A NONRESONANCE BETWEEN NON-CONSECUTIVE EIGENVALUES OF SEMILINEAR ELLIPTIC EQUATIONS: VARIATIONAL METHODS

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Abstract

We study the solvability of the problem

\[-\Delta u = f(x, u) + h \text{ in } \Omega ; \quad u = 0 \text{ on } \partial \Omega\]

when the nonlinearity \( f \) is assumed to lie asymptotically between two non-consecutive eigenvalues of \( -\Delta \). We show that this problem is nonresonant.

Key words: Eigenvalue, resonance, nonresonance, variational method.
1. Introduction

In this paper, we will examine the existence of a solution of the problem:

\[
\begin{aligned}
-\Delta u &= f(x,u) + h \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

\(\Delta u = \text{div}(\nabla u), f : \Omega \times \rightarrow \) is assumed to be a Carathéodory function such that

\[(f_0) \quad \text{and} \quad m_R(x) = \sup_{|s| \leq R} |f(x,s)| \in L^2(\Omega) \quad \text{for each } R > 0\]

\(h \in L^2(\Omega).\)

We are interested in the conditions to be imposed on \(f\) and on the primitive \(F, F(x,s) = \int_0^s f(x,t) \, dt\) in order to have the nonresonance i.e. the solvability of (1.1) for every \(h\) in \(L^2(\Omega)\).

First we introduce some notations, the inequality

\(\alpha(x) \leq \beta(x)\) means that \(\alpha(x) \leq \beta(x)\) for a.e. \(x \in \Omega\) with a strict inequality \(\alpha(x) < \beta(x)\) holding on subset of \(\Omega\) of positive measure. \(\lambda_i < \lambda_{i+1} < \lambda_{i+2}\) are the consecutive eigenvalues of the problem

\[-\Delta u = \lambda u \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial \Omega\]

\(E(\lambda_i)\) is the subspace of \(H^1_0(\Omega)\) spanned by the eigenfunctions corresponding to \(\lambda_i\). \(\|\cdot\|\) denotes the norm in \(H^1_0(\Omega)\) induced by the inner product \(<u,v> = \int_\Omega \nabla u \nabla v; u,v \in H^1_0(\Omega)\).

**Theorem 1.1.** Assume \((f_0)\) and

\[(H_1) \quad \lambda_i \leq l(x) = \lim_{|s| \to \infty} \frac{f(x,s)}{s} \leq k(x) = \lim_{|s| \to +\infty} \frac{f(x,s)}{s} \leq \lambda_{i+2} \text{ uniformly for a.e. } x \in \Omega\]

for some \(j\) such that \(2 \leq j \leq i\), \(\lambda_j\) is not simple i.e. \(\dim E(\lambda_j) \geq 2\)

\[(H_2) \quad L(x) = \lim_{|s| \to +\infty} \frac{2F(x,s)}{s^2} \geq \lambda_{i+1} \text{ uniformly for a.e. } x \in \Omega\]

then for any given \(h \in L^2(\Omega)\), there exists a weak solution of (1.1).
Remark: We can replace $(H_3)$, by one of the following conditions

1) \( \lim_{\|v\| \to +\infty, v \in \oplus_{1 \leq p \leq i+1} E(\lambda_p)} F(x, v(x)) - \frac{1}{2} \lambda_{i+1} v^2 = +\infty. \)

2) \( K(x) = \limsup_{|s| \to +\infty} \frac{2F(x, s)}{s^2} \leq \lambda_{i+1}. \)

3) \( \int_{v>0} (L_+(x) - \lambda_{i+1}) v^2 + \int_{v<0} (L_-(x) - \lambda_{i+1}) v^2 > 0; \ v \in E(\lambda_{i+1}) \) and \( v \neq 0 \) and \( L_\pm(x) = \liminf_{s \to \pm \infty} \frac{2F(x, s)}{s^2} \geq \lambda_{i+1}. \)

4) \( \int_{v>0} (\lambda_{i+1} - L_+(x)) v^2 + \int_{v<0} (\lambda_{i+1} - L_-(x)) v^2 > 0; \ v \in E(\lambda_{i+1}) \) and \( v \neq 0 \) and \( K_\pm(x) = \limsup_{s \to \pm \infty} \frac{2F(x, s)}{s^2} \leq \lambda_{i+1} \)

these limits are taken uniformly for a.e. \( x \in \Omega \)

Corollary 1.1. Assume \((f_0), (H_1), (H_2)\) and

\( (H_4) \) \( F(x, s) \leq 0 \) for \( |s| \leq \delta \) (\( \delta > 0 \))

\( (H_5) \) \( F(x, s) \leq As^2 + B \ A, B \in \)

then if \( h = 0 \) the problem (1.1) possesses a nontrivial solution.

Corollary 1.2. Assume \((f_0), (H_1), (H_2)\) and

\( (H_6) \) \( F(x, t + s) \geq F(x, t) + F(x, s) + B(x); B(\cdot) \in L^1(\Omega) \)

\( (H_7) \) \( \lim_{\|v\| \to +\infty, v \in E(\lambda_{i+1})} \int_{\Omega} F(x, v(x)) - \frac{1}{2} \lambda_{i+1} v^2 dx = +\infty \)

then the problem (1.1) possesses a solution for any given \( h \) in \( E(\lambda_{i+1})^\perp. \)

Where \( E(\lambda_{i+1})^\perp = \{ h \in L^2(\Omega) : \int_{\Omega} h \varphi = 0 \ \forall \varphi \in E(\lambda_{i+1}) \} \)

This generalizes many of the existing results for doubly resonant problems. see e.g. [3]; [7]; [4]; . . . Our approach to the problem (1.1) is variational and uses the well-known saddle point theorem of P. Rabinowitz.
2. Preliminary Lemmas

From the conditions \((f_0)\) and \((H_1)\), it follows that there exists constants \(a, A > 0\) and functions \(b(\cdot) \in L^2(\Omega), B(\cdot) \in L^1(\Omega)\) such that

\[
(1) \quad |f(x, s)| \leq a|s| + b(x)
\]

and

\[
(2) \quad |F(x, s)| \leq A|s|^2 + B(x)
\]

hence, the functional

\[
\Phi(u) = \frac{1}{2} \int_\Omega |
abla u|^2 - \int_\Omega F(x, u) - \int_\Omega hu
\]

is well defined, lower semi-continuous and of class \(^1\) on the Sobolev space \(H^1_0(\Omega)\) with derivative \(\Phi'(u)\) given by

\[
\Phi'(u)w = \int_\Omega \nabla u \nabla w - \int_\Omega f(x, u)w - \int_\Omega hw.
\]

for all \(u, w \in H^1_0(\Omega)\), thus the critical points of \(\Phi\) are precisely the weak solutions of (1.1).

Let \((u_n) \subset H^1_0(\Omega)\) be an unbounded sequence, such that

\[
(3) \quad \Phi(u_n) \text{ is bounded and } \Phi'(u_n) \to 0.
\]

Defining \((v_n)\) by \(v_n = \frac{u_n}{\|u_n\|}\), we have \(\|v_n\| = 1\) and, passing to a subsequence (still denoted by \((v_n)\)), we may assume

\[
\begin{align*}
    v_n & \rightharpoonup v \text{ weakly in } H^1_0(\Omega) \\
    v_n & \to v \text{ strongly in } L^2(\Omega) \\
    v_n(x) & \to v(x) \text{ a.e. } x \in \Omega \\
    |v_n| & \leq z(x) \text{ where } z(.) \in L^2(\Omega).
\end{align*}
\]

Assuming \((f_0)\) and \((H_1)\), we obtain that the sequence \(\left(\frac{f(x, u_n)}{\|u_n\|}\right)\) is bounded in \(L^2(\Omega)\), so we may assume that

\[
(4) \quad \frac{f(x, u_n)}{\|u_n\|} \to \tilde{f} \text{ weakly in } L^2(\Omega).
\]
An easy calculation (see [4]) shows that

\[
(5) \quad l(x) \leq \frac{\tilde{f}(x)}{v(x)} \leq k(x) \quad \text{if } v(x) \neq 0
\]

and

\[
(6) \quad \tilde{f}(x) = 0 \quad \text{if } v(x) = 0.
\]

Let us define

\[
m(x) = \begin{cases} 
\frac{\tilde{f}}{v(x)} & \text{if } v(x) \neq 0 \\
\frac{1}{2}(l(x) + k(x)) & \text{if } v(x) = 0.
\end{cases}
\]

Then by (5) and (6), we have

\[
(7) \quad \tilde{f}(x) = m(x)v(x) \quad \text{and} \quad l(x) \leq m(x) \leq k(x)
\]

**Lemma 2.1.** Assume \((f_0)\) and \((H_1)\), then \(v\) is a nontrivial solution of the following problem

\[
(1.2) \quad -\Delta u = m(.)u \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial \Omega
\]

**Proof.** Using (3) we have \(|\Phi'(u_n)w| \leq \varepsilon_n\|w\|\) for all \(w \in H^1_0(\Omega)\), where \(\varepsilon_n \to 0\), therefore

\[
\frac{|\Phi'(u_n)w|}{\|u_n\|^2} = \left|1 - \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} v_n - \frac{1}{\|u_n\|} \int_{\Omega} hv_n\right| \leq \varepsilon_n
\]

hence, by (4) and the fact that \(v_n \to v\) in \(L^2(\Omega)\), we obtain \(\int_{\Omega} \tilde{f}v = 1\), so that \(v \neq 0\).

On the other hand, for any \(w \in H^1_0(\Omega)\) we have

\[
\frac{|\Phi'(u_n)w|}{\|u_n\|} = \left|\int_{\Omega} \nabla v_n \nabla w - \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} w - \frac{1}{\|u_n\|} \int_{\Omega} hw\right| \leq \varepsilon_n \|w\|
\]

passing to the limit, we conclude

\[
\int_{\Omega} \nabla v \nabla w - \int_{\Omega} \tilde{f}w = 0 \quad \forall w \in H^1_0(\Omega)
\]
that is
\[\int_{\Omega} \nabla v \nabla w - \int_{\Omega} m(x) vw = 0 \quad \forall w \in H^1_0(\Omega)\]
in other words \(v\) is a weak solution of (1.2), moreover \(v \neq 0\). So the proof of lemma 2.1 is complete.

**Lemma 2.2.** Assume \((f_0), (H_1)\) and \((H_2)\) then the functional \(\Phi\) satisfies the Palais-Smale condition (PS), that is whenever \((u_n) \subset H^1_0(\Omega)\) is a sequence such that \(\Phi(u_n)\) is bounded and \(\Phi'(u_n) \to 0\) then \((u_n)\) possesses a convergent subsequence.

**Proof :** Let \((u_n) \subset H^1_0(\Omega)\) be such that \(|\Phi(u_n)| \leq c, \Phi'(u_n) \to 0\). Since \(\Phi'(u) = u - T(u)\) where \(T\) is a compact operator from \(H^1_0(\Omega)\) to \(H^{-1}(\Omega)\) \((T(u)w = \int_{\Omega} f(x,u)w - \int_{\Omega} hw)\), in order to show that \((u_n)\) has a convergent subsequence it suffices to show that \((u_n)\) is bounded. Suppose by contradiction that
\[(8) \quad \|u_n\| \to +\infty\]

Let \(v_n = \frac{u_n}{\|u_n\|}\), then, as we observed in (7) and lemma 2.1 there exists a subsequence of \((v_n)\) (still denoted by \((v_n)\)) such that \(v_n \rightharpoonup v\) in \(H^1_0(\Omega)\), \(v_n \to v\) strongly in \(L^2(\Omega)\) and \(v\) is a nontrivial solution of the problem
\[-\Delta u = m(.)u \quad \text{in } \Omega ; \quad u = 0 \quad \text{on } \partial \Omega\]
where
\(l(x) \leq m(x) \leq k(x)\)
so we conclude that
\[(9) \quad 1 \in \sigma(-\Delta, m(.))\]
and
\[(10) \quad \lambda_i \leq m(.) \leq \lambda_{i+2}.\]

On the other hand, by (10) and the strict monotonicity of \(\lambda_i(m)\) we deduce
\[\lambda_i(\lambda_i(1)) > \lambda_i(m(.)) \quad \text{and} \quad \lambda_{i+2}(m(.)) > \lambda_{i+2}(\lambda_{i+2}(1))\]
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hence

\[ 1 > \lambda_i(m(.)) \text{ and } \lambda_{i+2}(m(.)) \geq 1 \]

that is

\[ (11) \quad \lambda_i(m(.)) < 1 < \lambda_{i+2}(m(.)). \]

it follows from (9) and (11) that

\[ (12) \quad 1 = \lambda_{i+1}(m(.)). \]

In view of the variational characterization of \( \lambda_{i+1} \) we have

\[ (13) \quad 1 = \sup_{F_{i+1}} \inf \left\{ \int_{\Omega} m(.)u^2 : \|u\| = 1 , u \in F_{i+1} \right\} \]

where \( F_{i+1} \) varies over all \( i+1 \)-dimensional subspace of \( H^1_0(\Omega) \).

On the other hand, we claim that there exists \( \varepsilon > 0 \) such that

\[ (14) \quad \inf \left\{ \int_{\Omega} m(.)u^2 : \|u\| = 1 , u \in \oplus_{1 \leq p \leq i} E(\lambda_p) \right\} \geq 1 + \varepsilon \]

Indeed, suppose (14) is false, then there exists a sequence \( (u_n) \) in \( \oplus_{1 \leq p \leq i} E(\lambda_p) \), \( \|u_n\| = 1 \) and a sequence \( (\varepsilon_n) \) in \( \mathbb{R}^+ \) such that

\[ (15) \quad \varepsilon_n \rightarrow 0 \quad \text{and} \quad \int_{\Omega} m(.)u_n^2 \leq 1 + \varepsilon_n. \]

Since \( \|u_n\| = 1 \) and \( \dim \oplus_{1 \leq p \leq i} E(\lambda_p) < \infty \), we deduce

\[ (16) \quad u_n \rightarrow u, \quad \|u\| = 1 \text{ and } \|u\|^2 \leq \lambda_i \int_{\Omega} u^2. \]

Passing to the limit in (15) we conclude

\[ (17) \quad \int_{\Omega} m(x)u^2 \, dx \leq 1 \]

combining (10), (16) and (17) we obtain

\[ 1 = \|u\|^2 = \lambda_i \int_{\Omega} u^2 = \int_{\Omega} mu^2 \]
hence

\begin{equation}
(18) \quad u \in E(\lambda_i) \text{ and } \int_{\Omega} (m - \lambda_i)u^2 = 0.
\end{equation}

Since \( m(.) \geq \lambda_i \) and \( u \in E(\lambda_i) \) satisfies the unique continuation principle, (18) implies that \( m(.) = \lambda_i \) a.e. \( x \in \Omega \), which contradicts \((H_1)\) and shows that (15) can not occur. To complete the proof of lemma 2.2, let \( F \subset \oplus_{1 \leq p \leq i} E(\lambda_p) \) such that \( \text{dim} F = i + 1 \) (it is possible by hypothesis \((H_2)\)), it follows from (14) that

\[
\inf \left\{ \int_{\Omega} mu^2, \|u\| = 1, u \in F \right\} \geq 1 + \varepsilon
\]

which contradicts (13), so (8) can not occur and the proof of lemma 2.2 is complete.

Let us take the decomposition \( H^1_0(\Omega) = V \oplus W \) where \( V \) is the subspace spanned by the eigenfunctions corresponding to \( \lambda_j, j = 1, \ldots, i+1 \), and \( W = V^\perp \). It is easy to see that \((H_1)\) implies

\begin{equation}
K(x) = \limsup_{|s| \to +\infty} \frac{2F(x,s)}{s^2} \leq \lambda_{i+2}
\end{equation}

**Lemma 2.3.** Assume \((f_0), (H_1)\) and \((H_3)\), then we have

i) \( \lim_{\|v\| \to +\infty; v \in V} \Phi(v) = -\infty \)

ii) \( \lim_{\|w\| \to +\infty; w \in W} \Phi(w) = +\infty \)

**Proof.** Combining \((H_3)\) and (19) the above results follows.

**3. Proof Of The Main Results**

**Proof of theorem 1.1.** We can easily see that the functional \( \Phi : \)

\[
\Phi(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} F(x,u) - hu \, dx
\]

is weakly lower semicontinuous. Therefore, since \( \Phi|_W \) is coercive, (lemma2.3 ii)) the infimum \( \beta = \inf \Phi|_W > -\infty \) is attained. Taking
\( \alpha < \beta \) by 1) of lemma 2.3 there exists \( R > 0 \) such that \( \Phi(v) \leq \alpha \) for all \( v \in V \) with \( \|v\| \geq R \). Finally since \( \Phi \) satisfies the Palais-Smale condition, (lemma 2.2) we can apply the saddle point theorem of P. Rabinowitz to conclude the existence of a critical point \( u_0 \in H_0^1 \) of \( \Phi \), so the proof is complete.

**Proof of corollary 1.1.** To show the corollary 1.1, it suffices to show:

1) there exists \( \rho > 0 \) such that

\[
\Phi(u) \geq \alpha > 0 \text{ if } \|u\| = \rho, u \in H_0^1(\Omega)
\]

and 2) By hypotheses \((H_1)\) and \((H_2)\) we deduce

\[
\lim_{|t| \to +\infty} \Phi(t\varphi_1) = -\infty
\]

\( \varphi_1 \) is a \( \lambda_1 \) normalized eigenfunction.

Combining (20), (21) and lemma 2.2 we can apply the Mountain-Pass theorem to conclude that \( \Phi \) has a critical value \( \Phi(u_0) \) with \( \Phi(u_0) \geq \alpha > 0 \).

**Proof of corollary 1.2.** It is easy to show from \((H_1)\), \((H_6)\) and \((H_7)\) that

\[
\lim_{\|v\| \to +\infty, v \in V} \Phi(v) = -\infty
\]

On the other hand we have

\[
\lim_{\|w\| \to +\infty, w \in W} \Phi(w) = +\infty
\]

so by lemma 2.2 , (20) and (21) we deduce the result.
References


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