Abstract

We study the solvability of the problem

$$-\Delta_p u = g(x,u) + h \quad \text{in} \; \Omega; \; u = 0 \quad \text{on} \; \partial \Omega,$$

when the nonlinearity $g$ is assumed to lie asymptotically between 0 and the second eigenvalue $\lambda_2$ of $-\Delta_p$. We show that this problem is nonresonant.

Key words Eigenvalue, nonresonance, p-laplacian, variational approach.
1. Introduction

In this paper we consider nonresonant problems of the form

\[
\begin{cases}
-\Delta_p u = g(x,u) + h & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain, \(\Delta_p = \text{div} (|\nabla u|^{p-2} \nabla u)\) denotes the p-laplacian, \(h \in W^{-1,p'}(\Omega)\) and \(g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}\) is a Carathéodory function such that

\[g_0 R(x) = \sup_{|s| \leq R} |g(x, s)| \in L^{p'}(\Omega) \text{ for each } R > 0.\]

We are interested in the conditions to be imposed on \(g\) and on the primitive \(G (G(x,s) = \int_0^s g(x,t) \, dt)\) in order to have the nonresonance i.e. the solvability of (1.1) for every \(h\) in \(W^{-1,p'}(\Omega)\).

First we introduce some notations.

\(\lambda_1(m), \lambda_2(m)\) denote the first and the second eigenvalue of the weighted nonlinear eigenvalue problem

\[
-\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega,
\]

where \(m(.) \in L^{\infty}(\Omega)\) is a weight function which is positive on subset of positive measure. \(\lambda_1(\text{resp } \lambda_2)\) denotes \(\lambda_1(1)\) (resp \(\lambda_2(1)\)).

It is known that \(\lambda_1(m) > 0\) is a simple eigenvalue, \(\varphi_1\) the normalized \(\lambda_1\)-eigenfunction does not change sign in \(\Omega\) and \(\sigma(-\Delta_p, m(\cdot)) \cap [\lambda_1(m), \lambda_2(m)] = \emptyset\), where \(\sigma(-\Delta_p)\) is the spectrum of \(-\Delta_p\) (cf [2], [4]).

The inequality \(\alpha(x) \leq \beta(x)\) means that \(\alpha(x) \leq \beta(x)\) for a.e. \(x \in \Omega\) with a strict inequality \(\alpha(x) < \beta(x)\) holding on subset of positive measure. \(||.||\) denotes the norm in \(W_0^{1,p}(\Omega)\), \(||.||_p\) denotes the norm in \(L^p(\Omega)\). \(E(\lambda_1)\) is the subspace of \(W_0^{1,p}(\Omega)\) spanned by \(\varphi_1\) and \(E(\lambda_1)^\perp = \{h \in W^{-1,p'}(\Omega) : \int_{\Omega} h \varphi_1 = 0\}\).  

Now we are ready to present the main results, let us consider the hypotheses

\[(H_1) \quad k(x) = \limsup_{|s| \to +\infty} \frac{g(x,s)}{|s|^{p-2}s} < \lambda_2.\]

\[(H_2) \quad \liminf_{|s| \to +\infty} \frac{g(x,s)}{|s|^{p-2}s} = 0.\]
Nonresonance below the second eigenvalue ...

\( (H_3) \quad \lambda_1 \leq l_+(x) = \liminf_{|s| \to +\infty} \frac{g(x,s)}{|s|^{p-2}s}. \)

\( (H_4) \quad \lambda_1 \leq L_+(x) = \liminf_{|s| \to +\infty} \frac{pG(x,s)}{|s|^p}. \)

\( (H_5) \quad \int_{\Omega} G(x,t\varphi_1(x)) \, dx - \frac{|t|^p}{p} \to +\infty \quad \text{as} \quad |t| \to +\infty. \)

All these limits are taken uniformly for a.e. \( x \in \Omega. \)

**Theorem 1.1.** Assume \((H_1), (H_2), (H_3), (H_4)\) and \((H_5)\), then for any given \( h \in E(\lambda_1)^\perp \), the problem \((1.1)\) possesses a nontrivial solution.

**Remark 1.1.** We can replace \((H_3)\) by the following condition of Landesman-Lazer type

\[ \int_{v>0} (L_+(x) - \lambda_1)|v|^p > 0; \quad v \in E(\lambda_1) \setminus \{0\}. \]

In the nonlinear case \((p \neq 2)\), when the potential \( G \) satisfies \( \limsup_{|s| \to +\infty} \frac{pG(x,s)}{|s|^p} < \lambda_2 \), problems of nonresonance has been studied by just a few authors, a contribution in this direction is [3] where the authors studied the case when the perturbation \( g \) stays asymptotically between \( \lambda_1 \) and \( \lambda_2 \).

**2. Preliminary results**

From the conditions \((g_0), (H_1), (H_2)\) and \((H_3)\) it follows that there exists constant \( a > 0 \) and function \( b(.) \in L^p(\Omega) \) such that

\[ |g(x,s)| \leq a|s|^{p-1} + b(x), \quad (1) \]

then the critical points \( u \in W_0^{1,p}(\Omega) \) of the \( C^1 \) functional

\[ I(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p - \int_{\Omega} G(x,u(x)) - \int_{\Omega} hu \]

are the weak solutions of the problem \((1.1)\).

To get a critical point of \( I \), we will apply the following version of the Mountain-Pass theorem which is proved in [9], with condition \((C)\).
**Theorem 2.1.** Let $I \in C^1(X, \mathbb{R})$ satisfying condition $(PS)$, $\beta \in \mathbb{R}$ and let $Q$ be a closed connected compact subset such that $\partial Q \cap (-\partial Q) \neq \emptyset$. Assume that

1) $\forall K \in A_2$ there exists $v_k \in K$ such that $I(v_k) \geq \beta$ and $I(-v_k) \geq \beta$.
2) $\alpha = \sup I_{\partial Q} < \beta$.
3) $\sup I_Q < +\infty$.

Then $I$ has a critical value $c \geq \beta$.

Recall that $A_2 = \{ K \subset X : K \text{ is compact, symmetric and } \gamma(K) \geq 2 \}$, $\gamma(K)$ denotes the genus of $K$.

**Remark 2.1.** The condition $(C)$ is clearly implied by the Palais-Smale condition $(PS)$.

Let $(u_n) \subset W^{1,p}_0(\Omega)$ be an unbounded sequence such that

$$I'(u_n) \to 0 \text{ and } I(u_n) \text{ is bounded}$$

defining $v_n = \frac{u_n}{\|u_n\|}$ and $g_n(x) = \frac{g(x, u_n)}{\|u_n\|^{p-1}}$. Passing to a subsequence still denoted by $(v_n)$ (resp $(g_n)$), we may assume that

$v_n \rightharpoonup v \text{ weakly in } W^{1,p}_0(\Omega)$.

$v_n(x) \to v(x) \text{ a.e. } x \in \Omega$.

$|v_n(x)| \leq z(x) \quad z(.) \in L^p(\Omega)$.

$g_n \rightharpoonup \tilde{g} \text{ weakly in } L^{p'}(\Omega)$.

**Lemma 2.1.** Assume $(H_1)$, $(H_2)$ and $(H_3)$, then we have

1) $\|v\| = 1$ and $-\Delta_p v = m(.)|v|^{p-2}v$ where $0 \leq m(.) < \lambda_2$.
2) $v(x) > 0 \text{ p.p. } x \in \Omega$. 

Proof. By (1), we have

\[ I'(u_n) = -\Delta_p u_n - g(x, u_n) - h, \]

then

\[ -\Delta_p v_n = \frac{I'(u_n)}{\|u_n\|^{p-1}} + g_n + \frac{h}{\|u_n\|^{p-1}}, \]  

(3)

hence

\[ \lim_{n \to +\infty} < -\Delta_p v_n, v_n - v > = 0. \]  

(4)

Since \(-\Delta_p\) is of type \(S^+\), from (4) we conclude

\[ v_n \to v \quad \text{strongly in} \quad W_0^{1,p}(\Omega), \]

so that

\[ \|v\| = 1. \]  

(5)

Passing to the limit in (3), we obtain

\[ -\Delta_p v = \tilde{g}, \]  

(6)

hence (5) and (6) give

\[ \int_{\Omega} \tilde{g} v = 1. \]  

(7)

Let us define

\[ m(x) = \begin{cases} \frac{\tilde{g}}{|v|^{p-2}v} & \text{if} \quad v \neq 0 \\ \frac{1}{2\lambda_2} & \text{if} \quad v = 0 \end{cases} \]

Combining the hypotheses \((H_1), (H_2)\) and \((H_3)\), we show that

\[ 0 \leq m(x) < \lambda_2, \]  

(8)

and

\[ \tilde{g} = 0 \quad \text{if} \quad v(x) = 0. \]  

(9)

(The results (8) and (9) are standard cf [6] e.g.)

Using (6), we have

\[ -\Delta_p v = m(x)|v|^{p-2}v. \]  

(10)

To complete the proof of Lemma 2.1, we need to show that \(v > 0\) p.p. \(x \in \Omega\).

From (7), (8) and (10) we deduce that

\[ m(.) \in L^\infty(\Omega), \quad 0 \leq m(.) \]  

(11)
and

\[ 1 \in \sigma(-\Delta_p, m(.)). \]  \hspace{1cm} (12)

In view of (8) and the strict monotonicity of \( \lambda_2 \) (cf [4]) we get

\[ \lambda_2(m(.)) > \lambda_2(\lambda_2(1)), \]

that is

\[ \lambda_2(m(.)) > 1. \]  \hspace{1cm} (13)

Combining (11), (12), (13) and the fact that

\[ \sigma(-\Delta_p, m(.)) \cap [\lambda_1(m), \lambda_2(m)[ = \emptyset, \]

we conclude

\[ 1 = \lambda_1(m) \text{ and } v \in E(\lambda_1(m)) \setminus \{0\}, \]  \hspace{1cm} (14)

hence \( v \) does not change sign in \( \Omega \). Assume that \( v < 0 \), then we have

\[ u_n(x) = \|u_n\|v_n \to -\infty \quad p.p. \ x \in \Omega, \]  \hspace{1cm} (15)

from (7) and (8), we deduce

\[ \tilde{g} \leq 0. \]  \hspace{1cm} (16)

On the other hand

\[ \int_\Omega \frac{g(x, u_n(x))}{\|u_n\|^{p-1}} = \int_\Omega \frac{g(x, u_n(x))}{|u_n|^{p-2}u_n} |v_n|^{p-2}v_n. \]

Using \((H_2)\) and (15), Fatou’s Lemma gives

\[ \liminf_{n \to +\infty} \int_\Omega \frac{g(x, u_n(x))}{\|u_n\|^{p-1}} \geq \int_\Omega \liminf_{n \to +\infty} \frac{g(x, u_n(x))}{|u_n|^{p-2}u_n} |v_n|^{p-2}v_n. \]

therefore

\[ \int_\Omega \tilde{g} \geq 0 \]

which contradicts (16) and show that \( v > 0 \ p.p. \ x \in \Omega \), then the proof of Lemma 2.1 is complete.

**Lemma 2.2.** Assume \((H_1), (H_2)\) and \((H_3)\), then

\[ m(.) = \lambda_1 \text{ p.p. } x \in \Omega. \]
Proof. Let \( A_0 = \{ x \in \Omega : m(x) < \lambda_1 \} \), combining \((H_1)\) and \((H_3)\) we get

\[
\frac{g(x, u_n(x))}{\|u_n\|^{p-1}} \geq (1 + \text{sign}(u_n))(\lambda_1 - \varepsilon)|v_n|^{p-2}v_n \\
+ (1 - \text{sign}(u_n))(\lambda_2 + \varepsilon)|v_n|^{p-2}v_n + 0(n).
\]

Then

\[
\int_{A_0} g_n \chi_{A_0} \geq (1 + \text{sign}(v_n))(\lambda_1 - \varepsilon)|v_n|^{p-2}v_n \chi_{A_0} \\
+ (1 - \text{sign}(u_n))(\lambda_2 + \varepsilon)|v_n|^{p-2}v_n \chi_{A_0} + 0(n),
\]

passing to the limit we conclude

\[
\int_{A_0} \tilde{g} \geq (\lambda_1 - \varepsilon) \int_{A_0} |v|^{p-2}v,
\]

hence

\[
\int_{A_0} (m(x) - \lambda_1)|v|^{p-2}v \geq 0.
\]

Since \( v > 0 \), then necessarily \( \text{mes}(A_0) = 0 \), so it follows that

\[
m(x) \geq \lambda_1 \text{ p.p. } x \in \Omega. \quad (17)
\]

If \( m(.) \geq \lambda_1 \), then by the strict monotonicity of \( \lambda_1 \), we have

\[
\lambda_1(m) < 1
\]

which contradicts \((14)\), hence \( m(.) = \lambda_1 \text{ p.p. } x \in \Omega \).

Lemma 2.3. Assume \((H_1), (H_2), (H_3)\) and \((H_4)\), then the functional \( I \) satisfies the Palais-Smale condition \((PS)\), that is whenever \( (u_n) \subset W_0^{1,p}(\Omega) \) is a sequence such that \( I(u_n) \) is bounded and \( I'(u_n) \to 0 \) then \( (u_n) \) possesses a convergent subsequence.

Proof. Remark that, using \((1)\) any bounded sequence \( (u_n) \) such that \( I'(u_n) \to 0 \) and \( I(u_n) \) is bounded possesses a convergent subsequence, so we will show that \( (u_n) \) is bounded.

Suppose by contradiction that \( \|u_n\| \to +\infty \). Then, as we observed in the previous Lemmas, a subsequence of \( (v_n) \) \( (v_n = \frac{u_n}{\|u_n\|}) \) still denoted by \( (v_n) \) is such that

\[
v_n \to v \text{ strongly in } W_0^{1,p}(\Omega),
\]
\[ \|v\| = \lambda_1 \int_{\Omega} |v|^p = 1 \text{ and } v > 0 \text{ p.p. } x \in \Omega. \] (18)

In view of \((H_2)\) and \((H_3)\), we obtain
\[
G(x, u_n(x)) \geq \frac{1}{2p} (1 + \text{sign}(u_n))(L_+(x) - \varepsilon)|u_n|^p + \frac{1}{2p} (1 - \text{sign}(u_n))(-\varepsilon)|u_n|^p + B_\varepsilon(x). \tag{19}
\]

Since \(I(u_n)\) is bounded below, we have
\[
\frac{1}{p} - \int_{\Omega} \frac{G(x, u_n(x))}{\|u\|^p} - \int_{\Omega} \frac{h v_n}{\|u_n\|^{p-1}} \geq \frac{M}{\|u_n\|^p} \quad (M \in \mathbb{R}). \tag{20}
\]

Combining (19) and (20) and passing to the limit we get
\[
1 - \int_{\Omega} L_+(x)|v|^p \geq 0,
\]

hence, by (18) we deduce
\[
\int_{\Omega} (\lambda_1 - L_+(x))|v|^p \geq 0, \tag{21}
\]
as \(v > 0 \text{ p.p. } x \in \Omega \) and \(L_+ \geq \lambda_1\), (21) can not occur, then \(I\) satisfies the condition \((PS)\). The proof is now complete.

3. Proof of theorem 1.1

Let \(A = \left\{ u \in W_0^{1,p}(\Omega) : \lambda_2(k(x)) \int_{\Omega} k(x)|u|^p \leq \int_{\Omega} |\nabla u|^p \right\}\), where \(k(x) = \limsup_{|s| \to +\infty} \frac{g(x, s)}{|s|^{p-2} s} \). Recall that \(\limsup_{|s| \to +\infty} \frac{pG(x, s)}{|s|^p} \leq k(x)\).

It is easy to see that \(A\) is nonempty and symmetric set. For \(u \in A\) we have
\[
I(u) \geq \frac{1}{p} \|u\|^p - \frac{1}{p} \int_{\Omega} (k(x) + \varepsilon)|u|^p - \|u\|_p\|h\|_{p'} - \|B_\varepsilon\|_1 \\
\geq \frac{1}{p} \mu \|u\|^p - \|u\|_p\|h\|_{p'} - \|B_\varepsilon\|_1,
\]
since \(\lambda_2(k(x)) > \lambda_2(1) = 1\), \(\mu = \left(1 - \frac{\varepsilon}{\lambda_2(k(x))} - \frac{\varepsilon}{\lambda_1}\right) > 0\),

then
\[
\lim_{\|u\| \to +\infty, u \in A} I(u) = +\infty,
\]
hence
\[
I|_A \geq \beta \quad \text{for some } \beta \in \mathbb{R}. \tag{22}
\]
Let \( K \subset W_0^{1,p}(\Omega) \) compact, symmetric and \( \gamma(K) \geq 2 \), we will show that
\[
K \cap A \neq \emptyset. \tag{23}
\]
Indeed, if \( 0 \in K \), then (23) is proved by setting \( v = 0 \). if \( 0 \notin K \), we consider \( \tilde{K} = \left\{ \frac{u}{\|u\|}, u \in K \right\} \). It is easy to see that \( \gamma(\tilde{K}) \geq 2 \), hence by the variational characterization of \( \lambda_2(k(x)) \):
\[
\frac{1}{\lambda_2(k(x))} = \sup_{K \in A_2} \min_{u \in K} \int_{\Omega} k(x)|u|^p,
\]
we have
\[
\min_{u \in K} \int_{\Omega} k(x)|u|^p \leq \frac{1}{\lambda_2(k(x))}.
\]
Since \( \tilde{K} \) is compact, there exists \( \tilde{v}_0 \in \tilde{K} \) such that
\[
\int_{\Omega} k(x)|\tilde{v}_0|^p \leq \frac{1}{\lambda_2(k(x))},
\]
(recall that \( \tilde{v}_0 = \frac{v_0}{\|v_0\|}, v_0 \in K \)),
then
\[
\lambda_2(k(x)) \int_{\Omega} k(x)|v_0|^p \leq \int_{\Omega} |\nabla v_0|^p,
\]
hence
\[
v_0 \in A \cap K. \tag{24}
\]
On the other hand, by the hypothesis \((H_5)\), we can easily see that
\[
\lim_{|t| \to +\infty} I(t\varphi_1) = -\infty. \tag{25}
\]
From this, there exists \( R_1 > 0 \) such that
\[
I(t\varphi_1) < \beta \quad \text{for} \quad |t| \geq R_1 \tag{26}
\]
where \( \varphi_1 \) is a normalized, \( \lambda_1 \)-eigenfunction.
Letting \( Q = \{t\varphi_1 : |t| \leq R_1\} \).
We have
\[
\sup_I_{|Q|} < +\infty \tag{27}
\]
and from (26), we conclude
\[
\sup_I_{|\partial Q|} < \beta. \tag{28}
\]
In view of Lemma 2.3, (22), (24), (27) and (28) we may apply Theorem 2.1, to conclude the existence of a critical point \( u_0 \in W_0^{1,p}(\Omega) \) of \( I \).
4. Exemple

Let \( g \) be a continuous function given by

\[
g(s) = \begin{cases} 
\beta s^{p-1} & \text{if } s \geq 0 \\
-\beta |s|^{p-1} & \text{if } 0 \geq s \geq -1 + \frac{1}{e} \\
-\beta e^n (n - \frac{1}{e^2})^{p-1} (s + n) & \text{if } s \in [-n, -n + \frac{1}{e^2}] (n \in \mathbb{N}^*) \\
\beta e^n (n + \frac{1}{e^2})^{p-1} (s + n) & \text{if } s \in [-n - \frac{1}{e^2}, -n] \\
-\beta |s|^{p-1} & \text{if } s \in [-(n + 1) + \frac{1}{e^2}, -n - \frac{1}{e^2}] 
\end{cases}
\]

where \( \lambda_1 < \beta < \lambda_2 \).

It is not difficult to see that

\[
k(x) = \limsup_{|s| \to +\infty} \frac{g(x, s)}{|s|^{p-2}s} = \beta < \lambda_2. \tag{29}
\]

\[
\liminf_{|s| \to +\infty} \frac{g(x, s)}{|s|^{p-2}s} = 0. \tag{30}
\]

\[
\lambda_1 \leq \liminf_{|s| \to +\infty} \frac{g(x, s)}{|s|^{p-2}s}. \tag{31}
\]

\[
\lambda_1 \leq \liminf_{|s| \to +\infty} \frac{pG(x, s)}{|s|^p}. \tag{32}
\]
and
\[
\int_{\Omega} G(x,t\varphi_1(x)) \, dx - \frac{|t|^p}{p} \geq \frac{\beta}{p\lambda_1} |t|^p - \frac{1}{p} |t|^p - \sum_{n \geq 1} 2\beta \left( n + \frac{1}{e^n} \right)^{p-1} \frac{1}{e^n} \\
\geq \frac{1}{p} |t|^p \left( \frac{\beta}{\lambda_1} - 1 \right) - I,
\]
where \( I = \sum_{n \geq 1} 2\beta \left( n + \frac{1}{e^n} \right)^{p-1} \frac{1}{e^n} \in \mathbb{R} \).
So the hypotheses of Theorem 1.1 are satisfied.

References


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