Ostrowski-Sugeno fuzzy inequalities

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ABSTRACT

We present Ostrowski-Sugeno fuzzy type inequalities. These are Ostrowski-like inequalities in the context of Sugeno fuzzy integral and its special properties are investigated. Tight upper bounds to the deviation of a function from its Sugeno-fuzzy averages are given. This work is greatly inspired by [3] and [1].

RESUMEN

Presentamos desigualdades de Ostrowski-Sugeno de tipo fuzzy. Estas son desigualdades de tipo Ostrowski en el contexto de integrales fuzzy de Sugeno y se investigan sus propiedades especiales. Se entregan cotas superiores ajustadas para la desviación de una función de sus promedios fuzzy de Sugeno. Este trabajo está inspirado principalmente por [3] y [1].

Keywords and Phrases: Sugeno fuzzy, integral, function fuzzy average, deviation from fuzzy mean, fuzzy Ostrowski inequality.

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1 Introduction

The famous Ostrowski ([3]) inequality motivates this work and has as follows:

$$\left| \frac{1}{b-a} \int_a^b f(y) dy - f(x) \right| \leq \left(\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right) (b-a) \|f'\|_\infty,$$

where $f \in C'([a, b])$, $x \in [a, b]$, and it is a sharp inequality. One can easily notice that

$$\left(\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right) (b-a) = \frac{(x-a)^2 + (b-x)^2}{2(b-a)}.$$

Another motivation is author's article [1].

First we give a survey about Sugeno fuzzy integral and its basic properties. Then we derive a series of Ostrowski-like inequalities to all directions in the context of Sugeno integral and its basic important particular properties. We also give applications to special cases of our problem we deal with.

2 Background

In this section, some definitions and basic important properties of the Sugeno integral which will be used in the next section are presented.

Definition 2.1. (Fuzzy measure [5, 7]) Let Σ be a σ -algebra of subsets of X , and let $\mu : \Sigma \rightarrow [0, +\infty]$ be a non-negative extended real-valued set function. We say that μ is a fuzzy measure iff:

- (1) $\mu(\emptyset) = 0$,
- (2) $E, F \in \Sigma : E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity),
- (3) $E_n \in \Sigma$ ($n \in \mathbb{N}$), $E_1 \subset E_2 \subset \dots$, imply $\lim_{n \rightarrow \infty} \mu(E_n) = \mu(\cup_{n=1}^{\infty} E_n)$ (continuity from below);
- (4) $E_n \in \Sigma$ ($n \in \mathbb{N}$), $E_1 \supset E_2 \supset \dots$, $\mu(E_1) < \infty$, imply $\lim_{n \rightarrow \infty} \mu(E_n) = \mu(\cap_{n=1}^{\infty} E_n)$ (continuity from above).

Let (X, Σ, μ) be a fuzzy measure space and f be a non-negative real-valued function on X . We denote by $\mathcal{F}_+$ the set of all non-negative real valued measurable functions, and by $L_\alpha f$ the set: $L_\alpha f := \{x \in X : f(x) \geq \alpha\}$, the α -level of f for $\alpha \geq 0$.

Definition 2.2. Let (X, Σ, μ) be a fuzzy measure space. If $f \in \mathcal{F}_+$ and $A \in \Sigma$, then the Sugeno integral (fuzzy integral) [6] of f on A with respect to the fuzzy measure μ is defined by

$$(S) \int_A f d\mu := \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap L_\alpha f)), \quad (1)$$

where $\bigvee$ and $\wedge$ denote the sup and inf on $[0, \infty]$, respectively.

The basic properties of Sugeno integral follow:

Theorem 2.3. ([4, 7]) Let (X, Σ, μ) be a fuzzy measure space with $A, B \in \Sigma$ and $f, g \in \mathcal{F}_+$. Then

- 1) $(S) \int_A f d\mu \leq \mu(A)$;
- 2) $(S) \int_A k d\mu = k \wedge \mu(A)$ for a non-negative constant k ;
- 3) if $f \leq g$ on A , then $(S) \int_A f d\mu \leq (S) \int_A g d\mu$;
- 4) if $A \subset B$, then $(S) \int_A f d\mu \leq (S) \int_B f d\mu$;
- 5) $\mu(A \cap L_\alpha f) \leq \alpha \Rightarrow (S) \int_A f d\mu \leq \alpha$;
- 6) if $\mu(A) < \infty$, then $\mu(A \cap L_\alpha f) \geq \alpha \Leftrightarrow (S) \int_A f d\mu \geq \alpha$;
- 7) when $A = X$, $(S) \int_A f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(L_\alpha f))$;
- 8) if $\alpha \leq \beta$, then $L_\beta f \subseteq L_\alpha f$;
- 9) $(S) \int_A f d\mu \geq 0$.

Theorem 2.4. ([7, p. 135]) Let $f \in \mathcal{F}_+$, the class of all finite nonnegative measurable functions on (X, Σ, μ) . Then

- 1) if $\mu(A) = 0$, then $(S) \int_A f d\mu = 0$, for any $f \in \mathcal{F}_+$;
- 2) if $(S) \int_A f d\mu = 0$, then $\mu(A \cap \{x | f(x) > 0\}) = 0$;
- 3) $(S) \int_A f d\mu = (S) \int_A f \cdot \chi_A d\mu$, where χ_A is the characteristic function of A ;
- 4) $(S) \int_A (f + a) d\mu \leq (S) \int_A f d\mu + (S) \int_A a d\mu$, for any constant $a \in [0, \infty)$.

Corollary 2.5. ([7, p. 136]) Let $f, f_1, f_2 \in \mathcal{F}_+$. Then

- 1) $(S) \int_A (f_1 \vee f_2) d\mu \geq (S) \int_A f_1 d\mu \vee (S) \int_A f_2 d\mu$;
- 2) $(S) \int_A (f_1 \wedge f_2) d\mu \leq (S) \int_A f_1 d\mu \wedge (S) \int_A f_2 d\mu$;
- 3) $(S) \int_{A \cup B} f d\mu \geq (S) \int_A f d\mu \vee (S) \int_B f d\mu$;
- 4) $(S) \int_{A \cap B} f d\mu \leq (S) \int_A f d\mu \wedge (S) \int_B f d\mu$.

In general we have

$$(S) \int_A (f_1 + f_2) d\mu \neq (S) \int_A f_1 d\mu + (S) \int_A f_2 d\mu,$$

and

$$(S) \int_A a f d\mu \neq a (S) \int_A f d\mu, \text{ where } a \in \mathbb{R},$$

see [7, p. 137].

Lemma 2.6. ([7, p. 138]) $(S) \int_A f d\mu = \infty$ if and only if $\mu(A \cap L_\alpha f) = \infty$ for any $\alpha \in [0, \infty)$.

We need

Definition 2.7. ([2]) A fuzzy measure μ is subadditive iff $\mu(A \cup B) \leq \mu(A) + \mu(B)$, for all $A, B \in \Sigma$.

We mention the following result

Theorem 2.8. ([2]) If μ is subadditive, then

$$(S) \int_X (f + g) d\mu \leq (S) \int_X f d\mu + (S) \int_X g d\mu, \quad (2)$$

for all measurable functions $f, g : X \rightarrow [0, \infty)$.

Moreover, if (2) holds for all measurable functions $f, g : X \rightarrow [0, \infty)$ and $\mu(X) < \infty$, then μ is subadditive.

Notice here in (1) we have that $\alpha \in [0, \infty)$.

We have the following corollary.

Corollary 2.9. If μ is subadditive, $n \in \mathbb{N}$, and $f : X \rightarrow [0, \infty)$ is a measurable function, then

$$(S) \int_X n f d\mu \leq n (S) \int_X f d\mu, \quad (3)$$

in particular it holds

$$(S) \int_A n f d\mu \leq n (S) \int_A f d\mu, \quad (4)$$

for any $A \in \Sigma$.

Proof. By inequality (2). □

A very important property of Sugeno integral follows.

Theorem 2.10. If μ is subadditive measure, and $f : X \rightarrow [0, \infty)$ is a measurable function, and $c > 0$, then

$$(S) \int_A c f d\mu \leq (c + 1) (S) \int_A f d\mu, \quad (5)$$

for any $A \in \Sigma$.

Proof. Let the ceiling $\lceil c \rceil = m \in \mathbb{N}$, then by Theorem 2.3 (3) and (4) we get

$$(S) \int_A c f d\mu \leq (S) \int_A m f d\mu \leq m (S) \int_A f d\mu \leq (c + 1) (S) \int_A f d\mu,$$

proving (5). □

3 Main Results

From now on in this article we work on the fuzzy measure space $([a, b], \mathcal{B}, \mu)$, where $[a, b] \subset \mathbb{R}$, $\mathcal{B}$ is the Borel σ -algebra on $[a, b]$, and μ is a finite fuzzy measure on $\mathcal{B}$. Typically we take it to be subadditive.

The functions f we deal with here are continuous from $[a, b]$ into $\mathbb{R}_+$.

We make the following remark

Remark 3.1. Let $f \in C^1([a, b], \mathbb{R}_+)$, and μ is a subadditive fuzzy measure such that $\mu([a, b]) > 0$, $x \in [a, b]$. We will estimate

$$E := \left| (S) \int_{[a, b]} f(x) d\mu(t) - \mu([a, b]) \wedge f(x) \right| \quad (6)$$

(by Theorem 2.3 (2))

$$= \left| (S) \int_{[a, b]} f(t) d\mu(t) - (S) \int_{[a, b]} f(x) d\mu(t) \right|.$$

We notice that

$$f(t) = f(t) - f(x) + f(x) \leq |f(t) - f(x)| + f(x),$$

then (by Theorem 2.3 (3) and Theorem 2.4 (4))

$$(S) \int_{[a, b]} f(t) d\mu(t) \leq (S) \int_{[a, b]} |f(t) - f(x)| d\mu(t) + (S) \int_{[a, b]} f(x) d\mu(t), \quad (7)$$

that is

$$(S) \int_{[a, b]} f(t) d\mu(t) - (S) \int_{[a, b]} f(x) d\mu(t) \leq (S) \int_{[a, b]} |f(t) - f(x)| d\mu(t). \quad (8)$$

Similarly, we have

$$f(x) = f(x) - f(t) + f(t) \leq |f(t) - f(x)| + f(t),$$

then (by Theorem 2.3 (3) and Theorem 2.8)

$$(S) \int_{[a, b]} f(x) d\mu(t) \leq (S) \int_{[a, b]} |f(t) - f(x)| d\mu(t) + (S) \int_{[a, b]} f(t) d\mu(t),$$

that is

$$(S) \int_{[a, b]} f(x) d\mu(t) - (S) \int_{[a, b]} f(t) d\mu(t) \leq (S) \int_{[a, b]} |f(t) - f(x)| d\mu(t). \quad (9)$$

By (8) and (9) we derive that

$$\left| (S) \int_{[a, b]} f(t) d\mu(t) - (S) \int_{[a, b]} f(x) d\mu(t) \right| \leq (S) \int_{[a, b]} |f(t) - f(x)| d\mu(t). \quad (10)$$

Consequently it holds

$$\mathbb{E} \stackrel{(by (6), (10))}{\leq} (S) \int_{[a,b]} |f(t) - f(x)| d\mu(t)$$

(and by $|f(t) - f(x)| \leq \|f'\|_{\infty} |t - x|$)

$$\leq (S) \int_{[a,b]} \|f'\|_{\infty} |t - x| d\mu(t) \stackrel{(by (5))}{\leq} (\|f'\|_{\infty} + 1) (S) \int_{[a,b]} |t - x| d\mu(t). \quad (11)$$

We have proved the following Ostrowski-like inequality

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) d\mu(t) - \frac{\mu([a,b] \wedge f(x))}{\mu([a,b])} \right| \leq \frac{(\|f'\|_{\infty} + 1)}{\mu([a,b])} (S) \int_{[a,b]} |t - x| d\mu(t). \quad (12)$$

The last inequality can be better written as follows:

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])} \right) \right| \leq \frac{(\|f'\|_{\infty} + 1)}{\mu([a,b])} (S) \int_{[a,b]} |t - x| d\mu(t). \quad (13)$$

Notice here that $\left(1 \wedge \frac{f(x)}{\mu([a,b])} \right) \leq 1$, and $\frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) d\mu(t) \leq \frac{\mu([a,b])}{\mu([a,b])} = 1$, where $(S) \int_{[a,b]} f(t) d\mu(t) \geq 0$.

I.e. If $f : [a,b] \rightarrow \mathbb{R}_+$ is a Lipschitz function of order $0 < \alpha \leq 1$, i.e. $|f(x) - f(y)| \leq K|x - y|^{\alpha}$, $\forall x, y \in [a,b]$, where $K > 0$, denoted by $f \in \text{Lip}_{\alpha,K}([a,b], \mathbb{R}_+)$, then we get similarly the following Ostrowski-like inequality:

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])} \right) \right| \leq \frac{(K+1)}{\mu([a,b])} (S) \int_{[a,b]} |t - x|^{\alpha} d\mu(t). \quad (14)$$

We have proved the following Ostrowski-Sugeno inequalities:

Theorem 3.2. Suppose that μ is a fuzzy subadditive measure with $\mu([a,b]) > 0$, $x \in [a,b]$.

1) Let $f \in C^1([a,b], \mathbb{R}_+)$, then

$$\left| \frac{1}{\mu([a,b])} (S) \int_{[a,b]} f(t) d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a,b])} \right) \right| \leq \frac{(\|f'\|_{\infty} + 1)}{\mu([a,b])} (S) \int_{[a,b]} |t - x| d\mu(t). \quad (15)$$

2) Let $f \in \text{Lip}_{\alpha, \kappa}([a, b], \mathbb{R}_+)$, $0 < \alpha \leq 1$, then

$$\left| \frac{1}{\mu([a, b])} (S) \int_{[a, b]} f(t) d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a, b])} \right) \right| \leq \frac{(K+1)}{\mu([a, b])} (S) \int_{[a, b]} |t-x|^\alpha d\mu(t). \quad (16)$$

We make the following remark

Remark 3.3. Let $f \in C^1([a, b], \mathbb{R}_+)$ and $g \in C^1([a, b])$, by Cauchy's mean value theorem we get that

$$(f(t) - f(x)) g'(c) = (g(t) - g(x)) f'(c),$$

for some c between t and x ; for any $t, x \in [a, b]$.

If $g'(c) \neq 0$, we have

$$(f(t) - f(x)) = \left(\frac{f'(c)}{g'(c)} \right) (g(t) - g(x)).$$

Here we assume that $g'(t) \neq 0, \forall t \in [a, b]$. Hence it holds

$$|f(t) - f(x)| \leq \left\| \frac{f'}{g'} \right\|_\infty |g(t) - g(x)|, \quad (17)$$

for all $t, x \in [a, b]$.

We have again as before (see (11))

$$\begin{aligned} E &\leq (S) \int_{[a, b]} |f(t) - f(x)| d\mu(t) \stackrel{(by (17))}{\leq} \\ &(S) \int_{[a, b]} \left\| \frac{f'}{g'} \right\|_\infty |g(t) - g(x)| d\mu(t) \stackrel{(by (5))}{\leq} \\ &\left(\left\| \frac{f'}{g'} \right\|_\infty + 1 \right) (S) \int_{[a, b]} |g(t) - g(x)| d\mu(t). \end{aligned} \quad (18)$$

We have established the following general Ostrowski-Sugeno inequality:

Theorem 3.4. Suppose that μ is a fuzzy subadditive measure with $\mu([a, b]) > 0$, $x \in [a, b]$. Let $f \in C^1([a, b], \mathbb{R}_+)$ and $g \in C^1([a, b])$ with $g'(t) \neq 0, \forall t \in [a, b]$. Then

$$\left| \frac{1}{\mu([a, b])} (S) \int_{[a, b]} f(t) d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a, b])} \right) \right| \leq \frac{\left(\left\| \frac{f'}{g'} \right\|_\infty + 1 \right)}{\mu([a, b])} (S) \int_{[a, b]} |g(t) - g(x)| d\mu(t). \quad (19)$$

We give for $g(t) = e^t$ the next result

Corollary 3.5. Suppose that μ is a fuzzy subadditive measure with $\mu([a, b]) > 0$, $x \in [a, b]$. Let $f \in C^1([a, b], \mathbb{R}_+)$, then

$$\left| \frac{1}{\mu([a, b])} (S) \int_{[a, b]} f(t) d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a, b])} \right) \right| \leq \frac{\left(\left\| \frac{f'}{e^t} \right\|_{\infty} + 1 \right)}{\mu([a, b])} (S) \int_{[a, b]} |e^t - e^x| d\mu(t). \quad (20)$$

When $g(t) = \ln t$ we get the following corollary.

Corollary 3.6. Suppose that μ is a fuzzy subadditive measure with $\mu([a, b]) > 0$, $x \in [a, b]$ and $a > 0$. Let $f \in C^1([a, b], \mathbb{R}_+)$. Then

$$\left| \frac{1}{\mu([a, b])} (S) \int_{[a, b]} f(t) d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a, b])} \right) \right| \leq \frac{(\|tf'(t)\|_{\infty} + 1)}{\mu([a, b])} (S) \int_{[a, b]} \left| \ln \frac{t}{x} \right| d\mu(t). \quad (21)$$

Many other applications of Theorem 3.4 could follow but we stop it here.

We make the following remark.

Remark 3.7. Let $f \in [C([a, b], \mathbb{R}_+) \cap C^{n+1}([a, b])]$, $n \in \mathbb{N}$, $x \in [a, b]$. Then by Taylor's theorem we get

$$f(y) - f(x) = \sum_{k=1}^n \frac{f^{(k)}(x)}{k!} (y-x)^k + R_n(x, y), \quad (22)$$

where the remainder

$$R_n(x, y) := \int_x^y \left(f^{(n)}(t) - f^{(n)}(x) \right) \frac{(y-t)^{n-1}}{(n-1)!} dt; \quad (23)$$

here y can be $\geq x$ or $\leq x$.

By [1] we get that

$$|R_n(x, y)| \leq \frac{\|f^{(n+1)}\|_{\infty}}{(n+1)!} |y-x|^{n+1}, \quad \text{for all } x, y \in [a, b]. \quad (24)$$

Here we assume $f^{(k)}(x) = 0$, for all $k = 1, \dots, n$.

Therefore it holds

$$|f(t) - f(x)| \leq \frac{\|f^{(n+1)}\|_{\infty}}{(n+1)!} |t-x|^{n+1}, \quad \text{for all } t, x \in [a, b]. \quad (25)$$

Here we have again

$$\begin{aligned} E &\leq (S) \int_{[a,b]} |f(t) - f(x)| d\mu(t) \stackrel{(by \text{ Theorem 2.3 (3) and (25)})}{\leq} \\ &(S) \int_{[a,b]} \frac{\|f^{(n+1)}\|_{\infty}}{(n+1)!} |t-x|^{n+1} d\mu(t) \stackrel{(by (5))}{\leq} \\ &\left(\frac{\|f^{(n+1)}\|_{\infty}}{(n+1)!} + 1 \right) (S) \int_{[a,b]} |t-x|^{n+1} d\mu(t). \end{aligned} \quad (26)$$

We have derived the following high order Ostrowski-Sugeno inequality:

Theorem 3.8. Let $f \in [C([a, b], \mathbb{R}_+) \cap C^{n+1}([a, b])]$, $n \in \mathbb{N}$, $x \in [a, b]$. We assume that $f^{(k)}(x) = 0$, all $k = 1, \dots, n$. Here μ is subadditive with $\mu([a, b]) > 0$. Then

$$\begin{aligned} \left| \frac{1}{\mu([a, b])} (S) \int_{[a,b]} f(t) d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu([a, b])} \right) \right| &\leq \\ \frac{\left(\frac{\|f^{(n+1)}\|_{\infty}}{(n+1)!} + 1 \right)}{\mu([a, b])} (S) \int_{[a,b]} |t-x|^{n+1} d\mu(t), \end{aligned} \quad (27)$$

which generalizes (15).

When $x = \frac{a+b}{2}$ we get the following corollary

Corollary 3.9. Let $f \in [C([a, b], \mathbb{R}_+) \cap C^{n+1}([a, b])]$, $n \in \mathbb{N}$. Assume that $f^{(k)}\left(\frac{a+b}{2}\right) = 0$, $k = 1, \dots, n$. Here μ is subadditive with $\mu([a, b]) > 0$. Then

$$\begin{aligned} \left| \frac{1}{\mu([a, b])} (S) \int_{[a,b]} f(t) d\mu(t) - \left(1 \wedge \frac{f\left(\frac{a+b}{2}\right)}{\mu([a, b])} \right) \right| &\leq \\ \frac{\left(\frac{\|f^{(n+1)}\|_{\infty}}{(n+1)!} + 1 \right)}{\mu([a, b])} (S) \int_{[a,b]} \left| t - \frac{a+b}{2} \right|^{n+1} d\mu(t). \end{aligned} \quad (28)$$

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