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ON THE HYPERBOLIC DIRICHLET TO NEUMANN FUNCTIONAL IN ABELIAN LIE GROUPS *

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Abstract

We prove the injectivity of the linearization of the hyperbolic Dirichlet to Neumann functional in a “small” compact neighborhood of the identity element e of an abelian Lie group G , under some suitable transversality condition.

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1. Introduction and Statement of the Result

Let $\overline{\Omega}$ be a compact manifold of dimension n with smooth boundary $\partial\Omega$ and let $\text{Met}(\overline{\Omega})$ denote the set of all Riemannian metrics g on $\overline{\Omega}$.

We consider the anisotropic wave equation

$$(1.1) \quad \begin{aligned} \square_g u &= \frac{\partial^2 u}{\partial t^2} - \Delta_g u = 0 \text{ in } \Omega \times (0, T), \\ u &= f \text{ on } \Gamma = \partial\Omega \times (0, T), \quad f \in C_0^\infty(\Gamma), \\ u &= \frac{\partial u}{\partial t} = 0 \text{ in } \Omega \times \{0\}. \end{aligned}$$

There is a unique solution to (1.1); hence we may define the hyperbolic Dirichlet to Neumann map as the linear operator $\Lambda_g : C_0^\infty(\Gamma) \rightarrow C^\infty(\Gamma)$, given by

$$(1.2) \quad \Lambda_g f = du \cdot \nu_g \Big|_\Gamma = \frac{\partial u}{\partial \nu_g} \Big|_\Gamma$$

where u is the unique solution to (1.1) and ν_g is the g -interior unit normal to $\partial\Omega$. The hyperbolic Dirichlet to Neumann Functional:

$$(1.3) \quad \begin{aligned} \Lambda : \text{Met}(\overline{\Omega}) &\longrightarrow \text{O}_p(\Gamma) \\ g &\mapsto \Lambda_g, \end{aligned}$$

where $\text{O}_p(\Gamma)$ denotes the space of all linear operators from $C_0^\infty(\Gamma)$ into $C^\infty(\Gamma)$, is known to be invariantly defined on the orbit obtained by the action over $\text{Met}(\overline{\Omega})$ of the group $\mathcal{D}$ of all diffeomorphism ψ of $\overline{\Omega}$, each of which restricts to the identity on $\partial\Omega$. A natural conjecture is that this is the only obstruction to the uniqueness of Λ . From the point of view of applications, an even more important problem is to give a method to reconstruct g from Λ_g .

The elliptic Dirichlet to Neumann map was treated by several authors and is closely related to the physical problem referred as Electrical Impedance Tomography, of determining the conductivity of a body from measurement of voltage potential and corresponding current fluxes at the boundary (see [4]).

For fixed g , we consider the following map:

$$(1.4) \quad \psi \in \mathcal{D} \xrightarrow{A_g} \psi_g^* \in \text{Met}(\overline{\Omega})$$

It is easy to see that the tangent space $T_I \mathcal{D}$ of $\mathcal{D}$ at the identity mapping I is the vector space $\Gamma_0(T\bar{\Omega})$ of all smooth vector fields on $\bar{\Omega}$ which vanish on $\partial\Omega$. On the other hand, the tangent space $T_g(\text{Met}(\bar{\Omega}))$ of $\text{Met}(\bar{\Omega})$ at g is the vector space $\Gamma(S^2\bar{\Omega})$ of all smooth sections of symmetric 2-tensors on $\bar{\Omega}$. We introduce on the spaces $\Gamma_0(T\bar{\Omega})$ and $\Gamma(S^2\bar{\Omega})$ the inner products

$$(1.5) \quad \langle X, Y \rangle = \int_{\bar{\Omega}} g(X, Y) v_g, \quad X, Y \in \Gamma_0(T\bar{\Omega}),$$

$$(1.6) \quad \langle \langle m, l \rangle \rangle = \frac{1}{n} \int_{\bar{\Omega}} \text{tr}(\hat{m} \circ \hat{l}) v_g, \quad m, l \in \Gamma(S^2\bar{\Omega}),$$

where v_g (resp., tr), denote the volume element (resp., the trace) associated to g and $\hat{m}$ is the unique linear map defined by

$$(1.7) \quad g(\hat{m}u, v) = m(u, v) \text{ for all } u, v \in \Gamma(T\bar{\Omega}).$$

Considerer as in [3], the formal linearizations of A_g at I and of Λ at g , respectively :

$$(1.8) \quad A'_g[I] := A'_g : \Gamma_0(T\bar{\Omega}) \longrightarrow \Gamma(S^2\bar{\Omega}),$$

and

$$(1.9) \quad \Lambda'_g : \Gamma(S^2\bar{\Omega}) \longrightarrow \text{O}_p(\Gamma).$$

Let $(A'_g)^*$ denote the formal adjoint of A'_g with respect to the inner products (1.5) and (1.6) and $\text{diam}_g(\Omega)$ the diameter of Ω in the metric g . In [3], the authors stated the following :

Conjecture 1.1 *Let $\Omega_0 \subset \Omega$ be a submanifold, $m \in \Gamma(S^2\bar{\Omega})$ have support in $\bar{\Omega}_0$ i.e., $m \in \Gamma_0(S^2\bar{\Omega})$ and assume that a) $\Lambda'_g(m) = 0$, b) $(A'_g)^*(m) = 0$ and c) $\text{diam}_g(\Omega_0) < T$ is sufficiently small that the exponential map for g is a global diffeomorphism. Then m is identically zero.*

As in [3] we refer to condition b) as the Transversality Condition. We remind that condition c) is necessary to avoid the appearance of caustics. The Tranversality Condition replace the harmonic hypothese used in [4].

The main results of this paper is:

Theorem 1.1 *Conjecture 1.1 holds if G be an abelian Lie group and $\overline{\Omega}$ is a compact neighborhood of G at the identity element e , and g is an invariant metric on G and $m \in \Gamma_0(S^2\overline{\Omega})$.*

We are able to obtain an generalization Cardozo-Mendoza Theorem 1.1 [3] as corollary of Theorem 1.1.

Corollary 1.1 *Conjecture 1.1 holds if $\overline{\Omega}$ is a bounded domain of $\mathbb{R}^n$, and g is a metric on $\mathbb{R}^n$ and $m \in \Gamma_0(S^2\overline{\Omega})$.*

Corollary 1.2 *Conjecture 1.1 holds if $\overline{\Omega}$ is a bounded domain of the Torus T^n , and g is an invariant metric on T^n and $m \in \Gamma_0(S^2\overline{\Omega})$.*

Corollary 1.3 *Conjecture 1.1 holds if $\overline{\Omega}$ is a bounded domain of the product $\mathbb{R}^n \times T^m$ and g is an invariant metric on $\mathbb{R}^n \times T^m$ and $m \in \Gamma_0(S^2\overline{\Omega})$.*

Remark 1.1 *In [3], the authors proved that Conjecture 1.1 holds if $\overline{\Omega}$ is a bounded domain of $\mathbb{R}^n$, $n \geq 2$ and g is the Euclidean metric. They also proved the conjecture when $n=2$ and g is near the Euclidean metric in the C^3 -topology. In [1] they prove that Conjecture 1.1 holds if $\overline{\Omega}$ is a bounded domain of the hyperbolic space (resp., n -sphere) and g is the canonical metric in this spaces. In [2], the authors proved the uniqueness conjecture for the case when the manifold is a sufficiently small bounded domain of $\mathbb{R}^3$, under suitable geometric conditions and the metric g is C^3 -close to Euclidean metric. We shall make use of the invariant formulas for $A'_g, (A'_g)^*$ and Λ'_g proved in Section 2 [3].*

2. Proof of Theorem 1.1

Let G be an abelian Lie group, and $\text{Lie}(G)$ the algebra formed by the set of all left invariant vector fields on G , and $\overline{\Omega}$ is a compact neighborhood of G at the identity element e , and $\Gamma_0(S^2\overline{\Omega})$ denote the vector space of all smooth sections of symmetric 2-tensors on G which are supported on $\overline{\Omega}$.

Proof of Theorem 1.1 Let $x \in \Omega$, $v \in \text{Lie}(G)$. The geodesic with initial tangent vector $(x, v) \in TG \cong G \times \text{Lie}(G)$ is given by $\gamma(t) = x \exp(tv)$. Since $x \in \Omega$ there exists an element $A \in \text{Lie}(G)$ such that $x = \exp(A)$.

Let $\{E_\mu\}_{\mu=1,\dots,n}$ a orthonormal frame field in $\Gamma(TG)$ then we can define a orthonormal system $\{\tilde{E}_\mu, V_\mu\}_{\mu=1,\dots,n}$ in $\Gamma(TTG)$ as follows. We consider the diagram

$$\begin{array}{ccc} TG & \xrightarrow{\tilde{E}_\mu, V_\mu} & TTG \\ \pi \downarrow & & \downarrow \pi \\ G & \xrightarrow{E_\mu} & TG \end{array}$$

where $\tilde{E}_\mu(x, v) = (x, v, E_\mu, 0)$ and $V_\mu(x, v) = (x, v, 0, E_\mu)$. Let $x = (x_1, \dots, x_n)$ the canonical coordinates system (of the first kind) of G around the identity e associated to the orthonormal frame field $\{E_\mu\}_{\mu=1,\dots,n}$ and $v = (v_1, \dots, v_n)$ the canonical coordinates of $\text{Lie}(G)$. Now we consider the function $F_t : TG \longrightarrow \mathbb{R}$ defined by

$$F_t(x, v) = m(x \exp(tv))(v, v)$$

and the operator

$$\mathcal{L} = \sum_{\mu=1}^n \tilde{E}_\mu V_\mu \in \text{End}(C^\infty(TG)).$$

We obtain

$$(2.1) \quad \mathcal{L}F_t = (-t)\Delta_g F_t + 2 \sum_{\mu, \beta=1}^n v_\mu E_\beta m_{\mu\beta}(\gamma),$$

where Δ_g denotes the Laplace-Beltrami operator associated to the metric g and $m_{\mu\beta} = m(E_\mu, E_\beta)$.

On the other hand, the Transversality Condition satisfied by m means that

$$(2.2) \quad E_1(m_{\mu 1}) + \dots + E_n(m_{\mu n}) = 0, \quad \mu = 1, \dots, n$$

Using (2.1) and (2.2) we conclude that

$$(2.3) \quad \mathcal{L}F_t = (-t)\Delta_g F_t.$$

We define

$$(2.4) \quad H^k(x, v) = \int_{-\infty}^{\infty} (-t)^k F_t(x, v) dt,$$

for all $k = 0, 1, \dots$

Using (2.3), it is easy to show that

$$(2.5) \quad \mathcal{L}H^k = \Delta_g H^{k+1}, \quad k = 0, 1, 2, \dots$$

If we fix v , since m is supported in $\overline{\Omega}$, it follows that $H^k(\cdot, v)$ for all $k = 1, 2, \dots$ vanish on a non-empty open subset of G . From Proposition 3.1 in [3], $H^0 = 0$ and hence H^1 , and in fact all H^k are harmonic functions. Therefore H^k is identically zero for all $k = 0, 1, 2, \dots$.

Observe that $t \rightarrow F_t$ is supported in some closed interval $[a, b]$ containing zero. Since the subalgebra generated by the family of functions $\{(-t)^k\}_{k=0,1,\dots}$ is dense in $C([a, b], \mathbb{R})$, we obtain that $t \rightarrow F_t$ is identically zero. If we take $t = 0$ and recall that m is symmetric, we conclude that m is identically zero. ■

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