The beauty of the unexpected

Is Nature predictable? Can we know with certainty how physical and engineering systems evolve? Can unexpected complexity come from simplicity? These are some of the questions which arise when we discover that chaos is ubiquitous in our day-to-day life.

The laws and phenomena in Nature are described by means of differential or difference equations, which are approximations of the real world that allow us to describe the behaviour of the past and future states of a system by using the current state. In chaotic systems, the solution of a system of differential or difference equations may exhibit a random pattern. That is, the measurement of the current system state leads to unpredictability of the future system states, in addition to uncertainty with respect to the past states. Consequently, chaos reveals the unexpected evolution of a system. Generally speaking, a chaotic system can be defined as a deterministic system, described by nonlinear differential or difference equations which exhibit an apparently random dynamic behaviour.

Models describing chaotic dynamics do not necessarily need to be complex. Chaotic behaviour may appear in deterministic nonlinear equations which are extraordinarily simple. For this reason, it is easy to find chaotic dynamics in physical systems around us, or in systems we have built. This encourages us to think that chaos is the norm, and not the exception, in the behaviour of nonlinear systems.

We are within a solar system whose planetary motion may be unpredictable. We build electronic circuits that can exhibit, for some operation conditions, chaotic behaviour; for instance, Chua’s circuit. We can observe the behaviour of a pendulum which evolves as we expected until an event occurs, and after that, its behaviour becomes unpredictable. Some numerical methods also present chaotic phenomena; one of the simplest examples is Newton’s algorithm for computing the roots of a third-order polynomial. Feedback control systems or control systems —that is, systems designed to ensure that devices behave correctly, eliminating all negative behaviour— can also exhibit this irregular behaviour: for instance, an integral-type controller with a saturation element or an integral-type controller with cubic-type nonlinearity. Most of the phenomena and systems in our day-to-day life exhibit some kind of ‘hidden’ chaotic dynamics, such as acoustic systems, lasers, phenomena in fluids, mechanical vibrations, chemical reactions, biological membranes, our heart, population dynamics, ionic currents in the cornea of our eyes, or economy models.

The unexpected complexity of simple phenomena

The characteristics of chaotic behaviour in the different processes mentioned above show an integral characteristic of chaos: despite its complexity, it is based on specific and well identified facts which can be easily detected.

There are two main kinds of characteristics of chaotic dynamics: those concerning the dynamical evolution, and those concerning the geometry or shape of the state space. The state space is the set of all possible system states or system configurations.

The behaviour of a dynamical system is determined by what is called the orbits of the system, which are a representation of the system evolution, and they are described by the solutions of an equation modelling the system behaviour. There are three main signs of deterministic chaotic dynamics:

1. **Sensitivity to initial conditions.** This makes the orbits of close points behave independently, coming closer and moving further away in an unpredictable way. This also means that two orbits beginning at the same system state can evolve in very different ways, even when their initial values are very close.
2. **Onset of periodic orbits with different periods.** There is a mathematical tool that can show clearly this second sign of chaos, and illustrates the transition from determinism to chaos in a system. This tool is **Feigenbaum’s diagram**, which represents graphically how a system becomes chaotic. In particular, Feigenbaum’s diagram shows how a system variable changes as long as the parameter which originates the chaos in the system changes. The onset of chaos is typically associated with systems whose mathematical model has a parameter that changes. For a range of values of this parameter, the system will present chaotic behaviour. The point of change is called the bifurcation point. Feigenbaum’s diagram shows the period doubling route to chaos and the existence of multiple periods. It gives the value of the system parameter that marks the beginning of chaos. Moreover, Feigenbaum’s diagram has a fractal structure. In other words, within it, there are parts which are a reduced-size copy of the whole. The existence of periodic orbits of multiple periods makes chaotic systems exhibit a special frequency spectrum, similar to random signals (noise).

3. **Mixing behaviour.** The orbit starting at any point visits most of the points in a certain region of the phase space without ‘filling up’ the region. This makes chaotic systems generate ‘attractors’ with a fractal structure, as will be mentioned below.

To put it in a nutshell, chaos is the consequence of two processes: geometrical stretching and folding of the phase space. If we considered the phase space as the dough that a baker kneads, the recipe for chaos would be, on the one hand, to stretch the dough, and on the other hand, to fold it. The stretching action makes initially close points move further away in the future. The iterative repetition of this causes the sensitivity to initial conditions. The folding of the phase space results in the mixing behaviour of orbits, in such a way that by repeating this action successively, the information of the initial state is lost, and as mentioned previously, uncertainty in the dynamical behaviour arises.

Finally, another integral characteristic of chaotic systems is the existence of unusual and complex geometrical structures in the system phase space diagram. They are the **strange attractors**. That is, the asymptotic structures towards which the orbits of a chaotic system converge.

Simple nonlinear systems typically used as paradigms of chaos, since they capture all the essential characteristics of chaotic systems, are:

1. May’s logistic map which describes the growth of an insect population in a closed ecosystem. This is a one-dimensional discrete system with extraordinary value for the study of chaos.
2. Lorenz’s system, which gives rise to the famous Lorenz’s attractor, a mathematical model for thermal convection that has extraordinary value for climate models.
3. Rössler’s system, which is one of the simplest autonomous continuous-time systems exhibiting a chaotic attractor (called Rössler’s attractor).

**Taking advantage of chaos in engineering systems**

There is a wide variety of systems in different engineering areas which exploit the properties of chaotic behaviour. Some control systems in mechanical engineering, mechatronics, bioelectronics and mobile communications are examples.

Two kinds of processes can be highlighted in the control engineering applications: the control of chaos and what is called the anti-control of chaos.

The control of chaos is based on eliminating chaos. This can be done, mainly: 1) by forcing the chaotic system to converge to equilibrium at its stationary state, or 2) by transforming a chaotic or strange attractor into a periodic oscillation. This is carried out, for example, in electromechanical systems subject to mechanical vibrations. Another interesting application which brings together control engineering techniques and the analysis of dynamical systems is the biocentric control or control of biological systems, and as in the case of heart arrhythmia control, this takes advantage of
the chaos theory as well. It is well-known that heart arrhythmias are chaotic. Consequently, the way to have a healthy heart pattern is to eliminate any chaotic behaviour.

Conversely, in some other cases, we want to generate chaotic dynamics or to transform one type of chaotic behaviour into another type of chaos: this is called the anti-control of chaos. Concerning this topic, there are experimental results on the study of diseases and disorders of the neural system. The geometry of the neural signals of a healthy individual has the shape of a chaotic attractor. For this reason, if we associated a behavioural pattern with a chaotic attractor, we could ‘steer the neural system to a desirable behaviour’. These studies are carried out experimentally in the analysis of mental disorders, or in diseases like epilepsy.

In communication applications, there are several advantages that the use of the properties of chaos can facilitate. One example is the security in voice-data transmissions, achieved by the use of chaotic signals. Another example is the increase of the number of users of the same channel by means of the use of a chaotic signal in the coding. This is a key element of mobile communications, and has been studied in the CDMA channel access method (Code Division Multiple Access). In the CDMA, the generation of codes for users is made by means of the codification of a chaotic variable. With this, the amount of codes is much greater than that obtained with other traditional code generators.

A world to discover

To sum up, chaos is universal and present is a wide variety of systems and applications.

The beauty of the chaos theory lies in its vicinity to the boundaries of the unknown, which is the same as saying, in the processes of our day-to-day world.

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