Abstract

It is known the classification of commutative power-associative nilalgebras of dimension ≤ 4 (see, [4]). In [2], we give a description of commutative power-associative nilalgebras of dimension 5. In this work we describe Jordan nilalgebras of dimension 6.

AMS Subject Classification : 17 C 10.

*Research supported by Conicyt-Chile, Proyecto Fondecyt-Líneas Complementarias 8990001.
1. Preliminaries

Let $A$ be a commutative algebra over a field $K$. If $x$ is an element of $A$, we define $x^1 = x$ and $x^{k+1} = x^k x$ for all $k \geq 1$.

$A$ is called power-associative, if the subalgebra of $A$ generated by any element $x \in A$ is associative. An element $x \in A$ is called nilpotent, if there is an integer $r \geq 1$ such that $x^r = 0$. If any element in $A$ is nilpotent, then $A$ is called a nilalgebra. Now $A$ is called a nilalgebra of nilindex $n \geq 3$, if $y^n = 0$ for all $y \in A$ and there is $x \in A$ such that $x^{n-1} \neq 0$.

If $B, D$ are subspaces of $A$, then $BD$ is the subspace of $A$ spanned by all products $bd$ with $b$ in $B$, $d$ in $D$. Also we define $B^1 = B$ and $B^{k+1} = B^k B$ for all $k \geq 1$. If there exists an integer $n \geq 2$ such that $B^n = 0$ and $B^{n-1} \neq 0$, then $B$ is nilpotent of index $n$.

$A$ is a Jordan algebra, if it satisfies the Jordan identity $x^2 (yx) = (x^2 y) x$ for all $x, y$ in $A$. It is known that any Jordan algebra is power-associative, and also that any finite-dimensional Jordan nilalgebra (of characteristic $\neq 2$) is nilpotent (see, [5]).

We will use the following result which we give in [2]:

**Proposition 1.1** If $A$ is a Jordan nilalgebra of nilindex $n \geq 3$ with $\dim_K(A) = m \geq n$, then $n - 2 \leq \dim_K(A^2) \leq m - 2$.

Throughout, $A$ will denote a commutative nilalgebra of nilindex $n \geq 3$ over a field $K$ of characteristic $\neq 2, 3$. We will denote by $\langle x_1, ..., x_j \rangle_K$ the subspace generated over $K$ by the elements $x_1, ..., x_j$ in $A$. Also we will denote by $\alpha, \beta, ..., etc.$, the elements of field $K$. If $x \in A$ with $x^{n-1} \neq 0$, then we will denote by $X$ the subspace $\langle x, x^2, ..., x^{n-1} \rangle_K$. It is clear that $x, x^2, ..., x^{n-1}$ are linearly independent and so $\dim_K(A^2) \geq n - 2$ and $\dim_K(A^3) \geq n - 3$.

2. COMMUTATIVE NILALGEBRAS OF NILINDEX 3 AND DIMENSION 6

In this section, $A$ will denote a commutative nilalgebra of nilindex 3. It is well known that a commutative nilalgebra of nilindex 3 is a Jordan algebra (see [6], page 114).

Since $x^3 = 0$ for all $x A$, then by linearization method we obtain that the following identities are valid in $A$:

$$(2.1) \quad x^2 y + 2 (xy) x = 0, \ (xy) z + (yz) x + (zx) y = 0$$
It is clear that the identity $x^4 = (x^2)^2 = 0$ is valid in $A$, which implies that for all $x, y, z$ in $A$ we have:

$$(2.2) \quad x^2(yx) = (x^2y)x = 0, \quad 2(xy)^2 + x^2y^2 = 0$$

**Lemma 2.1** If $(A^2)^2 \neq 0$, then $\dim_K(A) \geq 8$.

**Proof.** If $(A^2)^2 \neq 0$, then there exist elements $x, y \in A$ such that $x^2y^2 \neq 0$. We note first that using (1) and (2), we obtain that:

\[
x^2(yx^2) = 0, \quad x^2(x^2y^2) = -2((x^2y^2)x) = 0, \quad x^2y^2 + 2(y^2)x = 0 \quad \text{and} \quad x^2y^2 + 2(x^2y)x = 0.
\]

We will prove that the elements $y, x, x^2, yx^2, xy^2, y, x, y^2$ are linearly independent. Let $\alpha_1y + \alpha_2x + \alpha_3x^2 + \alpha_4y^2 + \alpha_5yx^2 + \alpha_6xy^2 + \alpha_7xy + \alpha_8x^2y^2 = 0$. Multiplying by $x^2$ we obtain that $\alpha_1yx^2 + \alpha_4x^2y^2 = 0$. Thus $0 = 2y(\alpha_1yx^2 + \alpha_4x^2y^2) = 2\alpha_1y(xy^2) = -\alpha_1x^2y^2$ implies $\alpha_1 = 0$. Clearly also $\alpha_4 = 0$. Similarly we prove that $\alpha_2 = \alpha_3 = 0$. Now we have that $\alpha_5yx^2 + \alpha_6xy^2 + \alpha_7xy + \alpha_8x^2y^2 = 0$. Multiplying by $x$ we get $\alpha_6x(xy^2) + \alpha_7x(xy) = 0$. Hence $0 = 2y(\alpha_6x(xy^2) + \alpha_7x(xy)) = -\alpha_6y(x^2y^2) - \alpha_7y(xy^2) = \frac{1}{2}\alpha_7x^2y^2$ which implies $\alpha_7 = 0$. Finally it is clear that $\alpha_6 = 0$, and also that $\alpha_5yx^2 + \alpha_8x^2y^2 = 0$ implies $\alpha_5 = \alpha_8 = 0$. This proves what we wanted.

**Lemma 2.2** If $A^4 \neq 0$, then $\dim_K(A) \geq 7$.

**Proof.** By Lemma 2.1, we can suppose that $(A^2)^2 = 0$. Since $A^4 \neq 0$, there exist elements $y, x, z$ in $A$ such that $z(xy^2) \neq 0$. Now using relation (1), we obtain that $2z((yx)x) = -z(xy^2) \neq 0$. We will prove that $y, x, z, yx^2, yx, x^2, z(xy^2)$ are linearly independent. Let (1): $\alpha_1y + \alpha_2z + \alpha_3y + \alpha_4x^2 + \alpha_5yx + \alpha_6x^2 + \alpha_7z(yx^2) = 0$. Multiplying by $yx^2$ we get $0 = \alpha_1y(xy^2) + \alpha_3z(yx^2) = -\frac{1}{2}\alpha_1y^2x^2 + \alpha_3z(xy^2) = \alpha_3z(xy^2) = 0$ which implies $\alpha_3 = 0$. Multiplying (1) by $x^2$ we obtain $\alpha_1 = 0$. We note that using (1) we get $x(z(xy^2)) = -z(xy^2) = 0$ and $y(z(xy^2)) = -z(y(xy^2)) = (yx^2)(yz) = 0$. Similarly $z(y(xy^2)) = 0$. Now multiplying (1) by $2x$ we obtain $0 = 2\alpha_2x^2 + 2\alpha_5x(xy) = 2\alpha_2x^2 - \alpha_5y^2$. So $0 = y(2\alpha_2x^2 - \alpha_5y^2) = 2\alpha_2xy^2 - \alpha_5xy^2 = 2\alpha_2xy^2 + \frac{1}{2}\alpha_5y^2x^2 = 2\alpha_2x^2$. It is clear that also $\alpha_5 = 0$. Finally it is possible to prove that $\alpha_4y^2 + \alpha_6x^2 + \alpha_7z(yx^2) = 0$ implies $\alpha_4 = \alpha_6 = \alpha_7 = 0$. Therefore we conclude that $\dim_K(A) \geq 7$, as desired.

We see that Lemmas 2.1 and 2.2 imply the following result:
Corollary 2.3 If \( \dim_K(A) \leq 6 \), then \((A^2)^2 = A^4 = 0\).

Now if \( A^3 \neq 0 \), then there exist elements \( y, x \) in \( A \) such that \( yx^2 \neq 0 \). In this case it is easy to prove that \( y, x, yx^2, x^2, yx \) are linearly independent. Therefore we obtain the following result:

**Lemma 2.4** If \( A^3 \neq 0 \), then \( \dim_K(A) \geq 5 \).

We observe that when \( \dim_K(A) = 6 \), then by Proposition 1.1 we have that \( 1 \leq \dim_K(A^2) \leq 4 \). Moreover, if \( A^3 \neq 0 \) and \( \dim_K(A) = 6 \), then \( 3 \leq \dim_K(A) \leq 4 \).

**Proposition 2.5** If \( \dim_K(A) = 6 \), \( A^3 \neq 0 \) and \( \dim_K(A^2) = 4 \), then there exists a basis \( \{u_1, u_2, u_3, u_4, u_5, u_6\} \) of \( A \) such that \( u_1^2 = u_6 \), \( u_1u_2 = u_4 \), \( u_1u_5 = u_3 \), \( u_2^2 = u_5 \), \( u_2u_4 = -\frac{1}{2}u_3 \), all other products being zero.

**Proof**. We know that \((A^2)^2 = A^4 = 0\). Since \( A^3 \neq 0 \), then there exist \( y, x \) in \( A \) such that \( y, x, yx^2, yx, x^2 \) are linearly independent. Clearly \( y, x \) are not elements in \( A^2 \), and thus there exists \( z \in A \) such that \( \{y, x, yx^2, yx, x^2, z^2\} \) is a basis of \( A \). As \( z = \alpha_1 y + \alpha_2 x + \alpha_3 yx^2 + \alpha_4 yx + \alpha_5 x^2 + \alpha_6 z^2 \), then \( z^2 = (z - \alpha_0 z^2)^2 + \alpha_1 y + \alpha_2 x + \alpha_3 yx^2 + \alpha_4 yx + \alpha_5 x^2 + \alpha_6 z^2 \). From this we see that if \( y^2 \in \langle x^2, yx, yx^2 \rangle \), then \( z^2 \in \langle x^2, yx, yx^2 \rangle \), which is a contradiction. Hence \( y^2 \notin \langle x^2, yx, yx^2 \rangle \), and so \( \{y, x, yx^2, yx, x^2, y^2\} \) is a basis of \( A \). Since \( xy^2 \in A^2 \), then \( xy^2 = \alpha yx^2 + \beta yx + \gamma x^2 + \delta y^2 \). Multiplying by \( 2x \) we get \( 2\beta x(yx) + 2\delta xy^2 = -\beta yx^2 + 2\delta xy^2 = 0 \). Thus \( -\beta yx^2 + 2\delta (\alpha yx^2 + \beta yx + \gamma x^2 + \delta y^2) = 0 \), implies \( \beta = \delta = 0 \), and so \( xy^2 = \alpha yx^2 + \gamma x^2 \). But \( 0 = y(xy^2) = y(\alpha yx^2 + \gamma x^2) = \gamma yx^2 \) implies \( \gamma = 0 \), and therefore \( xy^2 = \alpha yx^2 \). Finally, if we define \( u_1 = y + \alpha x \), \( u_2 = x \), \( u_3 = yx^2 \), \( u_4 = y + \alpha x \), \( u_5 = x^2 \), \( u_6 = y^2 + 2\alpha yx + \alpha^2 x^2 \), we get \( u_1^2 = u_6 \), \( u_1u_2 = u_4 \), \( u_1u_5 = u_3 \), \( u_2^2 = u_5 \), \( u_2u_4 = -\frac{1}{2}u_3 \), all other products zero.

**Proposition 2.6** If \( \dim_K(A) = 6 \), \( A^3 \neq 0 \) and \( \dim_K(A^2) = 3 \), then there exists a basis \( \{u_1, u_2, u_3, u_4, u_5, u_6\} \) of \( A \) such that \( u_1u_2 = u_5 \), \( u_1u_6 = u_4 \), \( u_2^2 = u_6 \), \( u_2u_3 = -\beta u_6 \), \( u_2u_5 = -\frac{1}{2}u_4 \), \( u_3^2 = \delta u_4 \), \( u_3u_5 = \beta u_4 \), all other products being zero.

**Proof**. We know that \((A^2)^2 = A^4 = 0\). Since \( A^3 \neq 0 \), there exist \( y, x \) in \( A \) such that \( y, x, yx^2, yx, x^2 \) are linearly independent, and thus there exists an element \( z \in A \) such that \( \{y, x, yx^2, yx, x^2\} \) is a basis of \( A \). As \( y^2 \in A^2 \), then \( y^2 = \sigma_1 yx^2 + \sigma_2 yx + \sigma_3 x^2 \). If \( y_0 = y - \frac{1}{2}\sigma_2 x - \frac{1}{2}\sigma_1 x^2 \) we
obtain that $y_0^2 = (\sigma_3 + \frac{1}{2}\sigma_2^2)x^2$, and so $0 = y_0^2 = (\sigma_3 + \frac{1}{4}\sigma_2^2)y^2$ which implies $\sigma_3 + \frac{1}{4}\sigma_2^2 = 0$. Thus $y_0^2 = 0$ and clearly $\{y_0, x, z, y_0x^2, y_0x, x^2\}$ is a basis of $A$. Since $zx \in A^2$, then $zx = \alpha_1 y_0 x^2 + \alpha_2 y_0 x + \alpha_3 x^2$. If $z_0 = z + 2\alpha_1 y_0 x - \alpha_2 y_0 - \alpha_3 x$, we get that $\{y_0, x, z_0, y_0x^2, y_0x, x^2\}$ is a basis of $A$ with $z_0 x = 0$. Let $y_0 z_0 = \beta_1 y_0 x^2 + \beta_2 y_0 x + \beta_3 x^2$. If $z_1 = z_0 - \beta_1 x^2$, we obtain that $\{y_0, x, z_1, y_0x^2, y_0x, x^2\}$ is a basis of $A$ with $z_1 x = 0$ and $y_0 z_1 = \beta_2 y_0 x + \beta_3 x^2$. Now $0 = y_0^2 z_1 = -2y_0(y_0 z_1) = -2y_0(\beta_2 y_0 x + \beta_3 x^2) = \beta_2 y_0^2 x - 2\beta_3 y_0 x^2 = -2\beta_3 y_0 x^2$ implies $\beta_3 = 0$. Therefore we can suppose that in the basis $\{y, x, z, yx^2, yx, x^2\}$ of $A$, we have $y^2 = 0$, $zx = 0$ and $yz = \beta yx$. Let $z^2 = \delta yx^2 + \varepsilon yx + \theta x^2$. Now we have that: $z(yx) = -x(zy) - y(xz) = -x(zy) = -\beta x(yx) = \frac{1}{2}\beta yx^2$, $0 = 4(xz)z = -2xz^2 = -2x(\delta yx^2 + \varepsilon yx + \theta x^2) = -2\varepsilon x(yx) = \varepsilon yx^2$ implies $\varepsilon = 0$, and $\theta yx^2 = y(\delta yx^2 + \varepsilon yx + \theta x^2) = yz^2 = -2(yz)z = -2\beta(yzx) = -\beta yx^2$ implies $\theta = -\beta$. Thus $z^2 = \delta yx^2 - \beta^2 x^2$. Finally, if we define: $u_1 = y$, $u_2 = x$, $u_3 = z - \beta x$, $u_4 = yx^2$, $u_5 = yx$, $u_6 = x^2$, we obtain that $u_1 u_2 = u_5$, $u_1 u_6 = u_4$, $u_2^2 = u_6$, $u_2 u_3 = -\beta u_6$, $u_2 u_5 = -\frac{1}{2}u_4$, $u_2^3 = \delta u_4$, $u_3 u_5 = \beta u_4$, and other products zero.

We note that when $\dim_K(A) = 6$, then Proposition 1.1 implies $1 \leq \dim_K(A^2) \leq 4$. Suppose moreover that $A^3 = 0$ and $\dim_K(A^2) = 4$. Then there exists a subspace $A_0$ of $A$ such that $A = A_0 \oplus A^2$. Since $\dim_K(A_0) = 2$ and $A^2 = A_0^2$ we conclude that $\dim_K(A^2) \leq 3$, a contradiction. Therefore $\dim_K(A) = 6$ and $A^3 = 0$ imply $1 \leq \dim_K(A^2) \leq 3$.

**Proposition 2.7** Suppose that $\dim_K(A) = 6$, with $\dim_K(A^2) = 3$ and $A^3 = 0$.

(a) If for all $x, y \in A$ we have that $x^2, y^2, xy$ are linearly dependent, then there exist a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of $A$ such that $u_1^2 = u_4$, $u_1 u_2 = \frac{1}{8} \delta_1 \varepsilon_1^{-1} u_4 + 2\varepsilon u_5$, $u_1 u_3 = \frac{1}{4} \delta_1 \varepsilon_1^{-1} u_4 + \delta u_6$, $u_2^2 = u_5$, $u_2 u_3 = \varepsilon u_5 + \frac{1}{4} \varepsilon_1^{-1} u_6$, $u_3^2 = u_6$ with $\delta \neq 0$, all other products zero.

(b) If there exist elements $y, x$ in $A$ such that $x^2, y^2, xy$ are linearly independent, then there exist a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of $A$ such that $u_1^2 = \alpha_1 u_4 + \beta_1 u_5 + \gamma u_6$, $u_1 u_2 = \beta_5 u_5$, $u_1 u_3 = \alpha_0 u_4 + \beta_0 u_5 + \gamma_0 u_6$, $u_2^2 = u_4$, $u_2 u_3 = u_6$, $u_3^2 = u_5$, all other products zero.

**Proof.** To prove (a), we consider $x, y, z$ in $A$ such that $x^2, y^2, z^2$ are linearly independent. We will prove that $x, y, z, x^2, y^2, z^2$ are linearly independent. If $\delta_1 x + \delta_2 y + \delta_3 z + \delta_4 x^2 + \delta_5 y^2 + \delta_6 z^2 = 0$, then $\delta_1 x = -(\delta_2 y + \delta_3 z + \delta_4 x^2 + \delta_5 y^2 + \delta_6 z^2)$.
We have that 

\[ (u \cdot v, x^2, y^2, z^2) = \text{basis of } A. \]

By hypothesis \( yz < y^2, z^2 > K \), and so \( \delta_1 = 0 \). Similarly we prove that \( \delta_2 = \delta_3 = 0 \), and clearly \( \delta_4 = \delta_5 = 0 \). Therefore \( \{x, y, z, x^2, y^2, z^2\} \) is a basis of \( A \). By hypothesis \( xy = \alpha x^2 + \beta y^2, xz = \gamma x^2 + \delta z^2, yz = \varepsilon y^2 + \theta z^2, \) and also for all \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \) in \( K \), the vectors \( (\alpha_1 x + \alpha_2 y + \alpha_3 z)^2, \) \( (\alpha_1 x + \alpha_2 y + \alpha_3 z)(\beta_1 x + \beta_2 y + \beta_3 z), \) \( (\beta_1 x + \beta_2 y + \beta_3 z)^2 \) are linearly dependent.

We have that \( (\alpha_1 x + \alpha_2 y + \alpha_3 z)^2 = (\alpha_1^2 + 2\alpha_1\alpha_2 y + 2\alpha_1\alpha_3 z)x^2 + (\alpha_2^2 + 2\alpha_1\alpha_2 \beta + 2\alpha_2\alpha_3 \theta)z^2 + (\alpha_3^2 + 2\alpha_1\alpha_3 \delta)x^2 + (2\alpha_1\alpha_2^2 + 2\alpha_2^2 \beta + 2\alpha_2^2 \theta)z^2, \)

\( (\alpha_1 x + \alpha_2 y + \alpha_3 z)(\beta_1 x + \beta_2 y + \beta_3 z) = (\alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3 + \alpha_1 \beta_1 \gamma + \alpha_2 \beta_2 \gamma + \alpha_3 \beta_3 \gamma)x^2 + (\alpha_1 \beta_1 \alpha + \alpha_2 \beta_2 \alpha + \alpha_3 \beta_3 \alpha + \alpha_1 \beta_1 \beta + \alpha_2 \beta_2 \beta + \alpha_3 \beta_3 \beta)x^2 + (\alpha_1 \beta_1 \gamma + \alpha_2 \beta_2 \gamma + \alpha_3 \beta_3 \gamma)x^2 + (\alpha_1 \beta_1 \gamma + \alpha_2 \beta_2 \gamma + \alpha_3 \beta_3 \gamma)x^2, \)

and \( (\beta_1 x + \beta_2 y + \beta_3 z)^2 = (\beta_1^2 + 2\beta_1 \beta_2 \alpha + 2\beta_1 \beta_3 \beta + 2\beta_2 \beta_3 \beta)z^2 + (\beta_2^2 + 2\beta_1 \beta_2 \beta + 2\beta_2 \beta_3 \beta)x^2 + (\beta_3^2 + 2\beta_1 \beta_3 \beta + 2\beta_2 \beta_3 \beta)x^2. \)

We conclude that for all \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \) in \( K \), the vectors \( (\alpha_1^2 + 2\alpha_1\alpha_2 \beta + 2\alpha_2\alpha_3 \theta), (\alpha_1 \beta_1 + \alpha_2 \beta_2 \beta + \alpha_3 \beta_3 \beta + \alpha_1 \beta_1 \gamma + \alpha_2 \beta_2 \gamma + \alpha_3 \beta_3 \gamma), (\beta_1^2 + 2\beta_1 \beta_2 \alpha + 2\beta_1 \beta_3 \beta + 2\beta_2 \beta_3 \beta)z^2 + (\beta_2^2 + 2\beta_1 \beta_2 \beta + 2\beta_2 \beta_3 \beta)x^2 + (\beta_3^2 + 2\beta_1 \beta_3 \beta + 2\beta_2 \beta_3 \beta)x^2 \) in \( K^3 \) are linearly dependent, which implies that \( \beta = 2\delta \epsilon, \delta = 2\beta \theta, \theta = 2\gamma \beta \), \( \beta = 2\delta \epsilon, \delta = 2\beta \theta, \theta = 2\gamma \beta \). We observe that if \( 0 \in \{\alpha, \beta, \gamma, \delta, \epsilon, \theta\} \), then \( \alpha = \beta = \gamma = \delta = \epsilon = \theta = 0 \). In this case \((x + y)^2, (x + z)^2 \) and \((x + y)(x + z)\) are linearly independent, a contradiction. Therefore \( \alpha, \beta, \gamma, \delta, \epsilon, \theta \) are not zero and we get \( \alpha = \frac{1}{5} \delta, \gamma = \frac{1}{4} \delta, \theta = \frac{1}{3} \delta \). Finally, if we define \( u_1 = x, u_2 = y, u_3 = z, u_4 = x^2, u_5 = y^2, u_6 = z^2 \), we obtain \((a)\).

Suppose now that there exist \( x, y \) in \( A \) such that \( x^2, y^2, xy \) are linearly independent. In this case it is easy to prove that \( x, y, x^2, y^2, xy \) are linearly independent. Let \( u \) be an element in \( A \) such that \( \{u, x, y, x^2, y^2, xy\} \) is a basis of \( A \). Since \( ux \in A^2 \), then \( ux = \alpha x^2 + \beta y^2 + \gamma xy \). If \( u_0 = u - ax - by \), then \( u_0 x = \beta y^2 \). Finally, if we define \( u_1 = u_0, u_2 = x, u_3 = y, u_4 = x^2, u_5 = y^2, u_6 = xy \), we get \((b)\).

**Proposition 2.8** If \( \text{dim}_K(A) = 6 \), \( A^3 = 0 \) and \( \text{dim}_K(A^2) = 2 \), then there exists a basis \( \{u_1, u_2, u_3, u_4, u_5, u_6\} \) of \( A \) such that \( u_1^2 = \alpha_1 u_5 + \alpha_2 u_6, u_1 u_2 = \alpha_3 u_5 + \alpha_4 u_6, u_1 u_4 = \alpha_5 u_5 + \alpha_6 u_6, u_2^2 = \alpha_7 u_5 + \alpha_8 u_6, u_2 u_4 = \alpha_9 u_5 + \alpha_{10} u_6, u_3^2 = u_5, u_3 u_4 = u_6, u_4^2 = \varepsilon u_5 \), and other products zero.

**Proof.** It is possible to prove that there exist elements \( y, x \) in \( A \) such that \( x, y, x^2, yx \) are linearly independent, and \( y^2 = \varepsilon x^2 \) (see, [4]). We consider \( u, v \in A \) such that \( \{u, v, x, y, x^2, yx\} \) is a basis of \( A \). Since \( ux \) and \( vx \) are elements in \( A^2 \), then \( ux = \alpha x^2 + \beta y^2 + \gamma xy \). If \( u_0 = u - ax - by \) and \( v_0 = v - \gamma x - \delta y \), then \( \{u_0, v_0, x, y, x^2, yx\} \) is a basis of \( A \) with \( u_0 x = v_0 x = 0 \). If we define \( u_1 = u_0, u_2 = v_0, u_3 = x, u_4 = y, \)
Since \( \alpha x \) is easy to prove that \( 2(\alpha x) = 0 \), we obtain that \( u_1^2 = \alpha_1 u_5 + \alpha_2 u_6, u_1 u_2 = \alpha_3 u_5 + \alpha_4 u_6, u_1 u_4 = \alpha_5 u_5 + \alpha_6 u_6, u_2^2 = \alpha_7 u_5 + \alpha_8 u_6, u_2 u_4 = \alpha_9 u_5 + \alpha_10 u_6, u_3^2 = u_5, u_3 u_4 = u_6, u_4^2 = \varepsilon u_5, \) and other products being zero. \( \square \)

**Proposition 2.9** If \( \dim_K(A) = 6 \), \( A^3 = 0 \) and \( \dim_K(A^2) = 1 \), then there exists a basis \( \{u_1, u_2, u_3, u_4, u_5, u_6\} \) such that \( u_1^2 = u_6, u_2^2 = \beta u_6, u_3^2 = \gamma u_6, u_4^2 = \delta u_6, u_5^2 = \varepsilon u_6, \) all other products being zero.

**Proof.** There is an element \( u_1 \) in \( A \) such that \( u_1^2 \neq 0 \), and so \( A^2 = < u_1^2 >_K \).

We can write \( A \) as a direct sum \( A = Ku_1^2 \oplus A_0 \), where \( A_0 = Ku_1 \oplus W \) for some subspace \( W \). The map \( f : A_0 \times A_0 \to K \) defined by \( xy = f(x, y)u_1^2 \) for all \( x, y \) in \( A_0 \) is a symmetric bilinear form. It is known that there is a basis \( \{u_1, u_2, u_3, u_4, u_5\} \) of \( A_0 \) such that \( f(u_i, u_j) = 0 \), if \( i \neq j \). Finally, if \( u_6 = u_1^2 \) we have that \( \{u_1, u_2, u_3, u_4, u_5, u_6\} \) is a basis of \( A \) such that \( u_1^2 = u_6, u_2^2 = \beta u_6, u_3^2 = \gamma u_6, u_4^2 = \delta u_6, u_5^2 = \varepsilon u_6, \) all other products being zero. \( \square \)

3. JORDAN NILALGEBRAS OF NILINDEX 4 AND DIMENSION 6

In this section, \( A \) is a Jordan nilalgebra of nilindex 4 and dimension 6. Therefore the identities \( x^2(yx) = (x^2)y)x \) and \( x^4 = (x^2)2 = 0 \) are valid in \( A \). By linearization we obtain that also are valid in \( A \) the following identities:

(3.1) \( x^2 y^2 + 2(xy)^2 = 0 \)

(3.2) \( x^2(yx) = (x^2)y)x = 0 \)

In [3], we prove that any Jordan nilalgebra of nilindex \( n \geq 4 \) and dimension \( k \) with \( n + 1 \leq k \leq n + 2 \), is nilpotent of index \( n \). From this we conclude that \( A^4 = 0 \).

**Proposition 3.1** If \( (A^2)^2 \neq 0 \), then there exists a basis \( \{u_1, u_2, u_3, u_4, u_5, u_6\} \) of \( A \) such that \( u_1^2 = u_3, u_2^2 = u_4, u_6^2 = -\frac{1}{2}u_5, u_1 u_2 = u_6, u_3 u_4 = u_5, \) all other products being zero.

**Proof.** Since \( (A^2)^2 \neq 0 \), there exist \( x, y \in A \) such that \( x^2 y^2 \neq 0 \). We know that \( 2(xy)^2 = -x^2 y^2 \neq 0 \), \( A^4 = 0 \) and moreover \( A(A^2)^2 \subset AA^3 = A^4 = 0 \).

We will prove that \( x, y, x^2, y^2, x^2 y^2, xy \) are linearly independent. It is easy to prove that \( x^2, y^2, x^2 y^2, xy \) are linearly independent. Now if \( \alpha x + \beta y + \gamma x^2 + \delta y^2 + \varepsilon x^2 y^2 + \theta xy = 0 \), then \( \alpha x + \beta y = -(\gamma x^2 + \delta y^2 + \varepsilon x^2 y^2 + \theta xy = 0 \).
Proof. Since \( 2 \leq \dim_K (A^3) \leq 4 \), then \( 1 \leq \dim_K (A^3) \leq 3 \). Suppose that \( \dim_K (A^3) = 3 \). Then there exist elements \( y, z, u, v, x \) in \( A \) such that \( A^3 = \langle uy, vz, x^3 \rangle \). Clearly \( x^2 \notin A^3 \), and so \( A^2 = \langle x^2, uy, vz, x^3 \rangle \). Hence \( y^2 = \alpha x^2 + \beta uy + \gamma vz + \delta x^3 \) and \( z^2 = \alpha_0 x^2 + \beta_0 uy + \gamma_0 vz + \delta_0 x^3 \).

Since \( A^4 = 0 \), we obtain \( uy^2 = \alpha u x^2 \) and \( vz^2 = \alpha_0 v x^2 \) with \( \alpha \neq 0 \) or \( \alpha_0 \neq 0 \). Therefore \( A^2 = \langle x^2, uy, vz, x^3 \rangle \). Now it is easy to prove that \( u, v, x, x^2, uy, vz, x^3 \) are linearly independent, a contradiction. Therefore \( 1 \leq \dim_K (A^3) \leq 2 \), as desired. \( \square \)

By Proposition 3.1 we know that there is a unique nilalgebra such that \( (A^2)^2 \neq 0 \). In the following, we assume that \( (A^2)^2 = 0 \).

Proposition 3.3 Suppose that \( \dim_K (A^2) = 4 \) and \( \dim_K (A^3) = 2 \).

(a) If for all \( y, x \in A \) we have that \( yx^2, x^3 \) are linearly dependent, then there exists a basis \( \{u_1, u_2, u_3, u_4, u_5, u_6\} \) of \( A \) such that \( u_1^2 = u_6, u_1 u_2 = u_3, u_1 u_4 = \gamma u_2 + \delta u_3 + \varepsilon u_5 + \theta u_6, u_1 u_5 = u_6, u_2^2 u_4 = u_3, u_4^2 = u_5, u_4 u_5 = u_6 \), all other products being zero.

(b) If there exist elements \( y, x \) in \( A \) such that \( yx^2, x^3 \) are linearly independent, then there exists a basis \( \{u_1, u_2, u_3, u_4, u_5, u_6\} \) of \( A \) such that \( u_1^2 = u_6, u_1 u_2 = \alpha u_3 + \beta u_5 + \gamma u_6, u_1 u_3 = u_4, u_1 u_6 = \delta u_4 + \varepsilon u_5, u_2^2 = u_3, u_2 u_3 = u_5, u_2 u_6 = \theta u_4 + \sigma u_5 \), all other products being zero.
Proof. To prove (a), we consider an element $x \in A$ with $x^3 \neq 0$. By hypothesis, we have that for all $y \in A : yx^2 \in < x^3 >_K$. As $A^4 = 0$, we have that $J = < x^2, x^3 >_K$ is an ideal of $A$, and moreover $A^2$ is not a subset of $J$. Now if $y^3 \in J$ for all $y \in A$, then the quotient algebra $A/J$ is a nilalgebra of nilindex 3 with dim$_K(A) = 4$ and $A^3 \neq 0$ which is a contradiction, since by Lemma 2.4 we know that dim$_K(A) \geq 5$. Therefore there exists $y \in A$ such that $y^3 \notin < x^2, x^3 >_K$. By hypothesis $yx^2 = \alpha x^3$, $xy^2 = \beta y^3$. Now it is possible to prove that $x, y, x^2, y^3, y^2$ are linearly independent, and so $xy = \gamma_0 x^2 + \delta_0 x^3 + \varepsilon_0 y^2 + \theta_0 y^3$. By hypothesis for all $\gamma_1, \delta_1, \alpha_1, \beta_1$ in $K$, we have that the vectors $(\gamma_1 x + \delta_1 y)^3$, $(\alpha_1 x + \beta_1 y)(\gamma_1 x + \delta_1 y)^2$ are linearly dependent. Now we have that $xy^2 = \gamma_1 x + \delta_1 y)^3 = (\gamma_1^3 + 2\gamma_1^2 \delta_1 \gamma_0 + \gamma_1^2 \delta_1 \alpha + 2\gamma_1 \delta_1 \gamma_0^2 \alpha) x^3$. Therefore we can assume that $\gamma_1, \delta_0, \alpha_1, \beta_1$ in $K$ the vectors $(\gamma_1 x + \delta_1 y)^3, (\gamma_1 x + \delta_1 y)^2$ are linearly dependent. We conclude that for all $\gamma_1, \delta_0, \alpha_1, \beta_1, \delta_1 \gamma_0 \gamma_1 + \gamma_1^2 \beta \gamma_0 + 2\gamma_1 \delta_1 \gamma_0 \beta + \gamma_1^2 \gamma_0 \beta)$ and $\gamma_1^2 \alpha_1 + 2\gamma_1 \alpha_1 \delta_1 \gamma_0 + \gamma_1^2 \beta \alpha + 2\gamma_1 \delta_1 \beta \gamma_0 \beta + \gamma_1^2 \gamma_0 \beta)$ in $K^2$ are linearly dependent, which implies that $\alpha \beta = 1$. Finally, if $u_1 = x, u_2 = x^2, u_3 = x^3, u_4 = \beta y, u_5 = \beta^2 y^2, u_6 = \beta^3 y^3$, we obtain (a). To prove (b), we consider $y, x \in A$ such that $yx^2, x^3$ are linearly independent. Then $A^3 = < yx^2, x^3 >_K$ and $x^2, yx^2, x^3$ are linearly independent. As dim$_K(A^2) = 4$, there exists $z \in A$ such that $A^2 = < x^2, yx^2, x^3, z^2 >_K$. It is easy to prove that $\{y, x, x^2, yx^2, x^3, z^2\}$ is a basis of $A$. Now if $z = \alpha_1 y + \alpha_2 x + \alpha_3 x^2 + \alpha_4 y x^2 + \alpha_5 x^3 + \alpha_6 x^2$, then $z^2 - (\alpha_1^2 y^2 + 2\alpha_1 \alpha_2 xy + \alpha_2^2 y^2) \in A^3$ which implies $\alpha_1 \neq 0$. If $y_0 = \alpha_1 y + \alpha_2 x$, then $y_0^2 \notin < x^2, yx^2, x^3 >_K$. Therefore we can assume that there exist elements $y, x \in A$ such that $\{y, x, x^2, yx^2, x^3, y^2\}$ is a basis of $A$ with $yx = \alpha x^2 + \beta x^3 + \gamma y^2$, $y^3 = \delta y x^2 + \varepsilon x^3$ and $xy^2 = \theta y x^2 + \sigma x^3$. If we define $u_1 = y, u_2 = x, u_3 = x^2, u_4 = y x^2, u_5 = x^3, u_6 = y^2$, we obtain (b). 

Proposition 3.4 If dim$_K(A^2) = 4$ and dim$_K(A^3) = 1$, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of $A$ such that $u_1 = u_5, u_1 u_2 = u_6, u_1 u_5 = \beta u_4, u_1 u_6 = \gamma u_4, u_2 = u_3, u_2 u_3 = u_4, u_2 u_5 = \delta u_4, u_2 u_6 = \varepsilon u_4$, all other products being zero.

Proof. We consider $x \in A$ such that $x^3 \neq 0$. Since dim$_K(A^2) = 4$, there are $y, z \in A$ such that $A^2 = < x^2, x^3, y^2, z^2 >_K$. We have that $y, x, x^2, x^3, y^2, z^2$ are linearly independent. In fact: $\alpha_1 y + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 x^4 + \alpha_6 x^5$
\(\alpha_5 y^2 + \alpha_6 z^2 = 0\), then \(\alpha_1 y = -(\alpha_3 x + \alpha_4 x^2 + \alpha_4 x^3 + \alpha_5 y^2 + \alpha_6 z^2)\) which implies \(\alpha_1^2 y^2 = \alpha_1^2 x^2 + v\) with \(v \in A^3 = < x^3 >_K\). Hence \(\alpha_1 = \alpha_2 = 0\), and so \(\{y, x^2, y^2, x^3, z^2\}\) is a basis of \(A\). If \(z = \beta_1 y + \beta_2 x + \beta_3 y^2 + \beta_4 x^3 + \beta_5 y^2 + \beta_6 z^2\), then \(x^2 = (\beta_1 y^2 + 2\beta_2 y x + \beta_3 x^2) \in A^3\) which implies that \(yx \notin < x^2, x^3, y^2 >_K\), and therefore \(A^2 = < x^2, x^3, y^2, yx >_K\). We see that as \(yx^2 = \alpha x^3\), then \(x^2(y - \alpha x^2) = 0\). Therefore we can assume that in the basis \(\{y, x^2, x^3, y^2, yx\}\) of \(A\) we have that \(yx^2 = 0\), and moreover \(y^3 = \beta x^3, y(yx) = \gamma x^3, xy^2 = \delta x^3, x(yx) = \varepsilon x^3\). Finally, if we define \(u_1 = y, u_2 = x, u_3 = x^2, u_4 = x^3, u_5 = y^2, u_6 = yx\), we obtain our Proposition.

**Proposition 3.5** If \(\dim_K (A^2) = 3\) and \(\dim_K (A^3) = 2\), then there exists a basis \(\{u_1, u_2, u_3, u_4, u_5, u_6\}\) of \(A\) such that \(u_1 = \gamma_1 u_4 + \gamma_2 u_5 + \gamma_3 u_6, u_1 u_2 = \delta_1 u_4 + \delta_1 u_4 + \delta_2 u_5 + \delta_3 u_6, u_1 u_3 = \beta u_5, u_1 u_4 = \lambda_1 u_5 + \lambda_2 u_6, u_2 = \varepsilon_1 u_4 + \varepsilon_2 u_5 + \varepsilon_3 u_6, u_2 u_4 = u_5, u_3^2 = u_4, u_3 u_4 = u_6\), all other products being zero.

**Proof.** By Proposition 3.1, it is clear that \((A^2)^2 = 0\). We consider \(x \in A\) such that \(x^3 \neq 0\). We note that if \(I = < x^2, x^3 >_K\) is an ideal of \(A\), then \(yx^2 \in < x^3 >_K\) for all \(y \in A\), and \(A^3\) is not a subset of \(I\). If \(I\) is an ideal of \(A\) and \(z^3 \in I\) for all \(z \in A\), then the quotient algebra \(\overline{A} = A/I\) is of nilindex 3 with \(\overline{A^3} = 0\), which implies \(\dim_K(\overline{A^3}) \geq 5\), a contradiction. Hence if \(I\) is an ideal of \(A\), then there is \(y \in A\) such that \(y^3 \notin I\), and so \(A^2 = < x^2, x^3, y^3 >_K\). Since \(y^2 \in A^2 = < x^2, x^3, y^3 >_K\), then \(y^3 \in < yx^2 >_K < x^3 >_K\), a contradiction. Therefore we conclude that \(I = < x^2, x^3 >_K\) is not an ideal of \(A\) and so there exists an element \(y \in A\) such that \(yx^2, x^3\) are linearly independent. In this case it is possible to prove that \(y, x, x^2, yx^2, x^3\) are linearly independent, and thus \(A^2 = < x^2, yx^2, x^3 >_K\) and \(A^3 = < yx^2, x^3 >_K\). If \(yx = \beta_1 x^2 + \beta_2 yx^2 + \beta_3 x^3\), then \(yx^2 = \beta_1 x^2 + \beta_3 x^3 \in < x^3 >_K\) where \(x_0 = x - \beta x^2\). Thus we can suppose that \(yx = \beta_1 x^2 + \beta_3 x^3\), which implies \(y_0 x = 0\) where \(y_0 = y - \beta_1 x - \beta_3 x^2\). Therefore we can assume that \(y, x, x^2, yx^2, x^3\) are linearly independent with \(yx = 0\). Now it is easy to find an element \(z \in A\) such that \(\{z, y, x, x^2, yx^2, x^3\}\) is a basis of \(A\) with \(xz = \beta yx^2\). Moreover we have that \(z^2 = \gamma_1 x^2 + \gamma_2 yx^2 + \gamma_3 x^3, yz = \delta_1 x^2 + \delta_2 yx^2 + \delta_3 x^3, y^2 = \varepsilon_1 x^2 + \varepsilon_2 yx^2 + \varepsilon_3 x^3, z x^2 = \lambda_1 yx^2 + \lambda_2 x^3\). If we define \(u_1 = z, u_2 = y, u_3 = x, u_4 = x^2, u_5 = yx^2, u_6 = x^3\), we obtain our Proposition.

**Proposition 3.6** If \(\dim_K (A^2) = 3\) and \(\dim_K (A^3) = 1\), then there exists a basis \(\{u_1, u_2, u_3, u_4, u_5, u_6\}\) of \(A\) such that \(u_1^2 = \alpha_1 u_4 + \alpha_2 u_5 + \alpha_3 u_6\),
Clearly \((A^2)^2 = 0\). We consider \(x \in A\) with \(x^3 \neq 0\). Then \(A^3 = \langle x^3 \rangle_K\) and there is \(y \in A\) such that \(A^2 = \langle x^2, x^3, y^2 \rangle_K\). It is easy to show that \(y, x, x^2, x^3, y^2\) are linearly independent. It is easy to find an element \(z \in A\) such that \(\{z, y, x, x^2, x^3, y^2\}\) is a basis of \(A\) with \(zx = y^2\).

If we define \(u_1 = z, u_2 = y, u_3 = x, u_4 = x^2, u_5 = x^3, u_6 = y^2\), we obtain our Proposition.

**Proposition 3.7** If \(\dim_K(A^2) = 2\) and \(\dim_K(A^3) = 1\), then there exists a basis \(\{u_1, u_2, u_3, u_4, u_5, u_6\}\) of \(A\) such that \(u_1^2 = \alpha_1 u_5 + \alpha_2 u_6, u_1 u_2 = \beta_1 u_5 + \beta_2 u_6, u_1 u_3 = \gamma_1 u_5 + \gamma_2 u_6, u_1 u_4 = \alpha u_6, u_2^2 = \delta_1 u_5 + \delta_2 u_6, u_2 u_3 = \epsilon_1 u_5 + \epsilon_2 u_6, u_2 u_5 = \beta_6, u_3^2 = \lambda_1 u_5 + \lambda_2 u_6, u_3 u_5 = \gamma u_6, u_4^2 = u_5, u_4 u_5 = u_6\), all other products being zero.

**Proof.** We consider \(x \in A\) with \(x^3 \neq 0\). Then \(A^3 = \langle x^3 \rangle_K\) and \(A^2 = \langle x^2, x^3 \rangle_K\). It is easy find elements \(y, z, v \in A\) such that \(\{y, z, v, x, x^2, x^3\}\) is a basis of \(A\) with \(yx = zx = vx = 0\). Now we have that \(y^2 = \alpha x^2 + \alpha_2 x^3, yz = \beta_1 x^2 + \beta_2 x^3, yv = \gamma_1 x^2 + \gamma_2 x^3, yx^2 = \alpha x^3, z^2 = \delta_1 x^2 + \delta_2 x^3, zv = \epsilon_1 x^2 + \epsilon_2 x^3, vz = \beta x^3, v^2 = \lambda_1 x^2 + \lambda_2 x^3, vx^2 = \gamma x^3\). Finally, if \(u_1 = y, u_2 = z, u_3 = v, u_4 = x, u_5 = x^2, u_6 = x^3\), we obtain our Proposition.

### 4. JORDAN NILALGEBRAS OF NILINDEX k AND DIMENSION 6 WITH k ≥ 5

In [2], we describe Jordan nilalgebras of nilindex n and dimension \(n + 1\). In this work, we find the following results:

**Proposition 4.1** If \(A\) is a Jordan nilalgebra of nilindex 5 and dimension 6, \(\dim_K(A^2) = 4\) and \(\dim_K(A^3) = 2\), then there exists a basis \(\{u_1, u_2, u_3, u_4, u_5, u_6\}\) of \(A\) such that \(u_1^2 = \alpha u_2 + \gamma_2 u_4 + \gamma_3 u_5 + \gamma_4 u_6, u_1 u_2 = \beta_0 u_5 + \gamma_0 u_6, u_1 u_3 = u_2, u_1 u_4 = -2\beta u_6, 2u_2^2 = \beta(\alpha - 4\beta)u_6, u_2 u_3 = \beta u_5 + \gamma u_6, u_3^2 = u_4, u_3 u_4 = u_5, u_3 u_5 = u_6, u_4^2 = u_6\), all other products being zero. Moreover, if \(\beta = 0\) then \(\gamma_2 = \beta_0 = 0\), if \(\beta \neq 0\) and \(\alpha = 4\beta\) then \(\gamma_2 = -4\beta^2, \beta_0 = -2\beta^2\), if \(\beta \neq 0\) and \(\alpha \neq 4\beta\) then \(\alpha = -4\beta, \gamma_2 = -4\beta^2\) and \(\beta_0 = -6\beta^2\).
Proposition 4.2 If $A$ is a Jordan nilalgebra of nilindex 5 and dimension 6, $\dim_K(A^2) = 4$ and $\dim_K(A^3) = 3$, then there exists a basis \{u_1, u_2, u_3, u_4, u_5, u_6\} of $A$ such that $u_1 u_4 = u_2$, $u_1^2 = \lambda u_2 + \delta u_4 + \gamma u_5 + \varepsilon u_6$, $u_1 u_2 = \delta u_6$, $u_3^2 = u_4$, $u_3 u_4 = u_5$, $u_3 u_5 = u_6$, $u_4^2 = u_6$, all other products zero.

Proposition 4.3 If $A$ is a Jordan nilalgebra of nilindex 5 and dimension 6 and $\dim_K(A^2) = 3$, and then there exists a basis \{u_1, u_2, u_3, u_4, u_5, u_6\} of $A$ such that $u_1 u_3 = \alpha u_5$, $u_1^2 = \beta u_5 + \gamma u_6$, $u_2 u_3 = \alpha_0 u_5$, $u_2^2 = \beta_0 u_5 + \gamma_0 u_6$, $u_1 u_2 = \delta u_5 + \varepsilon u_6$, $u_3^2 = u_4$, $u_3 u_4 = u_5$, $u_3 u_5 = u_6$, $u_4^2 = u_6$, all other products zero.

In [1], the authors proved the following result:

Proposition 4.4 If $A$ is a Jordan nilalgebra of nilindex 6 and dimension 6, then there exists a basis \{u_1, u_2, u_3, u_4, u_5, u_6\} of $A$ such that $u_1^2 = \beta u_5 + \gamma u_6$, $u_1 u_2 = \alpha u_5$, $u_2^2 = u_3$, $u_2 u_3 = u_4$, $u_2 u_4 = u_5$, $u_2 u_5 = u_6$, $u_3^2 = u_5$, $u_3 u_4 = u_6$, all other products zero.

Moreover in this case it is possible to find five classes of algebras which are not isomorphic (see [1], Theorem 3).

Remark Finally, it is clear that there is a unique Jordan nilalgebra of nilindex 7 and dimension 6.

5. REFERENCES


Received : June, 2002

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