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A RADON NIKODYM THEOREM IN THE NON-ARCHIMEDEAN SETTING

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Abstract

In this paper we define the absolutely continuous relation between nonarchimedean scalar measures and then we give and prove a version of the Radon-Nykodym Theorem in this setting. We also define the nonarchimedean vector measure and prove some results in order to prepare a version of this Theorem in a vector case.

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1. Introduction and notations

In the classical case, it is known that the Radon-Nikodym Theorem involves real or complex valued measures and it becomes a property if the measures involved are vector valued measures (see [12] and [7]).

In this paper we give a non-archimedean version of the Radon-Nikodym Theorem for $\mathbf{K}$ -valued measures, where $\mathbf{K}$ is a non-archimedean field with a non-trivial valuation for which it is complete. In the last part of the paper, we introduce some definitions and statements which involve scalar valued measures.

Throughout the paper, X will denote a nonempty set, Ω a cover ring of subset of X and τ the topology generated by the ring Ω . As we know, any element of Ω is a clopen (closed and open) for this topology, Ω is a base of it and (X, τ) becomes to be a zero-dimensional topological space.

All of the following results in this section can be found in [10] and [11] and we will give a sketch of the proof of few of them.

Definition 1. A set function $\mu : \Omega \rightarrow \mathbf{K}$ is said to be a non-archimedean measure or simply measure if:

(i) μ is finitely additive, that is, for $A, B \in \Omega$, with $A \cap B = \emptyset$,

$$\mu(A) + \mu(B) = \mu(A \cup B)$$

(ii) $(U_\alpha)_{\alpha \in I}$ is a net in Ω with $U_\alpha \downarrow \emptyset$, and if for any $\alpha \in I$, we choose $V_\alpha \in \Omega$ with $V_\alpha \subset U_\alpha$, then

$$\lim_{\lambda} \mu(V_\lambda) = 0$$

(iii) if for each $a \in X$, there exists $U \in \Omega$, such that $a \in U$ and

$$\{\mu(V) : V \in \Omega; V \subset U\} \text{ is bounded.}$$

Definition 2. Let μ be a measure over a cover ring Ω of subsets of X and τ be the topology generated by Ω . For a τ -open $W \subset X$ and $a \in X$, we define:

$$\|W\|_\mu = \sup \{|\mu(U)| : U \in \Omega; U \subset W\}$$

$$\mathcal{N}_\mu(a) = \inf \left\{ \|W\|_\mu : W \text{ es } \tau\text{-open, } a \in W \right\}$$

These definitions satisfy the following properties (see [10] page 191):

Proposition 3. (i) If $W_1, W_2 \in \tau$, with $W_1 \subseteq W_2$, then $\|W_1\|_\mu \leq \|W_2\|_\mu$

(ii) If $W \in \tau$, then $\|W\|_\mu = \sup \{ \|U\|_\mu : U \in \Omega; U \subset W \}$

(iii) If $a \in X$, then $\mathcal{N}_\mu(a) = \inf \{ \|U\|_\mu : U \in \Omega; a \in U \}$

Lemma 4. The function $\mathcal{N}_\mu$ is τ -upper semicontinuous (u.s.c.).

Proof. Let us take $\epsilon > 0$ and consider $X_\epsilon = \{x \in X : \mathcal{N}_\mu(x) \geq \epsilon\}$. If $x \in X \setminus X_\epsilon$, then there exists $U \in \Omega$, $a \in U$, such that $\|U\|_\mu < \epsilon$. Now, for $z \in U$, $\mathcal{N}_\mu(z) \leq \|U\|_\mu < \epsilon$; hence $U \subset X \setminus X_\epsilon$. This proves that $\mathcal{N}_\mu$ is u.s.c. ■

For a linear space $\mathcal{F}$ of functions $f : X \rightarrow \mathbf{K}$, the collection $\Omega(\mathcal{F})$ given by

$$\Omega(\mathcal{F}) = \{U \subseteq X : f\chi_U \in \mathcal{F}, \forall f \in \mathcal{F}\}$$

is a cover ring of X . We will denote by $\tau(\Omega(\mathcal{F}))$ the corresponding topology.

Definition 5. We will say that $\mathcal{F}$ is a Wolfheze space if each $f \in \mathcal{F}$ is τ -continuous and for each $a \in X$, there exists $f \in \mathcal{F}$, with $f(a) \neq 0$.

Definition 6. A linear functional $I : \mathcal{F} \rightarrow \mathbf{K}$ will be called an integral if :

(I) for each net $(f_\alpha)_{\alpha \in \Gamma}$, $f_\alpha \in \mathcal{F}$ with $f_\alpha \downarrow 0$ and for any $\alpha \in \Gamma$,

$$\lim_{\alpha \in \Gamma} I(g_\alpha) = 0$$

for any net $(g_\alpha)_{\alpha \in \Gamma}$ in $\mathcal{F}$ with $|g_\alpha| \leq |f_\alpha|$.

Lemma 7. Fix $f \in \mathcal{F}$ and define

$$\begin{aligned} \mu_f : \Omega &\rightarrow \mathbf{K} \\ U &\mapsto \mu_f(U) = I(f\chi_U) \end{aligned}$$

Then, μ_f is a non-archimedean measure.

Proof. It follows easily from the fact that if $(U_\alpha)_{\alpha \in I}$ is a net in Ω with $U_\alpha \downarrow \emptyset$, then $f * f_\alpha = f * \chi_{U_\alpha} \downarrow 0$. ■

The next lemma gives a relation between those measures given in the previous lemma (see [10], page 193)

Lemma 8. *There exists a unique function $\mathcal{N}_I : X \rightarrow [0, \infty)$, such that*

$|f| \mathcal{N}_I = \mathcal{N}_{\mu_f}$ for each $f \in \mathcal{F}$. Even more, $\mathcal{N}_I$ is τ -u.s.c.

Let I be an integral over a Wolfheze space $\mathcal{F}$. We denote by $\Omega = \Omega(\mathcal{F})$, $\tau = \tau(\mathcal{F})$. For each $g : X \rightarrow \mathbf{K}$, we write

$$\|g\|_I = \sup \{|g(x)| \mathcal{N}_I(x) : x \in X\}$$

Then, we have:

Definition 9. *A function g is said to be I -integrable if for each $\delta > 0$, there exists $f \in \mathcal{F}$ such that,*

$$\|f - g\|_I < \delta$$

We denote by $L(I)$ the class of the integrable functions, that is,

$$L(I) = \{g : X \rightarrow \mathbf{K} : g \text{ is integrable}\}$$

This class is a linear space over the non-archimedean field $\mathbf{K}$.

The following theorem gives us a characterization of integrable functions.

Theorem 10. *A function $f : X \rightarrow \mathbf{K}$ is integrable if and only if f satisfies the following conditions:*

- (i) *f is $\tau(\mathcal{F})$ -continuous on each $X_t = \{x \in X : \mathcal{N}_I(x) \geq t\}$*
- (ii) *For each $\delta > 0$, there exists a $\tau(\mathcal{F})$ -compact P contained in some X_t such that $|f| \mathcal{N}_I < \delta$ off P .*

Proof. See [10], pages 195-96. ■

Theorem 11. *The space $L(I)$ is a Wolfheze space and $\mathcal{F}$ is $\|\cdot\|_I$ -dense in $L(I)$.*

Definition 12. *Let $\Theta : \Omega \rightarrow \mathbf{K}$ be set function and μ be an integral over $\mathcal{F}$. For each $a \in X, \alpha \in \mathbf{K}, c \in]0, 1[$ and $r \in \mathbf{R}$, we write*

1. $\lim_{U \rightarrow \alpha} \Theta(U) = \alpha$, if

$$(\forall \epsilon > 0) (\exists V \in \Omega; a \in V) (U \subset V; U \in \Omega \Rightarrow |\Theta(U) - \alpha| < \epsilon)$$

2. $\lim_{U \rightarrow a} \Theta(U) = \alpha$, if

for each $\epsilon > 0$, there exists a neighborhood V of a , such that for all $U \in \Omega$, with $U \subset V$, and $|\mu(U)| \geq c\mathcal{N}_\mu(a)$ we have

$$|\Theta(U) - \alpha| < \epsilon$$

3. $\lim_{U \rightarrow a} \Theta(U) = \alpha$, if

$$\overline{\lim}_{U \rightarrow a} \Theta(U) = \alpha, \quad \forall c \in]0, 1[$$

4. $\overline{\lim}_{U \rightarrow a} |\Theta(U)| = r$, if

$$(\forall \epsilon > 0)$$

$$(\exists U \in \Omega; a \in U) (r - \epsilon \leq \sup \{|\Theta(V)| : V \in \Omega; V \subset U\} \leq r + \epsilon)$$

The proof of the next lemmas follows directly from the definitions (see [11], page 80).

Lemma 13. For a measure $\mu : \Omega \rightarrow \mathbf{K}$ and $a \in X$ we have

$$1. \mathcal{N}_\mu(a) = \overline{\lim}_{U \rightarrow a} |\mu(U)|$$

$$2. \mathcal{N}_\mu(a) = 0 \iff \lim_{U \rightarrow a} \mu(U) = 0$$

Lemma 14. If $\Theta : \Omega \rightarrow \mathbf{K}$ is additive and $0 < c < 1$, then

$$\lim_{U \rightarrow a} \Theta(U) = 0 \iff \lim_{U \rightarrow a} \Theta(U) = 0$$

2. The Radon-Nikodym Theorem

Throughout this section X will be a set and $\mathcal{F}$ a Wolfheze space of functions from X into $\mathbf{K}$ with the assumption that $\mathcal{F}$ contains the constant functions. It follows from this assumption that $\mathcal{X}_U \in \mathcal{F}$ for all $U \in \Omega(\mathcal{F})$. Since $1 \in \mathcal{F}$, we have that if $I : \mathcal{F} \rightarrow \mathbf{K}$ is an integral, then $\mu : \Omega(\mathcal{F}) \rightarrow \mathbf{K}$ defined by $\mu(U) = I(\mathcal{X}_U)$ is a non-archimedean measure. In the sequel, we will use Greek letter to denote integral. We start with the following lemma [11]:

Lemma 1. *If $\mu : \mathcal{F} \rightarrow \mathbf{K}$ is an integral, $f \in L(\mu)$ and $a \in X$, then*

$$LIM_{U \rightarrow a} [\mu_f(U) - f(a)\mu(U)] = 0$$

Definition 2. *Let μ, ν be two measures over a cover ring Ω . We will say that ν is absolutely continuous with respect to μ , $\nu \ll \mu$, if for each $a \in X$ there exists $\alpha \in \mathbf{K}$, such that*

$$\mathcal{N}_{\nu-\alpha\mu}(a) = 0$$

Example 3. *Let $\mu : \mathcal{F} \rightarrow \mathbf{K}$ be an integral functional. Let $j : X \rightarrow \mathbf{K}$ be a μ -integrable function. We define*

$$\begin{aligned} j\mu : \mathcal{F} &\rightarrow \mathbf{K} \\ f &\mapsto j\mu(f) = \mu(fj) = \int fj d\mu \end{aligned}$$

It is well defined since $fj \in L(\mu)$ and is an integral. Note that its corresponding measure $j\mu : \Omega(\mathcal{F}) \rightarrow \mathbf{K}$ is the following:

$$j\mu(U) = \mu(\mathcal{X}_U j) = \mu_j(U)$$

(see Lemma 7). We claim that $j\mu$ is absolutely continuous with respect to μ . In fact, by Lemma 1

$$\begin{aligned} LIM_{U \rightarrow a} [j\mu(U) - j(a)\mu(U)] &= LIM_{U \rightarrow a} [\mu_j(U) - j(a)\mu(U)] \\ &= 0 \end{aligned}$$

and then, by Lemma 13-(2),

$$\mathcal{N}_{\mu_j-j(a)\mu}(a) = 0.$$

Theorem 4. (*The Radon-Nikodym Theorem*)

Let ν, μ be two measures defined on $\Omega(\mathcal{F})$. If $\nu \ll \mu$, then there exists a μ -integrable function j such that

$$\nu = j\mu$$

Let $a \in X$. Since $\nu \ll \mu$, there exists $\alpha_a \in \mathbf{K}$ such that

$$\mathcal{N}_{\nu - \alpha_a \mu}(a) = 0$$

Using Lemma 13, we can prove the uniqueness of α_a . Thus, we define the function j by:

$$\begin{aligned} j &: X \rightarrow \mathbf{K} \\ a &\mapsto j(a) = \begin{cases} \alpha_a & , \text{ if } \mathcal{N}_\mu(a) > 0 \\ 0 & , \text{ if } \mathcal{N}_\mu(a) = 0 \end{cases} \end{aligned}$$

The first step is to prove that j is μ -integrable. To do that we will use Theorem 10, that is, we have to prove that (i) $j|_{X_t}$ is continuous and (ii) for each $\delta > 0$, there exists a compact set P of X contained in some $X_t = \{x \in X : \mathcal{N}_\mu(x) \geq t\}$ such that $|j| \mathcal{N}_\mu < \delta$ off P . Take $t > 0$ and $a \in X_t$. By the assumption, $\mathcal{N}_{\nu - j(a)\mu}(a) = 0$ or, equivalently, $\lim_{U \rightarrow a} (\nu - j(a)\mu)(U) = 0$. Thus, if $\epsilon > 0$ is given, there exists $U_0 \in \Omega$, with $a \in U_0$, such that

$$|(\nu - j(a)\mu)(V)| < \epsilon, \quad \forall V \in \Omega, V \subset U_0$$

Take $b \in U_0 \cap X_t$; hence $\mathcal{N}_\mu(b) \geq t$. Since $\lim_{U \rightarrow b} (\nu - j(b)\mu)(U) = 0$, there exists $U_1 \in \Omega$, with $b \in U_1$, such that

$$|(\nu - j(b)\mu)(V)| < \epsilon, \quad \forall V \in \Omega, V \subset U_1$$

Thus, if $V \subset U_0 \cap U_1 = U_2$ and $V \in \Omega$, then

$$\begin{aligned}
 |j(a) - j(b)|t &\leq |j(a) - j(b)|\mathcal{N}_\mu(b) \\
 &\leq |j(a) - j(b)|\|U_2\|_\mu \\
 &= \sup\{|j(a) - j(b)|\mu(V) : V \in \Omega, V \subset U_2\} \\
 &\leq \sup\{|\nu - j(b)\mu(V) + j(a)\mu(V) - \nu| : \\
 &\quad V \in \Omega, V \subset U_2\} \\
 &\leq \max\left\{\sup_{V \subset U_2} |(\nu - j(b)\mu)(V)|, \right. \\
 &\quad \left. \sup_{V \subset U_2} |(\nu - j(a)\mu)(V)|\right\} \\
 &\leq \epsilon
 \end{aligned}$$

therefore, $j|_{X_t}$ is continuous. To prove (ii), take $\delta > 0$ and $a \in X$, with $\mathcal{N}_\mu(a) > 0$. The set $P = \{x \in X : \mathcal{N}_\nu(x) \geq \delta\}$ is τ -compact in X . Since $\mathcal{N}_{\nu-j(x)\mu}(x) = 0$ for all $x \in X$ with $\mathcal{N}_\mu(x) > 0$ and

$$|\mathcal{N}_\nu(x) - \mathcal{N}_{j(x)\mu}(x)| \leq \mathcal{N}_{\nu-j(x)\mu}(x) = 0$$

we have

$$\mathcal{N}_\nu(x) = \mathcal{N}_{j(x)\mu}(x) = |j(x)|\mathcal{N}_\mu(x)$$

Then, for $x \notin P$, we have $|j(x)|\mathcal{N}_\mu(x) < \delta$. Now, we finish this part proving that there is $t > 0$ such that $P \subset X_t$. If $a \in P$, then there exists $U_a \in \Omega$, with $a \in U_a$, such that

$$|(\nu - j(a)\mu)(V)| < \frac{\delta}{2}, \quad \forall V \in \Omega, V \subset U_a$$

Thus,

$$P \subset \bigcup_{a \in P} U_a$$

and since P is compact, there exist $a_1, a_2, \dots, a_n \in P$ such that

$$P \subset \bigcup_{i=1}^n U_{a_i}$$

Now, denote by $M = \max\{|j(a_1)|, |j(a_2)|, \dots, |j(a_n)|\}$ and define $t = \delta M^{-1}$. We claim that $P \subset X_t$; in fact, since

$$\begin{aligned}
 |\nu(V)| &\leq |\nu(V) - j(a_i)\mu(V) + j(a_i)\mu(V)| \\
 &\leq \max\{|\nu(V) - j(a_i)\mu(V)|, |j(a_i)\mu(V)|\} \\
 &\leq \max\left\{\frac{\delta}{2}, \delta t^{-1}|\mu(V)|\right\}
 \end{aligned}$$

Then, by Lemma 13, we have

$$\begin{aligned}\delta \leq \mathcal{N}_\nu(a) &= \overline{LIM}_{V \rightarrow a} |\nu(V)| \\ &\leq \max \left\{ \frac{\delta}{2}, \delta t^{-1} \overline{LIM}_{V \rightarrow a} |\mu(V)| \right\} \\ &\leq \max \left\{ \frac{\delta}{2}, \delta t^{-1} \mathcal{N}_\mu(a) \right\}\end{aligned}$$

and, therefore

$$\mathcal{N}_\mu(a) \geq t$$

To prove that $\nu = j\mu$, we consider the measure

$$\begin{aligned}\nu - j\mu : \Omega &\rightarrow \mathbf{K} \\ U &\rightarrow (\nu - j\mu)(U) = \nu(U) - \int \mathcal{X}_U j d\mu\end{aligned}$$

Fix $a \in X$, then

$$\overline{LIM}_{U \rightarrow a} |(\nu - j\mu)(U)| = \overline{LIM}_{U \rightarrow a} \left| \nu(U) - \int \mathcal{X}_U j d\mu \right|$$

Now,

$$\begin{aligned}|\nu(U) - \int \mathcal{X}_U j d\mu| &= |\nu(U) - j(a)\mu(U) + j(a)\mu(U) - \int \mathcal{X}_U j d\mu| \\ &\leq \max \{ |\nu(U) - j(a)\mu(U)|, \\ &\quad |j(a)\mu(U) - \int \mathcal{X}_U j d\mu| \}\end{aligned}$$

Therefore,

$$\begin{aligned}&\overline{LIM}_{U \rightarrow a} \left| \nu(U) - \int \mathcal{X}_U j d\mu \right| \leq \\ &\max \left\{ \overline{LIM}_{U \rightarrow a} |\nu(U) - j(a)\mu(U)|, \overline{LIM}_{U \rightarrow a} \left| j(a)\mu(U) - \int \mathcal{X}_U j d\mu \right| \right\} = 0\end{aligned}$$

Thus,

$$\overline{LIM}_{U \rightarrow a} \left| \nu(U) - \int \mathcal{X}_U j d\mu \right| = 0$$

and then,

$$\mathcal{N}_{\nu-j\mu}(a) = 0$$

Since $a \in X$ is arbitrary,

$$\mathcal{N}_{\nu-j\mu}(a) = 0, \forall a \in X$$

Therefore,

$$\nu - j\mu = 0$$

or,

$$\nu = j\mu$$

■

3. Non-archimedean vector measures

In this section we will briefly analyze E -valued measures, where E is a normed space over the field $\mathbf{K}$. As in the previous sections, X will be a nonempty set, Ω a cover ring of X and E a Banach space, with $E \neq \{\theta\}$ over a non-archimedean field $\mathbf{K}$ with a nontrivial valuation and complete. As we know, the function $\|\cdot\|_0 : E \rightarrow \mathbf{R}$ defined by

$$\|x\|_0 = \sup \{|x'(x)| : x' \in E'; \|x'\| \leq 1\}$$

is equivalent to the original norm of E .

Definition 1. Let $m : \Omega \rightarrow E$ a finitely additive and bounded set function. We will say that m is a non-archimedean vector measure or, simply, a vector measure if for each net $\{U_\lambda\}_{\lambda \in \Lambda}$ in Ω such that $U_\lambda \downarrow \emptyset$ and if, for each $\lambda \in \Lambda$, we choose $V_\lambda \subset U_\lambda$ and $V_\lambda \in \Omega$, then

$$\lim_{\lambda} m(V_\lambda) = 0$$

that is,

$$(\forall \varepsilon > 0) (\exists \lambda_0 \in \Lambda) (\lambda > \lambda_0 \Rightarrow \|m(V_\lambda)\|_0 < \varepsilon)$$

Definition 2. Let $m : \Omega \rightarrow E$ a finitely additive set function. For $U \in \Omega$, we define

$$\|U\|_m = \sup \{\|m(B)\|_0 : B \subset U; B \in \Omega\}$$

$$\mathcal{N}_m(x) = \inf \{\|U\|_m : U \in \Omega; x \in U\}$$

It is not difficult to prove that this function is $\tau(\Omega)$ -u.s.c.

Definition 3. Let $\Theta : \Omega \rightarrow E$ a set function, $\mu : \Omega \rightarrow \mathbf{K}$ a measure, $c \in]0, 1[$, $a \in X$ and $r \in \mathbf{R}$.

1. $\lim_{U \rightarrow a} \Theta(U) = e$ means

$$(\forall \epsilon > 0) (\exists U \in \Omega; a \in U) (V \subset U; V \in \Omega \Rightarrow \|\Theta(V) - e\|_0 < \epsilon)$$

2. $\lim_{\mu, c} \Theta(U) = e$ means

$$(\forall \epsilon > 0) (\exists U \in \Omega; a \in U) \\ (V \subset U; V \in \Omega; |\mu(V)| \geq c\mathcal{N}_\mu(a) \Rightarrow \|\Theta(V) - e\|_0 < \epsilon)$$

3. $\lim_\mu \Theta(U) = e$ means

$$\lim_{\mu, c} \Theta(U) = e$$

for all $c \in]0, 1[$

4. $\overline{\lim}_{U \rightarrow a} \|\Theta(U)\| = r$ means

$$(\forall \epsilon > 0) (\exists U \in \Omega; a \in U) \\ (r - \epsilon \leq \sup \{\|\Theta(V)\|_0 : V \in \Omega; V \subset U\} \leq r + \epsilon)$$

Lemma 4. Let $m : \Omega \rightarrow E$ a vector measure and $a \in X$. Then,

$$1. \mathcal{N}_m(a) = \overline{\lim}_{U \rightarrow a} \|m(U)\|$$

$$2. \mathcal{N}_m(a) = 0 \Leftrightarrow \lim_{U \rightarrow a} m(U) = \theta$$

Proof. In both cases, the proof follows easily from the definition and we omit them. ■

We will denote by $\mathcal{S}(X)$ the vector space spanned by $\langle \{\mathcal{X}_U : U \in \Omega\} \rangle$, that is, $f \in \mathcal{S}(X)$, if

$$f = \sum_{i=1}^n \alpha_i \mathcal{X}_{U_i}$$

where $\alpha_i \in \mathbf{K}$, $U_i \in \Omega$, and $\bigcup_{i=1}^n U_i = X$. We can assume that $\{U_i\}_1^n$ is pairwise disjoint.

Definition 5. Let m be a non-archimedean vector measure. We will define the E -valued linear operator $\int$ on $\mathcal{S}(X)$ by

$$\int \mathcal{X}_U dm = m(U)$$

Lemma 6. This operator satisfies

$$\left\| \int f dm \right\| \leq \|f\|_{\mathcal{N}}$$

Proof. It follows from the fact that

$$\begin{aligned} \left\| \int \mathcal{X}_U dm \right\| &= \|m(U)\| \\ &\leq \|U\|_m \\ &\leq \sup_{x \in U} \mathcal{N}(x) \\ &\leq \sup_{x \in X} |\mathcal{X}_U| \mathcal{N}(x) \\ &\leq \|\mathcal{X}_U\|_{\mathcal{N}} \end{aligned}$$

■

Definition 7. A function $f : X \rightarrow \mathbf{K}$ is said to be m -integrable if there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ of $\mathcal{S}(X)$ such that

$$\lim_{n \rightarrow \infty} \|f - f_n\|_{\mathcal{N}} = 0$$

We will denote by $L(m)$ the space of all m -integrable functions. Since $\mathcal{S}(X)$ is $\|\cdot\|_{\mathcal{N}}$ -dense in $L(m)$, we have that the linear operator $\int$ has a unique extension to $L(m)$ and this extension satisfies the condition:

$$\left\| \int f dm \right\| \leq \|f\|_{\mathcal{N}}, \quad \forall f \in L(m)$$

For $U \in \Omega$ and $e \in E$, we define the function $\mathcal{X}_U \otimes e$ by

$$\begin{aligned} \mathcal{X}_U \otimes e : X &\rightarrow E \\ x &\mapsto \mathcal{X}_U \otimes e(x) = \begin{cases} e & \text{if } x \in U \\ \theta & \text{if } x \notin U \end{cases} \end{aligned}$$

We denote by

$$\mathcal{S}(X, E) = \langle \{\mathcal{X}_U \otimes e : U \in \Omega; e \in E\} \rangle$$

Thus, $f \in \mathcal{S}(X, E)$ means

$$f = \sum_{i=1}^n \mathcal{X}_{U_i} \otimes e_i$$

Now, if $\mu : \Omega \rightarrow \mathbf{K}$ a measure and $f \in \mathcal{S}(X, E)$, then we define the set function

$$\begin{aligned} \mu \otimes f : \Omega &\rightarrow E \\ U &\mapsto \mu \otimes f(U) = \sum_{i=1}^n \mu(U_i \cap U) e_i \end{aligned}$$

Lemma 8. $\mu \otimes f$ is a vector measure and satisfies the following property

$$\mathcal{N}_{\mu \otimes f}(a) = \|f(a)\| \mathcal{N}_{\mu \otimes f(a)}(a)$$

Proof. Trivially, $\mu \otimes f$ is finitely additive. To see $\mu \otimes f$ is bounded, take $U \in \Omega$. Then,

$$\|U\|_{\mu \otimes f} = \sup \{ \|\mu \otimes f(V)\|_0 : V \in \Omega; V \subset U \}$$

Now, since

$$\|\mu \otimes f(V)\|_0 = \sup \{ |x'(\mu \otimes f(V))| : \|x'\| \leq 1 \}$$

where,

$$\begin{aligned} x'(\mu \otimes f(V)) &= x' \left(\sum_{i=1}^n \mu(U_i \cap V) e_i \right) \\ &= \sum_{i=1}^n \mu(U_i \cap V) x'(e_i) \end{aligned}$$

we have,

$$\begin{aligned} |x'(\mu \otimes f(V))| &\leq \max_{1 \leq i \leq n} |\mu(U_i \cap V)| |x'(e_i)| \\ &\leq \max_{1 \leq i \leq n} |\mu(U_i \cap V)| \|e_i\|_0 \\ &\leq M \max_{1 \leq i \leq n} \|U_i \cap V\|_\mu \\ &\leq M \|\bigcup_{i=1}^n U_i \cap V\|_\mu \\ &= M \|U\|_\mu \end{aligned}$$

where $M = \sup \{\|e_i\|_0 : i = 1, 2, \dots, n\}$.

From this,

$$\|U\|_{\mu \otimes f} \leq M \|U\|_\mu$$

and since μ is a measure, we have

$$\{\|U\|_{\mu \otimes f} : U \in \Omega\} \text{ is bounded.}$$

Now, consider a net $\{U_\lambda\}_{\lambda \in \Lambda}$ in Ω such that $U_\lambda \downarrow \emptyset$ and choose $V_\lambda \subset U_\lambda$ with $V_\lambda \in \Omega$ for any $\lambda \in \Lambda$. Then,

$$\begin{aligned} \lim_\lambda \mu \otimes f(V_\lambda) &= \lim_\lambda \sum_{1 \leq i \leq n} \mu(U_i \cap V_\lambda) e_i \\ &= \sum_{1 \leq i \leq n} \lim_\lambda \mu(U_i \cap V_\lambda) e_i \\ &= \theta \end{aligned}$$

since μ is measure and $\{U_i \cap V_\lambda\}_{\lambda \in \Lambda}$ is a net in Ω with $U_i \cap V_\lambda \subset U_\lambda$ for each λ . This proves that $\mu \otimes f$ is a vector measure

For the second part, let $a \in X$; hence there exists a unique $i \in \{1, 2, 3, \dots, n\}$ such that $a \in U_i$, where $f = \sum_{i=1}^n \mathcal{X}_{U_i} \otimes e_i$. By Lemma 4, we have

$$\mathcal{N}_{\mu \otimes f}(a) = \overline{LIM}_{U \rightarrow a} \|\mu \otimes f(U)\|$$

Thus, if $U \in \Omega$, with $a \in U$, is small enough, that is, $U \subset U_i$ for some i , then

$$\mu \otimes f(U) = \mu(U) e_i$$

and therefore,

$$\begin{aligned} \mathcal{N}_{\mu \otimes f}(a) &= \overline{LIM}_{U \rightarrow a} \|\mu(U) e_i\| \\ &= \|e_i\| \overline{LIM}_{U \rightarrow a} |\mu(U)| \\ &= \|f(a)\| \mathcal{N}_\mu(a) \end{aligned}$$

■

Definition 9. Let $m : \Omega \rightarrow E$ and $\mu : \Omega \rightarrow \mathbf{K}$ two non-archimedean measures. We will say that m is absolutely continuous with respect to μ if:

$$(\forall a \in X) (\exists e \in E) (\mathcal{N}_{m-\mu \otimes e}(a) = 0)$$

Lemma 10. *For any $f \in \mathcal{S}(X, E)$, $\mu \otimes f$ is absolutely continuous with respect to μ .*

Proof. Let $a \in X$; hence, $a \in U_i$, where $f = \sum_{i=1}^n \mathcal{X}_{U_i} \otimes e_i$. Now, if $U \in \Omega$, with $a \in U$, is small enough, then

$$\mu \otimes f(U) = \mu(U)e_i$$

where $f(a) = e_i$. Thus,

$$\mu \otimes f(U) - \mu \otimes f(a)(U) = 0$$

and then,

$$LIM_{U \rightarrow a} [\mu \otimes f(U) - \mu \otimes f(a)(U)] = 0$$

Therefore, by Lemma 4, we get

$$\mathcal{N}_{\mu \otimes f - \mu \otimes f(a)}(a) = 0$$

■

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