INTRODUCCIÓN

En adición a la posibilidad de moldear la respuesta de un sistema, el uso del feedback permite otras propiedades que deben ser satisfechas, como robustez, la capacidad de un sistema para mantener el desempeño en presencia de incertidumbres y perturbaciones. Normalmente, los parámetros de la dinámica en abierto no son bien conocidos, y en muchas aplicaciones prácticas el controlador debe producir un desempeño aceptable no solo para las condiciones de diseño nominales, sino también dentro de un rango aceptable alrededor de esta nominal. Por lo tanto, cualquier metodología de diseño de un sistema de control debe incluir estas propiedades como objetivos en el proceso de diseño. Es entonces un problema fundamental en la teoría del control y, como el uso de controladores robustos, un campo de investigación activo. Habitualmente esta aproximación es la resolución directa del problema como en los métodos H∞ o LMI [4-7]. En el caso general y para sistemas lineales de gran escala, la solución del problema de control óptimo es proporcional a las leyes de control como función de todas las variables del sistema. Esta solución no es tan conveniente en aplicaciones prácticas aunque presenta la característica deseable que la estabiliza. Control estructuralmente constreñido, es decir, formación de una ley de control como función de estados locales, o de todos los salidas de subsistemas, o de solo variables locales, ha sido ampliamente investigado. En particular, en [3] se da una solución a este problema a través de un proceso simple que genera una ley de control con restricciones. En este artículo se traduce el problema de robustez a un marco de control óptimo como en [1] y se resuelve a través de una técnica que considera las restricciones estructurales, como la alimentación de solo estados locales para un control local [3].

APPROACH TO ROBUST DESIGN

Sea el sistema representado por el modelo de espacio de estado siguiente

\begin{equation}
\begin{array}{c}
x(t) = A(x) x + Bu \\
y(t) = Cx
\end{array}
\end{equation}

Donde \( x \in \mathbb{R}^n, y \in \mathbb{R}^m, u \in \mathbb{R}^m \). A, B, y C son matrices de dimensiones apropiadas. \( p \in P \) es un parámetro incierto. En [1] se muestra que para un incertidumbre ajustada, es decir, el parámetro incierto en A está en el rango de B, ha sido ampliamente investigado. En particular, en [3] una solución a este problema se da a través de un proceso simple que genera una ley de control con restricciones. En este artículo se traduce el problema de robustez a un marco de control óptimo como en [1] y se resuelve a través de una técnica que considera las restricciones estructurales, como la alimentación de solo estados locales para un control local [3].

$P_0 \in P$ being a nominal value, there exists a mxn matrix $\phi(p)$ such that

$$A(p) - A(p_0) = B\phi(p)$$  \hspace{1cm} (2)

allows the determination of an asymptotically stable solution for all $p_0 \in P$ with a feedback law $u = -Kx$ for the system. To this end the LQR problem is stated as finding, for the nominal system

$$\dot{x} = A(p_0)x + Bu$$ \hspace{1cm} (3)

a control law $u = -Kx$ that minimizes the cost functional

$$J = \int (x^TFx + x^TQx + u^TRu)dt$$ \hspace{1cm} (4)

where $F$ is the following $\inf$ given by

$$F = \inf \{ F' : (\forall p \in P)F' \geq \phi(p)^T\phi(p) \}$$

If the uncertainty is unmatched, then it is decomposed into a sum of a matched and an unmatched component by projecting it into the range of $B$. The proof that the solution to this problem renders a stable feedback for all $p_0 \in P$ is omitted and can be found in [1].

**CONTROL CONSTRAINTS**

Suppose that the state and control are partitioned as

$$x^T = [x_1^T, ..., x_n^T] \quad \text{and} \quad u^T = [u_1^T, ..., u_m^T]$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$, represent an interconnected dynamic system composed of $n$ subsystems with local state and control variables. The constraint given by the partition above corresponds with a decentralized state feedback, that is

$$u(x)^T = [u_1(x_1)^T, ..., u_m(x_m)^T]$$

meaning that any local control only depends on each local state vector. This is tantamount to saying that the optimal feedback gain ought to be expressed as

$$K = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & k_n \end{bmatrix}$$ \hspace{1cm} (5)

where $k_i \in \mathbb{R}^{m \times n}$. This constraint can be written in general as

$$K = \text{diag}\{k_i\} = 0$$ \hspace{1cm} (6)

where $\text{diag}\{\cdot\}$ is used to denote a diagonal matrix. Equation (6) can be expressed as a matrix function

$$F(K) = 0$$ \hspace{1cm} (7)

representing the constraint law.

**Theorem 1 [3]**

Let $K \in \mathbb{R}^{m \times n}$ be such that

$$K + L = R^{-1}B^T S$$

where $L \in \mathbb{R}^{m \times n}$ is an arbitrary matrix and $S$ is a symmetric, definite positive solution matrix to the Generalized Riccati Equation

$$A^TS + SA - SBR^{-1}B^TS + Q + L^TRL = 0$$ \hspace{1cm} (8)

then $\sigma(A - BK) \in C^-.$

Proof: Since $S > 0$, consider $V(x) = x^TSx$ as a Lyapunov candidate associated to the closed loop system. Then

$$\dot{V}(x) = x^T((A - BK)^TS + S(A - BK))x$$

it follows then :

$$\dot{V}(x) = x^T(-Q - L^TRL - SBR^{-1}B^TS + L^TB^TS + SBL)x$$

$$= -x^T(Q + K^TBK)x$$

$$< 0$$

What has been done is really a parameterisation on $L \in \mathbb{R}^{m \times n}$. This matrix can be defined in order to get a feasible gain $K \in \Omega$. This choice should be done
considering that $K = R^{-1}B^TS - L \in \Omega$. Actually we are trying to minimize the norm of $L$ in view that with $L=0$ we have the optimal feedback gain of the LQR problem. That is, we must solve:

$$\min\{\|L\|_2^2 : (R^{-1}B^TS - L) \in \Omega\}$$

For the constraints under consideration, the solution to the minimization problem yields \[3\]

$$L = F(R^{-1}B^TP) \quad (9)$$

That is, $L$ is the matrix that accounts for the constraint to be imposed over the feedback control law (Equation (6)).

**CONSTRAINED ROBUST CONTROL**

If the robust control for a matched uncertainty and the constrained control are to be satisfied simultaneously, the problem is finding a feedback control law $u = \text{diag}\{k_i\}x$ for system (1) such that the following cost functional is minimized

$$J = \int_0^{\infty} (x^T(F + Q + L^TRL)x + u^TRu)dt$$

where $K = R^{-1}B^TP$, and $P > 0$ is a solution to the following generalized Ricatti equation:

$$A^TS + SA - SBBR^{-1}B^TS + F + L^TRL = 0 \quad (11)$$

For the case of unmatched uncertainty the problem is to find control laws $u = \text{diag}\{k_i\}x$ and $v = \text{diag}\{z_i\}x$ for the auxiliary system:

$$\dot{x} = Ax(p_0)x + Bu + r^{-1}(I-BB^T)v$$

such that the cost functional

$$J = \int_0^{\infty} (x^T(G + \rho^2H + Q + L^TRL)x + u^TRu + \rho^2v^Tv)dt$$

is minimum, where $\rho$ is some positive constant. The matrices $G \geq 0$ and $H \geq 0$ are given as:

$$G = \inf\left\{G : (\forall p \in P), \begin{array}{l} G \geq (A(p) - A(p_0))^TB^TB'(A(p) - A(p_0)) \\ G \geq r^2(A(p) - A(p_0))^T(A(p) - A(p_0)) \end{array}\right\}$$

$$H = \inf\left\{H : (\forall p \in P), \begin{array}{l} H \geq r^2(A(p) - A(p_0))^T(A(p) - A(p_0)) \end{array}\right\}$$

and the solution is obtained through the generalized Ricatti equation:

$$A^TS + SA - SBBR^{-1}B^TS + Q' = 0 \quad (14)$$

Where

$$Q' = Q + G + \rho^2H + L^TRL$$

It can be shown that for $V(x) < 0$ we must have $Q + L^TRL - 2\rho^2\text{diag}\{z_i\}^T\text{diag}\{z_i\} > 0$ that is $\dot{x} = A(p)x + B\text{diag}\{k_i\}x$ is asymptotically stable for all admissible uncertainties, and $u=\text{diag}\{k_i\}x$ is a solution to the robust decentralised problem. (See the appendix).

A general procedure to handle this kind of problems is the following:

1. Set $L=0$ in equation (11) or equation (14), and solve for the robust control problem without constraints.
2. Determine $L = F(K)$ for the chosen constraint.
3. Solve the Ricatti equation (equation (11), or equation (14)).
4. Test for convergence. For example, two consecutive values for the cost functional being less than a certain small positive value.
5. If 4 is not satisfied, go to step 2 with the calculated value for $P$ from step 3.
6. Determine the feasible gain $K \in \Omega$ through.

$$K = R^{-1}B^TS - L$$

**Example 1. Matched Uncertainty**

As an example of robust decentralised control for a matched uncertainty the linear model of a spring coupled pendula is presented in figure 1 [3]. The uncertainties are assumed to be the position of the spring along the rod, $h$, and its stiffness, $k$. It is known that the mathematical model of the pendula is given by:
$ML^2 \frac{d^2 \theta_1}{dt^2} = -Mg\sin \theta_1 + \frac{h}{L} (x_2 - x_1).$

$ML^2 \frac{d^2 \theta_2}{dt^2} = -Mg\sin \theta_2 + \frac{h}{L} (x_2 - x_1).$ (16)

Figure 1. Spring Coupled Inverted Pendula.

Where $g$ represents gravity acceleration and the meaning of the rest of the variables are evident from the figure. The uncertain values for $h$ and $k$ belong to a given interval as shown:

$h \in [2, 10]$

$k \in [0.5, 2.5]$

Furthermore $L=10$ mg, $g=9.81$ m/seg, $m_1=0.5$ kg, $m_2=0.6$ kg.

For a linear model of the pendula valid for the equilibrium point $(\pi, 0)$, figure 2 shows the migration of poles when the uncertain parameters vary within the given interval. The LQR closed loop poles migrate towards the imaginary axes as a function of the uncertain parameters. For a robust LQR solution it is evident that the real part of the poles are unaltered and the frequency excursions are minimal.

When the robust decentralised LQR strategy is applied, a real slightly dominant pole appears with no variation of its position for all the values of the parameters. The feedback gains and values for the cost functional in each case are:

**Robust LQR feedback gain:**

$Kra = \begin{bmatrix} 0.8345 & 0.9814 & 0.9762 & 0.8402 \\ 0.9307 & 0.8402 & 1.1828 & 1.2883 \end{bmatrix}$

**Robust LQR cost functional value:**

$Jra = 0.2273$

**Robust Decentralized Feedback gain:**

$Krda = \begin{bmatrix} 1.2295 & 1.6969 & 0 & 0 \\ 0 & 0 & 1.5645 & 1.8840 \end{bmatrix}$

**Robust Decentralized cost functional value:**

$Jrda = 0.3463$

The functional cost value is a 34.4% bigger than the robust LQR value so the decentralized solution is then a suboptimal one.

Figure 3 shows the results obtained from simulation for a non robust LQR control strategy for three different pair of parameters values:

$(k, h) \in [(a)(0.5, 2.0), (b)(1.5, 6.0), (c)(2.5, 10.0)]$

State variables $x_1$ and $x_3$ represent angles $\theta_1$ and $\theta_2$ respectively. Simulation starts at $t=0$, with initial condition $\theta_1=1$ rad. It can be seen that for the value $h=L$ and maximum spring coefficient (figure 3.c) both pendulums act in phase opposition. Clearly the robustness is related to the speed of response.

The Robust LQR and the Decentralised Robust solutions (figure 4 and figure 5) are much faster and less oscillatory. It is clear from figure 5 that the decentralised robust solution corresponds to an almost dominant real pole. For the solution of the Riccati equation the weighing matrix $Q$ was determined as comparable to the $F$ matrix for the matched uncertainty case and matrix $R$ was set as diagonal with values equal to 0.1.
Example 2. Unmatched Uncertainty

As an example of robust decentralised control for an unmatched uncertainty, the linear model of the same inverted coupled pendula but with an altered matrix B was used. The results obtained are as follows (see figure 6).

Clearly the solution to the decentralised robust case shows how the real part of the dominant poles stays almost constant but close to the imaginary axes if compared to the robust LQR solution. The simulation results for the LQR, Robust LQR and Decentralised Robust LQR are shown in figure 7, figure 8 and figure 9, respectively.

From figure 7 and figure 8 it is seen that the LQR solution is faster and with less frequency excursion than the robust case. Nonetheless they are almost comparable. For the decentralised robust solution we have a big difference in response speed and frequency excursion, even though they are almost constant for varying values of the pair (k,h).
Figure 8. Robust LQR for three different values of the pair (k,h).

Figure 9. Robust Decentralized solution for three different values of the pair (k,h).

The values of the L matrix, H and G matrices and the design parameters rho, beta and r for the unmatched uncertainty case are as follows:

\[
L = \begin{bmatrix}
0 & 0 & -0.1692 & 35.8924 \\
-0.1692 & 40.0509 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-0.0338 & 0 & 0.5006 & 0.0460 \\
0.4487 & 0 & 0 & 0.0534
\end{bmatrix}
\]

\[
G_r = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
H_r = \begin{bmatrix}
1.0844 & 0 & -1.0844 & 0 \\
0 & 0 & 0 & 0 \\
-1.0844 & 0 & 1.0844 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[\text{rho}=2, \text{ beta}=10 \text{ and } r=20.\]

The functional cost values are:

- Robust LQR: \(J_{rna} = 120.5972\)
- Decentralized Robust LQR: \(J_{rdna} = 539.3480\)

Robust Feedback Gain for controls \(u\) and \(v\) is:

\[
K_{rna} = \begin{bmatrix}
11.0211 & 0.1324 & -0.1228 & 9.5697 \\
-0.1228 & 9.6874 & 11.2020 & 1.8669 \\
0.0017 & 1.2650 & 0.1211 & -0.0087 \\
0.1196 & -0.0087 & 0.0233 & 1.4722
\end{bmatrix}
\]

Decentralised Robust Feedback Gain for controls \(u\) and \(v\) is

\[
K_{rdna} = \begin{bmatrix}
13.0989 & -2.7033 & 0 & 0 \\
0 & 0 & 13.9197 & 4.2731 \\
0 & 0 & 0 & 0 \\
0 & 6.5336 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6.6123
\end{bmatrix}
\]

**CONCLUSIONS**

A decentralized robust optimal control approach accounting for matched and unmatched uncertainties has been presented. The simulation results obtained from application of the methodology show that the approach gives good results. To the authors knowledge the optimal control approach to robust control design considering constraints as local states feedback for local control has not been reported yet. Work is under way to include output feedback or local outputs feedback for local control and robust compensators synthesis using this approach.

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**REFERENCES**

Theorem 2 [1]

Choosing \( r \) and \( \rho \) such that 
\[
Q + L^T RL - 2\rho^2 Z^T Z > 0
\]
(positive definite), then the solution
\[
u = \text{diag}\{k_i\} x
\]
is a solution to the robust decentralised problem.

Proof: \([A(p_0), B]\) is stabilizable, and \(G \geq 0, H \geq 0\). The solution to the LQR problem exists. Denote the solution as \(u = \text{diag}\{k_i\}\) and \(v = \text{diag}\{z_i\}\). We want to prove that \(u = \text{diag}\{k_i\}\) is a solution to the robust decentralised control problem. That is,
\[
x = A(p)x + B\text{diag}\{k_i\} x
\]
is asymptotically stable for all \(p \in \mathbb{P}\).

Define
\[
V(x_0) = \min_{w \in \mathbb{R}} \int_0^\infty (x^T(G + \rho^2 H + Q)x + u^T Ru + \rho^2 v^T v) dt
\]
to be the minimum cost of the optimal control from some initial state \(x_0\). The Hamilton-Jacobi-Bellman equation gives
\[
\min_{w \in \mathbb{R}} (x^T(G + \rho^2 H + Q)x + u^T Ru + \rho^2 v^T v + V_{x}^T x) = 0
\]
Since \(u = \text{diag}\{k_i\}\) and \(v = \text{diag}\{z_i\}\) is the optimal control
\[
x^T(G + \rho^2 H + Q)x + u^T Ru + \rho^2 v^T v + V_{x}^T x = 0 \quad (a)
\]
\[
2x^T \text{diag}\{k_i\}^T + V_{x}^T B = 0 \quad (b)
\]
\[
2\rho^2 x^T Z^T + V_{x}^T r^{-1}(I - B^*) = 0 \quad (c)
\]
where \(B^*\) is the pseudo inverse of \(B\).

Clearly \(V(x) > 0\) for \(x \neq 0\) and \(V(x) = 0\) for \(x = 0\).

Since \(\dot{V}(x) < 0\) for \(x \neq 0\), we have
\[
\dot{V}(x) = V_{x}^T \dot{x} = V_{x}^T (A(p)x + B\text{diag}\{k_i\} x)
\]
\[
= V_{x}^T (A(p)x + A(p_0)x - A(p_0)x + B\text{diag}\{k_i\} x)
\]
\[
+ r^{-1}(I - B^*)\text{diag}\{z_i\} x - r^{-1}(I - B^*)\text{diag}\{z_i\} x
\]
\[
= V_{x}^T (A(p)x + B\text{diag}\{k_i\} x + r^{-1}(I - B^*)\text{diag}\{z_i\} x)
\]
\[
+ V_{x}^T (A(p) - A(p_0))x - r^{-1}(I - B^*)\text{diag}\{z_i\} x
\]
From equation (15a)
\[
V_{x}^T (A(p_0)x + B\text{diag}\{k_i\} x + r^{-1}(I - B^*)\text{diag}\{z_i\} x) =
\]
\[
- \{x^T(G + \rho^2 H + Q + L^T RL)x + u^T Ru + \rho^2 v^T v
\]
but
\[
[A(p) - A(p_0)]x
\]
\[
= B B^* [A(p) - A(p_0)]x + (I - B^*) [A(p) - A(p_0)] x
\]
then
\[
V(x) = -\left[ x^T G x + x^T \rho^2 H x + x^T (Q + L^T R L) x + u^T R u + \rho^2 J^T y \right] \\
+ V_x^T (BB^\top \left[ A(p) - A(p_0) \right] x + (I - BB^\top) [A(p) - A(p_0)] x \\
- r^{-1}(I - BB^\top) \text{diag}[z_i(x)]
\]
and because from equation (15b)
\[
V_x^T = -2x^T \text{diag}[k_i]^T R B^\top
\]
and from equation (15c)
\[
2\rho^2 x^T Z^T = -V_x^T r^{-1}(I - BB^\top)
\]
Knowing that \(B^\top B = B^\top\), it follows that
\[
\dot{V}(x) = -x^T G x - x^T \rho^2 H x - x^T (Q + L^T R L) x - x^T \text{diag}[k_i]^T R \text{diag}[k_i] \\
- \rho^2 x^T \text{diag}[z_i]^T \text{diag}[z_i] x - 2x^T \text{diag}[k_i]^T R B^\top (A(p) - A(p_0)) x \\
+ 2\rho^2 x^T \text{diag}[z_i]^T \text{diag}[z_i] x
\]
Clearly
\[
2x^T \text{diag}[k_i]^T R B^\top (A(p) - A(p_0)) x = \\
2\rho^2 x^T Z^T R x + 2x^T \text{diag}[k_i]^T R B^\top (A(p) - A(p_0)) x
\]
so
\[
\dot{V}(x) = -x^T G x - x^T \rho^2 H x - x^T (Q + L^T R L) x \\
- x^T \text{diag}[k_i]^T R \text{diag}[k_i] - \rho^2 x^T \text{diag}[z_i]^T \text{diag}[z_i] x \\
- 2\rho^2 x^T Z^T R x - 2x^T \text{diag}[k_i]^T R B^\top (A(p) - A(p_0)) x \\
+ 2\rho^2 x^T \text{diag}[z_i]^T \text{diag}[z_i] x
\]
but
\[
\dot{x}^T \text{diag}[k_i]^T R \text{diag}[k_i] - 2x^T \text{diag}[k_i]^T R B^\top (A(p) - A(p_0)) x = \\
- x^T \text{diag}[k_i]^T R \text{diag}[k_i] - x^T (\text{diag}[k_i]^T \text{diag}[z_i] (A(p) - A(p_0))) \\
+ (\text{diag}[k_i]^T \text{diag}[z_i] (A(p) - A(p_0))) x + x^T (B^\top (A(p) - A(p_0)))^T \\
B^\top (A(p) - A(p_0)) x
\]
\[
= -x^T \left[ \text{diag}[k_i]^T R - B^\top (A(p) - A(p_0)) \right]^T \\
\left[ \text{diag}[k_i]^T R - B^\top (A(p) - A(p_0)) \right]\]  \\
+ x^T (B^\top (A(p) - A(p_0)))^T (B^\top (A(p) - A(p_0))) x
\]
\[
\leq x^T (B^\top (A(p) - A(p_0)))^T (B^\top (A(p) - A(p_0))) x \\
\text{and}
\]
\[
-\rho^2 x^T \text{diag}[z_i]^T \text{diag}[z_i] x - 2\rho^2 x^T r Z x = \\
- x^T \rho^2 \text{diag}[z_i]^T \text{diag}[z_i] - 2 \text{diag}[z_i]^T r \text{diag}[z_i] x = \\
- x^T (\rho^2 \text{diag}[z_i]^T - \rho^2 r \text{diag}[z_i])^T (\rho^2 \text{diag}[z_i]^T - \rho^2 r \text{diag}[z_i]) x \\
\leq \rho^2 x^T r \text{diag}[z_i]^T \text{diag}[z_i] x = \rho^2 x^T r \text{diag}[z_i]^T \text{diag}[z_i] x
\]
\[
\leq \rho^2 x^T H x
\]
Clearly then if \(\dot{V}(x) < 0\) we must have
\[
Q + L^T R L - 2\rho^2 \text{diag}[z_i]^T \text{diag}[z_i] > 0
\]
that is \(x = A(p)x + B \text{diag}[k_i] x\) is asymptotically stable for all admissible uncertainties, and
\[
u = \text{diag}[k_i] x
\]
is a solution to the robust decentralised problem.