number $\epsilon$, the pseudo-spectrum of the matrix pencils of the form $\lambda V - U$ is defined as the following closed set in the complex plane

$$\sigma_\epsilon(U, V) = \left\{ \lambda \in C : s_n(\lambda V - U) \leq \epsilon \right\}.$$

Let $U, V \in M_n(C)$ and $0 < \epsilon < 1$. The condition pseudo-spectrum of the matrix pencils $\lambda V - U$ is denoted by $\Sigma_\epsilon(U, V)$ and is defined as

$$\Sigma_\epsilon(U, V) = \left\{ \lambda \in C : s_n(\lambda V - U) \leq \epsilon \lambda s_1(\lambda V - U) \right\}.$$

Let $\epsilon$ be a small positive number. For an operator $U, V \in M_n(C)$, recall that the determinant spectrum of matrix pencils of the form $\lambda V - U$ is the set $d_\epsilon(U, V)$ and is defined as

$$d_\epsilon(U, V) = \left\{ \lambda \in C : |\det(\lambda V - U)| \leq \epsilon \right\}.$$

The analysis of eigenvalues and eigenvectors has had a great effect on mathematics, science, engineering, and many other fields. Then, there are countless applications for this type of analysis. The study of matrix pencils is by now a very thoughtful subject, with the notion of pseudospectrum playing a key role in the theory. However, matrix pencils play an important role in numerical linear algebra, perturbation theory, generalized eigenvalue problems. In this paper, we interest by a generalization of eigenvalues called generalized trace pseudo-spectrum for an element in the matrix
algebra to give more information about the matrix pencils of the form $\lambda V - U$. For more information on various details on the above concepts, properties and applications of pseudo-spectrum [2, 3, 6, 7], condition spectrum [1, 4, 5] and determinant spectrum [8]. Now, we introduce the new concept of the generalized trace pseudo-spectrum in the following definition.

**Definition 1.1.** For $\varepsilon > 0$, the generalized trace pseudo-spectrum of the matrix pencils of the form $\lambda V - U \in \mathcal{M}_n(\mathbb{C})$ is denoted by $\text{Tr}_\varepsilon(U, V)$ and is defined as

$$\text{Tr}_\varepsilon(U, V) = \sigma(U, V) \bigcup \{\lambda \in \mathbb{C} : |\text{Tr}(\lambda V - U)| \leq \varepsilon\}.$$ 

The generalized trace pseudoresolvent of the matrix pencils of the form $\lambda V - U$ is denoted by $\text{Tr}_\rho(U, V)$ and is defined as

$$\text{Tr}_\rho(U, V) = \rho(U, V) \bigcap \{\lambda \in \mathbb{C} : |\text{Tr}(\lambda V - U)| > \varepsilon\}.$$ 

The singular values of a the matrix pencil are important not only for their role in diagonalization but also for their utility in a variety of applications. Since $\text{Tr}_\varepsilon(U, V)$ use all the singular values of $\lambda V - U$ to get defined, it is expected to give more information about $U, V$ than pseudo-spectrum and condition spectrum. Since the definition use idea of "Trace" the generalization of eigenvalues defined above is named as generalized trace pseudo-spectrum. It is easily seen that the map

$$U \rightarrow \text{Tr}(U)$$

is continuous linear functional. Here, some important properties of the trace of $U, B \in \mathcal{M}_n(\mathbb{C})$ are

$$\text{Tr}(UB) = \text{Tr}(BU),$$
$$\text{Tr}(\alpha U) = \alpha \text{Tr}(U) \quad \text{with } \alpha \in \mathbb{C},$$
$$\text{Tr}(U + B) = \text{Tr}(U) + \text{Tr}(B).$$

An outline of this paper is the following. In Section 2, we focuses on a new description of the generalized trace pseudo-spectra. Not only do we give a characterization of the generalized trace pseudo-spectrum in the matrix algebra, but also we investigate the connection between generalized trace pseudo-spectrum and algebraic multiplicity of the eigenvalues. In Section 3, we give an analogue of the spectral mapping theorem for the generalized trace pseudo-spectrum in the matrix algebra.

## 2 Generalized trace pseudo-spectrum.

In this section, some relevant properties of the generalized trace pseudo-spectrum are discussed in detail. For $U, V \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$, the generalized trace pseudo-spectrum of the matrix pencils of the form $\lambda V - U$ is denoted by $\text{Tr}_\varepsilon(U, V)$ and is defined as

$$\text{Tr}_\varepsilon(U, V) = \sigma(U, V) \bigcup \{\lambda \in \mathbb{C} : |\text{Tr}(\lambda V - U)| \leq \varepsilon\}.$$
The generalized trace pseudo-resolvent of the matrix pencils of the form $\lambda V - U$ is denoted by $\text{Tr}_\rho(U, V)$ and is defined as
\[ \text{Tr}_\rho(U, V) = \rho(U, V) \cap \{ \lambda \in \mathbb{C} : |\text{Tr}(\lambda V - U)| > \varepsilon \} \]
while the generalized trace pseudo-spectral radius of the matrix pencils of the form $\lambda V - U$ is defined as
\[ \text{Tr}_\rho(U, V) := \sup \left\{ |\lambda| : \lambda \in \text{Tr}_\rho(U, V) \right\}. \]

Remark 2.1. Let $U, V \in \mathcal{M}_n(\mathbb{C})$. Then, if $V$ is nonsingular, then it is possible to reduce the generalized trace pseudo-spectrum to a standard trace pseudo-spectrum for the matrices $V^{-1}U$ or $UV^{-1}$, i.e.
\[ \text{Tr}_\rho(U, V) = \sigma(V^{-1}U, I) \cup \left\{ \lambda \in \mathbb{C} : |\text{Tr}(\lambda - V^{-1}U)| \leq \varepsilon \right\}, \]
or
\[ \text{Tr}_\rho(U, V) = \sigma(UV^{-1}, I) \cup \left\{ \lambda \in \mathbb{C} : |\text{Tr}(\lambda - UV^{-1})| \leq \varepsilon \right\}. \]

The following theorem gives some properties of the generalized trace pseudo-spectrum that follow in a straightforward manner from the definition of the generalized trace pseudo-spectrum.

Theorem 2.1. Let $U, V \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then,

(i) $\text{Tr}_0(U, V) = \bigcap_{\varepsilon > 0} \text{Tr}_\varepsilon(U, V)$.

(ii) If $0 < \varepsilon_1 < \varepsilon_2$, then $\text{Tr}_{\varepsilon_1}(U, V) \subset \text{Tr}_{\varepsilon_2}(U, V)$.

(iii) $\text{Tr}_\varepsilon(U, V)$ is a non-empty compact subset of $\mathbb{C}$.

(iv) If $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}\setminus\{0\}$, then $\text{Tr}_\varepsilon(\beta U + \alpha V, V) = \beta \text{Tr}_{\frac{\varepsilon}{|\text{Tr}(V)|}}(U, V) + \alpha$.

(v) $\text{Tr}_\varepsilon(\alpha V, V) = \left\{ \lambda \in \mathbb{C} : |\lambda - \alpha| \leq \frac{\varepsilon}{|\text{Tr}(V)|} \right\}$ for all $\lambda, \alpha \in \mathbb{C}$.

Proof. The proofs of items (i) and (ii) are clear from the definition of generalized trace pseudo-spectrum.

(iii) Using the continuity from $\mathbb{C}$ to $[0, \infty)$ of the map
\[ \lambda \to |\text{Tr}(\lambda V - U)|, \]
we get that $\text{Tr}_\varepsilon(U, V)$ is a compact set in the complex plane containing the eigenvalues of the matrix pencils $\lambda V - U$. 

(iv) In fact, it is well known
\[
\text{Tr}_\varepsilon(\beta U + \alpha V, V) = \left\{ \lambda \in \mathbb{C} : |\text{Tr}(\lambda V - \beta U - \alpha V)| \leq \varepsilon \right\} \\
= \left\{ \lambda \in \mathbb{C} : |\beta \text{Tr}\left(\frac{\lambda - \alpha}{\beta} V - U\right)| \leq \varepsilon \right\} \\
= \left\{ \lambda \in \mathbb{C} : |\text{Tr}\left(\frac{\lambda - \alpha}{\beta} V - U\right)| \leq \varepsilon |\beta| \right\}.
\]
Then, \( \lambda \in \text{Tr}_\varepsilon(\beta U + \alpha V, V) \). Thus, \( \frac{\lambda - \alpha}{\beta} \in \text{Tr}_\varepsilon(U, V) \). Hence, \( \lambda \in \beta \text{Tr}_\varepsilon(U, V) + \alpha \).

(v) Let \( \lambda \in \text{Tr}_\varepsilon(\alpha V, V) \), then
\[
|\text{Tr}(\lambda V - \alpha V)| = |\lambda - \alpha||\text{Tr}(V)| \leq \varepsilon.
\]
This means that \( \text{Tr}_\varepsilon(\alpha V, V) = \left\{ \lambda \in \mathbb{C} : |\lambda - \alpha| \leq \frac{\varepsilon}{|\text{Tr}(V)|} \right\} \) for all \( \lambda, \alpha \in \mathbb{C} \).

**Theorem 2.2.** Let \( U, V \in \mathcal{M}_n(\mathbb{C}) \) and \( \varepsilon > 0 \). Then,

(i) If \( U = ZBZ^{-1} \) and \( ZV = VZ \) for all nonsingular matrix \( Z \in \mathcal{M}_n(\mathbb{C}) \) we have,
\[
\text{Tr}_\varepsilon(U, V) = \text{Tr}_\varepsilon(B, V).
\]

(ii) If \( U = ZBZ^{-1} \) and \( V = ZKZ^{-1} \) for all nonsingular matrix \( Z \in \mathcal{M}_n(\mathbb{C}) \) we have,
\[
\text{Tr}_\varepsilon(U, V) = \text{Tr}_\varepsilon(B, K).
\]

(iii) The map \( T \rightarrow \text{Tr}_\varepsilon(U, V) \) is an upper semi continuous function from \( \mathcal{M}_n(\mathbb{C}) \) to compact subsets of \( \mathbb{C} \).

**Proof.** (i) Let \( \lambda \in \text{Tr}_\varepsilon(B, V) \), then
\[
|\text{Tr}(\lambda V - B)| = |\text{Tr}(\lambda V - Z^{-1}UZ)|,
\]
\[
= |\text{Tr}(\lambda Z^{-1}ZV - Z^{-1}UZ)|
\]
\[
= |\text{Tr}(Z^{-1}(ZV - UZ))|
\]
\[
= |\text{Tr}(Z^{-1}(\lambda V - UZ)| = |\text{Tr}(\lambda V - U)| \leq \varepsilon.
\]
It follows that, \( \lambda \in \text{Tr}_\varepsilon(U, V) \).

The proofs of items (ii) and (iii) follows immediately from Definition 1.1. Q.E.D.

The following example shows that the converse of the assertion (i) is not true.
Example 2.1. Let \( U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \), \( B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). Then, \( U \) and \( B \) are not similar and for \( \varepsilon > 0 \), we have
\[
\text{Tr}_\varepsilon(U, V) = \text{Tr}_\varepsilon(B, V) = \{ \lambda \in \mathbb{C} : |\lambda - 2| \leq \varepsilon \}.
\]

In the following, we obtain additional results on \( \text{Tr}_\varepsilon(U, V) \) that are useful in our analysis.

Theorem 2.3. Let \( U, V \in M_n(\mathbb{C}) \), \( \lambda \in \mathbb{C} \), and \( \varepsilon > 0 \). Then, there is \( D \in M_n(\mathbb{C}) \) such that
\[
|\text{Tr}(D)| \leq \varepsilon \quad \text{and} \quad \text{Tr}(\lambda V - U - D) = 0 \quad \text{if, and only if,} \quad \lambda \in \text{Tr}_\varepsilon(U, V).
\]

Proof. To see this, we suppose that there exists \( D \in M_n(\mathbb{C}) \) such that
\[
|\text{Tr}(D)| \leq \varepsilon \quad \text{and} \quad \text{Tr}(\lambda V - U - D) = 0.
\]
Then,
\[
|\text{Tr}(\lambda V - U)| = |\text{Tr}(D)| \leq \varepsilon.
\]
Thus, \( \lambda \in \text{Tr}_\varepsilon(U, V) \). Conversely, let \( \lambda \in \text{Tr}_\varepsilon(U, V) \). Then, we will discuss these two cases:

1\textsuperscript{st} case: If \( \lambda \in \text{Tr}_0(U, V) \), then it is sufficient to take \( D = 0_{n \times n} \).

2\textsuperscript{nd} case: \( \lambda \in \text{Tr}_\varepsilon(U, V) \setminus \text{Tr}_0(U, V) \). Then,
\[
|\text{Tr}(\lambda V - U)| \leq \varepsilon.
\]
Now, we consider
\[
D = \frac{\text{Tr}(\lambda V - U)}{n} I.
\]
It is easy to verify that, \( D \in M_n(\mathbb{C}) \) and
\[
|\text{Tr}(D)| = \left| \text{Tr} \left( \frac{\text{Tr}(\lambda V - U)}{n} I \right) \right| = \frac{|\text{Tr}(\lambda V - U)|}{n} \text{Tr}(I) \leq \varepsilon.
\]
Also, we have
\[
\text{Tr}(\lambda V - U - D) = \text{Tr} \left( \lambda V - U - \frac{\text{Tr}(\lambda V - U)}{n} I \right) = 0.
\]

Q.E.D.

Theorem 2.4. Let \( U, V \in M_n(\mathbb{C}) \) and \( \varepsilon > 0 \). Then,
\[
\text{Tr}_{\delta}(U, V) + \Theta_{\varepsilon} \subseteq \text{Tr}_{\varepsilon + \delta}(U, V),
\]
(1)
holds for \( \delta, \varepsilon > 0 \) with \( \Theta_{\varepsilon} \), denoting the closed disk in the complex plane centered at the origin with radius \( \frac{\varepsilon}{|\text{Tr}(V)|} \). If we take \( \delta = 0 \), we obtain an inner bound for \( \text{Tr}_\varepsilon(U, V) \), namely
\[
\text{Tr}_0(U, V) + \Theta_{\varepsilon} \subseteq \text{Tr}_\varepsilon(U, V).
\]
(2)
**Proof.** Let $\lambda \in \text{Tr}_g(U, V) + \Theta_{\epsilon}$. Then, there exists there exists $\lambda_1 \in \text{Tr}_g(U, V)$ and $\lambda_2 \in \Theta_{\epsilon}$ such that $\lambda = \lambda_1 + \lambda_2$. Therefore,

$$|\text{Tr}(\lambda_1 V - U)| \leq \delta$$

and

$$|\text{Tr}(\lambda V - U)| = |\text{Tr}((\lambda_1 + \lambda_2) V - U)| = |\text{Tr}(\lambda_2 V) + \text{Tr}(\lambda_1 V - U)|$$

$$\leq |\lambda_2| |\text{Tr}(V)| + |\text{Tr}(\lambda_1 V - U)|$$

$$\leq |\text{Tr}(V)| |\lambda_2| + |\text{Tr}(\lambda_1 V - U)| \leq \epsilon + \delta,$$

so that (1) holds. Finally, let $\delta = 0$, then the desired inclusion (2) is obtained. Q.E.D.

**Theorem 2.5.** Let $U, V \in \mathcal{M}_n(\mathbb{C})$ such that $UB = BU$ and $\epsilon > 0$. If $U$ is normal, then

$$\text{Tr}_\epsilon(U + B, V) \subseteq \sigma(U, V) + \text{Tr}_\epsilon(B, V).$$

**Proof.** We assume that $U$ is normal, so there exists a unitary matrix $Z \in \mathcal{M}_n(\mathbb{C})$ such that

$$Z^* U Z = \lambda_1 I_{n_1} \oplus \lambda_2 I_{n_2} \oplus \ldots \oplus \lambda_k I_{n_k}.$$  

The condition $UB = BU$ implies that

$$Z^* B Z = U_1 \oplus U_2 \ldots \oplus U_k$$

where, $U_i \in \mathcal{M}_{n_i}(\mathbb{C})$, $i = 1, \ldots, k$. Then,

$$\text{Tr}_\epsilon(U + B, V) = \text{Tr}_\epsilon(Z^* U Z + Z^* B Z, V) = \text{Tr}_\epsilon((\lambda_1 I_{n_1} + U_1) \oplus \ldots \oplus (\lambda_k I_{n_k} + U_k), V)$$

$$= \bigcup_{i=1}^{k} \text{Tr}_\epsilon(\lambda_i I_{n_i} + U_i, V) \subseteq \sigma(U, V) + \text{Tr}_\epsilon(B, V).$$

The proof is thus complete. Q.E.D.

**Remark 2.2.** Let $U, B$ and $V \in \mathcal{M}_n(\mathbb{C})$ and $\epsilon > 0$. Then, using Theorem 2.5, we obtain the following inequality,

$$\text{Tr}_{\epsilon}(U + B, V) \subseteq \tau(U, V) + \text{Tr}_{\epsilon}(B, V).$$
Theorem 2.6. Let \( U, B \) and \( V \in \mathcal{M}_n(\mathbb{C}) \) and \( \varepsilon > 0 \). Then,

(i) \( \text{Tr}_\varepsilon(UB, V) = \text{Tr}_\varepsilon(BU, V) \).

(ii) \( \text{Tr}_\varepsilon(U, V) + \text{Tr}_\varepsilon(B, V) \subseteq \text{Tr}_\varepsilon(U + B, V) \).

Proof. (i) Let \( \lambda \in \text{Tr}_\varepsilon(UB, V) \), then

\[
\varepsilon \geq |\text{Tr}(\lambda V - UB)| = |\text{Tr}(\lambda V) + \text{Tr}(-UB)| = |\text{Tr}(\lambda V) + \text{Tr}(-BU)| = |\text{Tr}(\lambda V - BU)|.
\]

Hence, \( \lambda \in \text{Tr}_\varepsilon(BU, V) \). Thus,

\( \text{Tr}_\varepsilon(UB, V) \subseteq \text{Tr}_\varepsilon(BU, V) \).

The conclusion can be obtained similarly to the first inclusion, then we deduce that

\( \text{Tr}_\varepsilon(BU, V) = \text{Tr}_\varepsilon(UB, V) \).

(ii) Let \( \lambda \in \text{Tr}_\varepsilon(U, V) + \text{Tr}_\varepsilon(B, V) \). Then, there exists

\( \lambda_1 \in \text{Tr}_\varepsilon(U, V) \) and \( \lambda_2 \in \text{Tr}_\varepsilon(B, V) \)

such that \( \lambda = \lambda_1 + \lambda_2 \). Therefore,

\( \text{Tr}(\lambda_1 V - U) \leq \frac{\varepsilon}{2} \) and \( \text{Tr}(\lambda_2 V - B) \leq \frac{\varepsilon}{2} \).

On the other hand,

\[
|\text{Tr}(\lambda V - U - B)| = |\text{Tr}(\lambda_1 V - U + \lambda_2 V - B)| \\
\leq |\text{Tr}(\lambda_1 V - U)| + |\text{Tr}(\lambda_2 V - B)| \\
\leq \varepsilon
\]

Then, \( \lambda \in \text{Tr}_\varepsilon(U + B, V) \). Q.E.D.

Theorem 2.7. Let \( U, V \in \mathcal{M}_n(\mathbb{C}) \) and \( N \in \mathcal{M}_n(\mathbb{C}) \) is a nilpotent matrix and \( \varepsilon > 0 \). Then,

\( \text{Tr}_\varepsilon(U + N, V) = \text{Tr}_\varepsilon(U, V) \).

Proof. Let \( \lambda \in \text{Tr}_\varepsilon(U + N, V) \), then \( |\text{Tr}(\lambda V - U - N)| \leq \varepsilon \). Since

\[
|\text{Tr}(\lambda V - U) - \text{Tr}(N)| \leq \varepsilon.
\]
Using the fact that the matrix trace vanishes on nilpotent matrices, therefore
\[ \lambda \in \text{Tr}_\varepsilon(U, V). \]

Hence,
\[ \text{Tr}_\varepsilon(U + N, V) \subseteq \text{Tr}_\varepsilon(U, V). \]

Let \( \lambda \in \text{Tr}_\varepsilon(U, V) \), then \(|\text{Tr}(\lambda V - U)| \leq \varepsilon\). Now, we can write for any \( \lambda \in \mathbb{C} \)
\[ |\text{Tr}(\lambda V - U)| = |\text{Tr}(\lambda V - U - N + N)| = |\text{Tr}(\lambda V - U - N) + \text{Tr}(N)|. \]

Because, \( \text{Tr}(N) = 0 \), it follows that \(|\text{Tr}(\lambda V - U - N)| \leq \varepsilon\). Consequently,
\[ \text{Tr}_\varepsilon(U, V) \subseteq \text{Tr}_\varepsilon(U + N, V). \]

Q.E.D.

### 3 Trace pseudospectral mapping Theorem

Let \( U, V \in \mathcal{M}_n(\mathbb{C}) \) and \( f \) be an analytic function defined on \( D \), an open set containing \( \text{Tr}_0(U, V) \). For each \( \varepsilon > 0 \), we define
\[ \varphi(\varepsilon) = \sup_{\lambda \in \text{Tr}_\varepsilon(U, V)} |\text{Tr}(f(\lambda)V - f(U))| \]
and suppose there exists \( \varepsilon_0 > 0 \) such that \( \text{Tr}_{\varepsilon_0}(f(U), V) \subseteq f(D) \). Then, for \( 0 < \varepsilon < \varepsilon_0 \) we define
\[ \phi(\varepsilon) = \sup_{\mu \in f^{-1}(\text{Tr}_\varepsilon(U, V)) \cap D} |\text{Tr}(\mu V - U)|. \]

**Lemma 3.1.** Let \( U, V \in \mathcal{M}_n(\mathbb{C}) \) and \( \varepsilon > 0 \), then \( \varphi(\varepsilon) \) and \( \phi(\varepsilon) \) are well defined, \( \lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0 \) and \( \lim_{\varepsilon \to 0} \phi(\varepsilon) = 0 \).

**Proof.** In the order to prove that \( \varphi(\varepsilon) \) is well defined, we define \( h : \mathbb{C} \to \mathbb{R}_+ \)
\[ h(\lambda) = |\text{Tr}(f(\lambda)V - f(U))| \]
Since \( h(\lambda) \) is continuous and \( \text{Tr}_\varepsilon(U, V) \) is a compact subset of \( \mathbb{C} \), then it is clear that
\[ \varphi(\varepsilon) = \sup \{ h(\lambda) : \lambda \in \text{Tr}_\varepsilon(U, V) \}. \]
We conclude, \( \varphi(\varepsilon) \) is well defined. Now, let assume that there exists \( \varepsilon_0 > 0 \) such that
\[ \text{Tr}_{\varepsilon_0}(f(U), V) \subseteq f(D). \]
We show that for $0 < \varepsilon < \varepsilon_0$, $\phi(\varepsilon)$ is well defined. Define $g : \mathbb{C} \rightarrow \mathbb{R}_+$, 

$$g(\mu) = |\text{Tr}(\mu V - U)|.$$ 

Since $g$ is continuous for all $\mu \in \mathbb{C}$, then $\phi(\varepsilon)$ is well defined. It is also clear that $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ are a monotonically non-decreasing function, $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ goes to zero as $\varepsilon$ goes to zero. Q.E.D.

**Theorem 3.1.** Let $U, V \in M_n(\mathbb{C})$ and let $f$ be an analytic function defined on $D$, an open set containing $\text{Tr}_0(U, V)$. Then, for each 

$$f(\text{Tr}_\varepsilon(U, V)) \subseteq \text{Tr}_{\varphi(\varepsilon)}(f(U), V),$$

where $\varphi(\varepsilon)$ defined above.

**Proof.** Let $\lambda \in \text{Tr}_\varepsilon(U, V)$. Then, using Lemma 3.1 we obtain that $\varphi(\varepsilon)$ is well defined and 

$$\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0.$$ 

Therefore, $h(\lambda) \leq \varphi(\varepsilon)$. Hence 

$$|\text{Tr}(f(\lambda)V - f(U))| := h(\lambda) \leq \varphi(\varepsilon).$$

Thus, $f(\lambda) \in \text{Tr}_{\varphi(\varepsilon)}(f(U), V)$. This means that 

$$f(\text{Tr}_\varepsilon(U, V)) \subseteq \text{Tr}_{\varphi(\varepsilon)}(f(U), V).$$

Q.E.D.

**Theorem 3.2.** Let $U, V \in M_n(\mathbb{C})$ and let $f$ be an analytic function defined on $D$, an open set containing $\text{Tr}_0(U, V)$. Then, for each 

$$\text{Tr}_\varepsilon(f(U), V) \subseteq f(\text{Tr}_{\varphi(\varepsilon)}(U, V)).$$

where $\phi(\varepsilon)$ defined above.

**Proof.** Let $\lambda \in \text{Tr}_\varepsilon(f(U), V)$. Then, using Lemma 3.1 we obtain the existence of $\varepsilon_0 > 0$ such that 

$$\text{Tr}_\varepsilon(f(U), V) \subseteq \text{Tr}_{\varepsilon_0}(f(U), V) \subseteq f(D).$$

Consider $\mu \in D$ such that $\lambda = f(\mu)$. Then $\mu \in f^{-1}(\text{Tr}_\varepsilon(U, V))$, hence 

$$g(\mu) \leq \phi(\varepsilon).$$

Therefore, 

$$|\text{Tr}(\mu V - U)| := g(\mu) \leq \phi(\varepsilon)$$

Thus, $\mu \in \text{Tr}_{\phi(\varepsilon)}(U, V)$. Then, $\lambda = f(\mu) \in f(\text{Tr}_{\phi(\varepsilon)}(U, V))$. This means that 

$$\text{Tr}_\varepsilon(f(U), V) \subseteq f(\text{Tr}_{\phi(\varepsilon)}(U, V)).$$

Q.E.D.
Corollary 3.1. Combining the two inclusions in Theorems 3.1 and 3.2 we get
\[ f(\text{Tr}_\varepsilon(U, V)) \subseteq \text{Tr}_{\varphi(\varepsilon)}(f(U), V) \subseteq f(\text{Tr}_{\varphi(\varepsilon)}(U, V)) \]
and
\[ \text{Tr}_\varepsilon(f(U), V) \subseteq f(\text{Tr}_{\varphi(\varepsilon)}(U, V)) \subseteq \text{Tr}_{\varphi(\varepsilon)}(f(U), V). \]

Here are some remarks.

Remark 3.1. (i) It will be clear from the proofs of Theorems 3.1 and 3.2 that the functions \( \varphi \) and \( \phi \) measure the sizes of the trace pseudo-spectra are optimal.

(ii) From the definitions of \( \varphi \) and \( \phi \), the set inclusions are sharp in the sense that the functions cannot be replaced by smaller functions.

(iii) In general, the spectral mapping theorem is not true for generalized trace pseudo-spectrum.

Example 3.1. Let \( \alpha, \beta \in \mathbb{C} \) with \( \alpha \neq \beta \neq 0 \) and let \( U = \begin{pmatrix} \alpha & 1 \\ \beta & 0 \end{pmatrix}, \ V = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \) and \( f(\lambda) = \lambda^2 \). Then \( f(U) = \begin{pmatrix} \alpha^2 & \alpha + \beta \\ 0 & \beta^2 \end{pmatrix} \). A direct computation shows that
\[ \text{Tr}_\varepsilon(f(U), V) = \{ \lambda \in \mathbb{C} : |2\lambda - \alpha^2| \leq \varepsilon - \beta^2 \} \]
and
\[ f(\text{Tr}_\varepsilon(U, V)) = \{ \lambda^2 \in \mathbb{C} : |2\lambda - \alpha^2| \leq \varepsilon - \beta^2 \}. \]

We can see for all \( \varepsilon > 0 \) that \( \text{Tr}_\varepsilon(f(U), V) \neq f(\text{Tr}_\varepsilon(U, V)) \).
References


