DIFFERENTIABILITY OF SOLUTIONS OF THE ABSTRACT CAUCHY PROBLEM

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Abstract

In this note we establish a result of differentiability for the mild solution of the inhomogeneous abstract Cauchy problem when the underlying space is reflexive.

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1. Introduction.

In this work we are concerned with regularity properties of solutions of the first order abstract Cauchy problem (in short, ACP). We refer the reader to [3, 10] for the theory of strongly continuous semigroup operators and the associated ACP.

Let $X$ be a Banach space endowed with a norm $\| \cdot \|$. Henceforth $T(t)$ is a strongly continuous semigroup of operators on $X$ with infinitesimal generator $A$.

The existence of solutions of the first order abstract Cauchy problem

\begin{align}
  x'(t) &= Ax(t) + h(t), \quad 0 \leq t \leq a, \\
  x(0) &= x_0,
\end{align}

it has been treated in several works. We only mention here the texts [3, 10] and the references cited therein. Similarly, the existence of solutions of the semilinear abstract Cauchy problem it has been discussed in [1, 9].

Assuming that $h : [0, a] \to X$ is integrable the function given by

\begin{align}
  x(t) &= T(t)x_0 + \int_0^t T(t - s)h(s) \, ds, \quad 0 \leq t \leq a,
\end{align}

is said mild solution of (1.1)-(1.2). In the case in which $h$ is continuous, the function $x(\cdot)$ is called a classic solution on $[0, a]$ of (1.1)-(1.2) if $x$ is a function of class $C^1$, $x(t) \in D(A)$ and (1.1) is verified.

The existence of classical solutions of (1.1)-(1.2) as well as some weaker forms of differentiability of solutions have been studied in a number of works. We refer the reader to [2, 8, 10, 12, 13, 14] and the references therein indicated.

The purpose of this note is to establish a new condition in order to the mild solution $x(\cdot)$ turn to be a classical solution.

Next the notation $C([0, a]; X)$ stands for the space of continuous functions from $[0, a]$ into $X$, whilst $BV([0, a]; X)$ represents the space of functions with bounded variation from $[0, a]$ into $X$. For a function $h \in BV([0, a]; X)$ we denote by $V(h)$ the variation of $h$ on $[0, a]$ and by $v(t, h)$ the variation of $h$ on $[0, t]$, for $0 \leq t \leq a$. Additional termi-
nology and notations are those generally used in functional analysis. In particular, $X^*$ denotes the dual space of $X$.

2. Results.

In this section $h$ denotes a continuous function of bounded variation on a fixed interval $[0, a]$, $a > 0$. We define the translation of $h$ by

$$ T_t h(s) = \begin{cases} h(s + t), & s \leq a - t, \\ h(a), & s \geq a - t, \end{cases} $$

for $t \geq 0$. Let $\mu(t, h) = V(T_t h - h)$.

We introduce the following condition for a function $h \in C([0, a]; X) \cap BV([0, a]; X)$.

$(H_0)$ $\mu(t, h) \to 0$, as $t \to 0^+$.

Initially we discuss some examples.

**Example 1.** If $h \in W^{1,1}([0, a]; X)$, then $\mu(t, h) \to 0$, $t \to 0^+$.

**Example 2.** Let $h : [0, 1] \to \mathbb{R}$ be the function defined in [4], Exercise 4.19. Let $E$ be a perfect nowhere dense set with measure 0 included in $[0, 1]$. Let $(a_k, b_k)$, $k \in \mathbb{N}$, be disjoint intervals such that $(0, 1) \setminus E = \bigcup_{k=1}^{\infty} (a_k, b_k)$ and let $\sum_{k=1}^{\infty} c_k$ be a convergent series of positive number with sum equal to 1. For each $x \in [0, 1]$ let

$$ I(x) = \{k : [a_k, b_k] \cap [0, x] \neq \emptyset\} $$

and define

(2.1) $h(x) = \sum_{k \in I(x)} c_k.$

It is clear that $h(0) = 0$, $h(1) = 1$. Moreover, $h$ is continuous and nondecreasing with $h' = 0$, a.e. Thus $h$ is a singular function. Now we establish that $h$ does not satisfy $(H_0)$. In fact, from (2.1) it follows easily that for each $t > 0$ and $0 \leq s \leq 1 - t$,

(2.2) $h(s + t) - h(s) = \sum_{k \in I} c_k.$
where $I = \{ k \in \mathbb{N} : a_k \in (s, s + t) \}$. In addition, for $n \in \mathbb{N}$ we can choose $t > 0$ small enough such that $\bigcup_{i=1}^{n} [a_i, b_i] \subseteq [0, 1 - t]$, $a_i + 3t < b_i$, and, for each $k = 1, \ldots, n$, $a_k - t \notin \bigcup_{i=1, i \neq k}^{n}[a_i, b_i]$.

Defining $\alpha_i = a_i - t/2$ and $\beta_i = a_i + 2t$ it follows from (2.2) that $h(\alpha_i + t) - h(\alpha_i) = c_i$ and $h(\beta_i + t) - h(\beta_i) = 0$. From this we obtain that

$$V(\mathcal{T}_t h - h) \geq \sum_{i=1}^{n} |(\mathcal{T}_t h - h)(\beta_i) - (\mathcal{T}_t h - h)(\alpha_i)|$$

$$= \sum_{i=1}^{n} |h(\beta_i + t) - h(\beta_i) - (h(\alpha_i + t) - h(\alpha_i))|$$

$$= \sum_{i=1}^{n} c_i$$

which implies that $\mu(t, h)$ does not converge to 0 as $t \to 0^+$. 

**Example 3.** Let $h : [0, 1] \to \mathbb{R}$ be the singular function defined in [6], Example 18.8. As above, $h(0) = 0$, $h(1) = 1$, $h$ is continuous and strictly increasing and $h' = 0$, a.e. We will show that this function satisfies the assumption $(H_0)$. Initially, for completeness we include here the construction carried out in [6].

Let $(t_n)_n$ be a sequence in $(0, 1)$. Set $F_1(0) = 0$, $F_1(1) = 1$, $F_1(\frac{1}{2}) = \frac{1 + t_1}{2}$ and define $F_1$ to be linear on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Suppose that $F_1, F_2, \ldots, F_n$ have been defined. Then define

$$F_{n+1} \left( \frac{k}{2^n} \right) = F_n \left( \frac{k}{2^n} \right), \text{ for } k = 0, 1, \ldots, 2^n,$$

$$F_{n+1} \left( \frac{2k+1}{2^{n+1}} \right) = \frac{1-t_{n+1}}{2} F_n \left( \frac{k}{2^n} \right) + \frac{1+t_{n+1}}{2} F_n \left( \frac{k+1}{2^n} \right),$$

for $k = 0, 1, \ldots, 2^n - 1$, and complete the definition of $F_{n+1}$ as a continuous linear function in the intervals $[\frac{k}{2^n+1}, \frac{k+1}{2^{n+1}}]$, for $k = 0, 1, \ldots, 2^{n+1} - 1$. It
is shown in [6] that \((F_n)_n\) is a nondecreasing sequence. Thus this sequence converges to a function \(h\) which satisfies the properties already mentioned. Applying now the Dini’s theorem ([11]) we obtain that the convergence of \((F_n)_n\) is uniform. Since \((T_tF_n - F_n)_n\) converges uniformly to \(T_th - h\), as \(n \to \infty\), and this convergence is also uniform on \(t \geq 0\), it is follows that \(\mu(t, F_n) \to \mu(t, h), \ n \to \infty,\) and this convergence is uniform on \(t\). In view of \(F_n\) is absolutely continuous, from Example 1 we infer that \(\mu(t, F_n) \to 0,\ \text{as} \ t \to 0^+\), which implies that \(\mu(t, h) \to 0,\ \text{as} \ t \to 0^+\).

To study the regularity of solutions of the abstract Cauchy problem (1.1)-(1.2) we begin by establishing some preliminary lemmas.

In the sequel we denote by \(M\) a positive constant such that \(\|T(t)\| \leq M, \ 0 \leq t \leq a\). Moreover, for a fixed \(h\), we use the notation

\[
(2.3) \quad u(t) = \int_0^t T(t - s)h(s) \, ds,
\]

Lemma 2.1. Assume that \(X\) is a reflexive space. Let \(T(\cdot)\) be a strongly continuous semigroup of operators on \(X\) and let \(h : [0, a] \to X\) be a continuous function of bounded variation which satisfies the assumption \((H_0)\). Then the Riemann-Stieltjes integral

\[
w(t) = \int_0^t T(t - s) \, ds \, h = \int_0^t T(s) \, ds \, h(t - s)
\]

exists in the weak topology and define a continuous function \(w : [0, a] \to X\).

Proof. Let \(\Lambda : X^* \to \mathcal{C}'\) be defined by

\[
\Lambda(x^*) = \int_0^t < T(t - s)^*x^*, \, ds \, h >.
\]

The Riemann-Stieltjes integral in the above expression exists because \(T(\cdot)^*x^*\) is a continuous function ([10]) and \(h\) has bounded variation ([7]). Moreover, \(\Lambda\) is linear and

\[
|\Lambda(x^*)| \leq M\|x^*\|V(h).
\]

Consequently, \(\Lambda \in X^{**}\) and in view of that \(X\) is reflexive we infer the existence of \(w(t) \in X\) such that \(\Lambda(x^*) = < x^*, w(t) >\), for all \(x^* \in X^*\).
On the other hand, for $t < 1$ and $\tau$ small enough, from the relations

$$w(t + \tau) - w(t) = \int_{0}^{t+\tau} T(s) \, ds \, h(t + \tau - s) - \int_{0}^{t} T(s) \, ds \, h(t - s)$$

we deduce that

$$\|w(t + \tau) - w(t)\| \leq M\mu(\tau, h) + Mv(\tau, h).$$

Since $\mu(\tau, h) \to 0$, $\tau \to 0$, because the condition $(H_0)$ holds and $v(\tau, h) \to 0$, $\tau \to 0$, by the Proposition I.2.9 in [7]) the previous estimation shows that $w(\cdot)$ is right continuous at $t$. Similarly, one can prove that $w$ is left continuous at $t > 0$.

Next we denote by $\chi_E$ the characteristic function of a set $E$.

Lemma 2.2. Let $h : [0, a] \to X$ be the step function $h = \sum_{i=1}^{n} x_{i} \chi_{I_{i}}$, where $I_{i}$ are intervals and $\{I_{1}, \cdots, I_{n}\}$ is a partition of $[0, a]$. Then the function $u$ given by (2.3) is piecewise smooth, $u(t) \in D(A)$, $Au(\cdot)$ is continuous on $[0, a]$ and $u'(t) = Au(t) + h(t)$, $t \notin P$, where $P$ denotes the set formed by the extreme points of intervals $I_{i}$, $i = i, \cdots, n$.

Proof. Applying the linearity of $u$ in terms of $h$, it is sufficient to prove the assertion for a function $h = x\chi_I$ where $I$ is an interval of type $[t_1, t_2]$. In fact, in this case, $u(t)$ is given by

$$u(t) = \begin{cases} 
0, & 0 \leq t \leq t_1, \\
\int_{0}^{t-t_1} T(s)x \, ds, & t_1 \leq t \leq t_2, \\
\int_{t-t_2}^{t} T(s)x \, ds, & t_2 \leq t.
\end{cases}$$
From the properties of semigroups we infer that $u(t) \in D(A)$ and that

$$Au(t) = \begin{cases} 
0, & 0 \leq t \leq t_1, \\
T(t-t_1)x - x, & t_1 \leq t \leq t_2, \\
T(t-t_1)x - T(t-t_2)x, & t_2 \leq t.
\end{cases}$$

This shows that $Au(\cdot)$ is continuous. Moreover, it is immediate to verify that $u'(t) = Au(t) + h(t)$, $t \neq t_1, t_2$.

Now we can prove the main result of this note.

**Theorem 2.1.** Assume that $X$ is a reflexive space and let $h$ be a continuous function of bounded variation on $[0, a]$ which satisfies assumption $(H_0)$. Let $x_0 \in D(A)$. Then the mild solution of (1.1)-(1.2) is a classical solution.

**Proof.** We consider a sequence $(h_n)_n$ of step functions, where each $h_n$ is given by

$$h_n = \sum_{i=1}^{n} h(t_i) \chi_{I_i}.$$  

In this expression we have denoted $I_i = [t_{i-1}, t_i)$, $i = 1, \ldots, n-1$, and $I_n = [t_{n-1}, t_n]$, where the points $t_i$ have been chosen as $t_i = \frac{a}{n}i$, $i = 0, 1, \ldots, n$.

It is clear that the sequence $(h_n)_n$ converge uniformly to $h$. Let $u_n$ be the function given by (2.3), with $h_n$ instead of $h$. Then, $u_n \to u$, $n \to \infty$, uniformly on $[0, a]$. Moreover, by Lemma 2.2 we have that $u_n(t) \in D(A)$ and, if we fix $0 \leq t \leq a$ and $n \in \mathbb{N}$, then $t \in I_k$, for some $k = 1, \ldots, n$. From our definitions we can write

$$Au_n(t) = A \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} T(t-s)h(t_i) \, ds + A \int_{t_{k-1}}^{t} T(t-s)h(t_k) \, ds$$

$$= \sum_{i=1}^{k-1} [T(t-t_{i-1}) - T(t-t_i)]h(t_i) + [T(t-t_{k-1}) - I]h(t_k)$$

$$= \sum_{i=1}^{k-1} T(t-t_{i-1})[h(t_i) - h(t_{i-1})] + T(t-t_{k-1})$$
\[ [h(t) - h(t_{k-1})] + T(t - t_{k-1})[h(t_k) - h(t)] + T(t) h(0) - h(t_k) \]

so that
\[
\|Au_n(t)\| \leq MV(h) + (M + 1)\|h\|\infty. 
\]

This shows that \((Au_n(t))_n\) is a bounded sequence. Consequently, there is a subsequence which converges to \(z(t) \in X\) in the weak topology. Moreover, from (2.4) it follows that
\[
z(t) = w(t) - h(t) + T(t) h(0).
\]

An standard argument shows that the full sequence \((Au_n(t))_n\) converges to \(w(t)\). As \(A\) is a closed operator this implies that \(u(t) \in D(A)\) and \(z(t) = Au(t)\).

An application of Lemma 2.1 yields that \(Au(\cdot)\) is a continuous function. On the other hand, from Lemma 2.2 we have
\[
u'_n(t) = Au_n(t) + h_n(t), \quad n \in \mathbb{N}, \quad t \neq i/n, \quad i = 1, \ldots, n - 1,
\]
so that for each \(x^* \in X^*\) we obtain
\[
<x^*, u_n(t) > = \int_0^t <x^*, Au_n(s) + h_n(s) > ds
\]
and taking limit as \(n \to \infty\), it follows that
\[
<x^*, u(t) > = \int_0^t <x^*, Au(s) + h(s) > ds
\]
which implies that
\[
u(t) = \int_0^t Au(s) ds + \int_0^t h(s) ds.
\]
This shows that \(u(\cdot)\) is a function of class \(C^1\) that satisfies (1.1)-(1.2).

A similar result holds for the second order abstract Cauchy problem ([5]).
References


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