GREEN’S FUNCTION OF DIFFERENTIAL EQUATION WITH FOURTH ORDER AND NORMAL OPERATOR COEFFICIENT IN HALF AXIS

M. BAYRAMOGLU

and

K. OZDEN KOKLU

Yildiz Technical University, Turkey

Abstract

Let $H$ be an abstract separable Hilbert space. Denoted by $H_1 = L_2(0, \infty; H)$, the all functions defined in $[0, \infty)$ and their values belongs to space $H$, which $\int_0^\infty \|f(x)\|_H^2 \, dx < \infty$. We define inner product in $H_1$ by the formula:

$$(f, g)_{H_1} = \int_0^\infty (f, g)_H \, dx \quad f(x), g(x) \in H_1,$$

$H_1$ forms a separable Hilbert space$[3]$ where $\| \cdot \|_H$ and $(\cdot, \cdot)_H$ are norm and scalar product, respectively in $H$.

In this study, in space $H_1$, it is investigated that Green’s function (resolvent) of operator formed by the differential expression

$$y^{IV} + Q(x)y, \quad 0 \leq x < \infty,$$

and boundary conditions

$$y'(0) - h_1 y(0) = 0,$$

$$y''(0) - h_2 y'(0) = 0,$$

where $Q(x)$ is a normal operator mapping in $H$ and invers of it is a compact operator for every $x \in [0, \infty)$. Assume that domain of $Q(x)$ is independent from $x$ and resolvent set of $Q(x)$ belongs to $|\arg \lambda - \pi| < \varepsilon$ $(0 < \varepsilon < \pi)$ of complex plane $\lambda$, $h_1$ and $h_2$ are complex numbers. In addition assume that the operator function $Q(x)$ satisfies the Titchmarsh-Levitan conditions.
1. INTRODUCTION

In this work, Green’s function (resolvent) of operator $L$ in $H_1 = L_2 (0, \infty; H)$ space is investigated. Here, operator $L$ is formed by the differential expression

\[ y^{IV} + Q(x)y, \quad 0 \leq x < \infty \]  

and the boundary conditions

\[ y'(0) - h_1 y(0) = 0, \quad y'''(0) - h_2 y''(0) = 0 \]

where $Q(x)$ is a normal operator for every $x \in [0, \infty)$ in $H$ and inverse of it is a compact operator, $h_1, h_2$ are arbitrary complex numbers. In the case of $Q^*(x) = Q(x), h_1 = h_2$ and regular behavior of $Q(x)$ in infinity, Green function of operator $L$ investigated in (Albayrak, Bayramoglu)[1].

Green’s function of Sturm-Liouville equation given in $(-\infty, \infty)$ with unbounded self-adjoint operator coefficient was first investigated by B.M. Levitan [11].

In the space of $L_2 (-\infty, \infty; H)$, Green’s function and the asymptotic behaviour for the number of the eigenvalues of the operator formed by differential expression $(-1)^n y^{(2n)} + \sum_{j=2}^{2n} Q_j(x)y^{(2n-j)}$ was obtained by M. Bayramoglu 1971 [4], where $Q_j(x)$ ($j = 2, \ldots, 2n$) are the self-adjoint operators in $H$. Later on many studies (Boymatov, K.Ch.[6], Otelbayev, M.[12], Aslanov, G.I.[2], Kleyman, E.G.[8], Saito, Y.[15]) were published in this subject. Wide reference of these studies is given in (Kostyuchenko, A.G., Sargsyan, I.S. [10], Otelbayev, M. [13]). The main reference related to Green’s function for the ordinary differential equation is the book Stakgold, I. [16].

2. DETERMINATION OF THE PROBLEM

In $H_1 = L_2 (0, \infty; H)$, let consider the differential expression

\[ y^{IV} + Q(x)y + \mu y \]

with boundary conditions
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\[(4)\]
\[y'(0) - h_1 y(0) = 0\]

\[(5)\]
\[y'''(0) - h_2 y''(0) = 0\]

where \(\mu \geq 0\) is a real number. It is assumed that \(Q(x)\) is a normal operator mapping in \(H\) for every \(x \in [0, \infty)\) and satisfies the following specification called Titchmarsh-Levitan conditions:

1) Let \(Q(x)\) be a normal operator for each \(x \in [0, \infty)\) in \(H\) with domain \(D(Q(x)) \equiv D\) independent from \(x\), \(\mathcal{D} = H\) (Here \(\mathcal{D}\) shows the closure of \(D\)).

2) Let \(Q^{-1}(x)\) be compact operator for every \(x \ (Q^{-1}(x) \in \sigma_\infty)\) and \(1 \leq |\alpha_1(x)| \leq |\alpha_2(x)| \leq ... \leq |\alpha_n(x)| \leq ...\) where \(\alpha_1(x), \alpha_2(x), \alpha_3(x), ...\), are the eigenvalues of \(Q(x)\).

3) Assume that resolvent set of \(Q(x)\) included domain

\[S_\varepsilon = \{\lambda : \pi - \varepsilon < \arg \lambda < \pi + \varepsilon, \ 0 < \varepsilon < \pi, \ \varepsilon = const\}\]

of complex plane \(\lambda\).

4) Suppose that function \(F(x) = \sum_{k=1}^{\infty} \frac{1}{|\alpha_k(x)|^{7/4}}\) belongs to \(L(0, \infty)\):

\[\int_0^\infty F(x)dx < \infty\]

5) \(\left\|Q^{-1/4}(x).Q^{1/4}(s)\right\| \leq c\) and \(\left\|Q^{1/4}(x).Q^{-1/4}(s)\right\| \leq c\) while \(|x - s| \leq 1\), \(c = constant\)

6) Assume that \(\left\|Q^{-a}(x)|Q(s) - Q(x)|\right\| \leq c|x - s|\) while \(|x - s| \leq 1\), where \(c = constant\) and \(0 < a < \frac{5}{4}\).

Different constants are denoted with \(c\).

Let \(B(H)\) be a Banach space whose elements are bounded operators mapping in \(H\) \([7]\). \(G(x, s; \mu)\) function which belongs to \(B(H)\) for \(0 \leq x, s < \infty\) and satisfies the following conditions is called Green’s function of (3)-(5).

1) The operator function \(G(x, s; \mu)\) itself and its two partial derivative according to \(s\) are continuous functions for variables \(x\) and \(s\) \((0 \leq x, s \leq \infty)\).
2) When \( s \neq x \) third derivative of \( G(x, s; \mu) \) for \( s \) is continuous.

3) \( \frac{\partial G}{\partial s^3}(x, x + 0, \mu) - \frac{\partial G}{\partial s^3}(x, x - 0, \mu) = I \quad (I \text{ is identity operator in } H) \)

4) When \( s \neq x \), \( \frac{\partial G}{\partial s^4}(x, s; \mu) + G(x, s; \mu)Q(s) + \mu G(x, s; \mu) = 0 \)

5) \( \frac{\partial G}{\partial s^3}(x, s; \mu)|_{s=0} - h_1 G(x, s; \mu)|_{s=0} = 0, \)
\( \frac{\partial G}{\partial s^4}(x, s; \mu)|_{s=0} - h_2 \frac{\partial G}{\partial s^2}(x, s; \mu)|_{s=0} = 0. \)

According to parametrics method, the operator function \( G(x, s; \mu) \) will be found as a solution of integral equation given by

\[
G(x, s; \mu) = r(x - s)g(x, s; \mu) - \int_0^\infty \left\{ r^{(IV)}(x - \xi)g(x, \xi; \mu) + 4r''(x - \xi)g'(x, \xi; \mu) + 6r''(x - \xi)g''(x, \xi; \mu) + 4r'(x - \xi)g'''(x, \xi; \mu) \right\} G(\xi, s; \mu) d\xi
\]

(6) \( + r(x - \xi)g(x, \xi; \mu) \{ Q(\xi) - Q(x) \} G(\xi, s; \mu) d\xi \)

where \( r(u) = \begin{cases} 1 & |u| \leq \rho \\ 0 & |u| \geq 2\rho, \quad 0 < \rho < \frac{1}{2} \end{cases} \)

is any fixed sufficiently smooth function and

\[
g(x, s; \mu) = \frac{\sqrt{2}}{8} \alpha^{-3} (1 + i) e^{-\alpha \frac{\sqrt{2}}{2}|x-s|} \]
\[
- \frac{\sqrt{2}}{8} \alpha^{-3} (-1 + i) e^{-\alpha \frac{\sqrt{2}}{2}|x-s|}
\]
\[
+ \frac{1}{\alpha^2 i} \left( \frac{1}{2} (1 + i) h_2 + \frac{\sqrt{2}}{4} \alpha^{-1} (1 + i) h_1 h_2 - \frac{\sqrt{2}}{4} \alpha^{-1} (1 + i) + \frac{1}{2} (1 + i) h_1 \right) \frac{1}{\alpha^2 (h_1 h_2 - \sqrt{2} \alpha (h_1 + h_2) - 2\alpha^2)} e^{-\alpha \frac{\sqrt{2}}{2}(1+i)(x+s)}
\]
\[
+ \left[ \frac{1}{\alpha^2 i} \frac{1}{2} h_2 + \frac{\sqrt{2}}{4} \alpha (1 + i) + \frac{1}{2} h_1 \right] e^{\alpha \frac{\sqrt{2}}{2}[(x+s) + (x-s)]}
\]
\[
+ \left[ \frac{1}{\alpha^2 i} \frac{1}{2} h_1 - \frac{\sqrt{2}}{4} \alpha i + \frac{1}{2} h_2 \right] e^{\alpha \frac{\sqrt{2}}{2}[(x+s) - (x-s)]}
\]
\[ g(x, s; \mu) = \sum_{n=1}^{\infty} \sqrt{\alpha_n(x) + \mu} e_n \]

where \( e_1(x), e_2(x), \ldots, e_n(x), \ldots \) are the orthonormalized eigenvectors corresponding to eigenvalues \( \alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x), \ldots \) of \( Q(x) \). Branch of the \( \sqrt{\alpha_j(x) + \mu} \) is defined with \( |\arg(\alpha_n(x) + \mu)| < \pi \). Note that the operator function \( g(x, s; \mu) \) is founded by writing \( \alpha = \sqrt[4]{Q(x) + \mu I} \) instead of \( \alpha \) in Green function defined with differential expression

\[ y^{IV} + \alpha^4 y \quad (\alpha > 0) \]

and boundary conditions

\[ y^{(j)}(0) - h_{(j+1)/2}y^{(j-1)}(0) = 0, \quad j = 1, 3 \]

in space \( L_2(0, \infty) \). Let’s say that

\[ g = \sum_{i=1}^{6} g_i. \]

It will be shown that integral equation (6) has only one solution and this solution is Green’s function of the problem (3)-(5).

Equation (6) will be investigated in these spaces: \( X_2, X_3^{(1)}, X_4^{(-1/4)}, X_5 \). These are shown that they are Banach spaces and given by Levitan, B.M. [11].

Consider that integral Eq.(6) is in the space \( X_2 \). Let us show that this equation has one solution in \( X_2 \) for \( \mu >> 0 \) (\( \mu \) is a big enough positive value) and the solution can be found by successive approximation method.

For this it is enough to show that \( g(x, s; \mu) \in X_2 \) and the operator of

\[ NA = \int_0^\infty \left\{ r^{(IV)}(x - \xi)g(x, \xi; \mu) + 4r^{m}(x - \xi)g'(x, \xi; \mu) \\
+ 6r''(x - \xi)g''(x, \xi; \mu) + 4r'(x - \xi)g'''(x, \xi; \mu) \\
+ r(x - \xi)g(x, \xi; \mu) |Q(\xi) - Q(x)| \right\} A(\xi, s; \mu) d\xi \]

is constriction operator in \( X_2 \) for \( \mu >> 0 \).
**Lemma 1:** If operator function $Q(x)$ satisfies the conditions 4-) and 6-) for $\mu >> 0$, operator $N$ is constriction operator in the space $X_2$.

**Proof:** If it is shown that the norm of operator $N$ for $\mu >> 0$ are small enough, it is demonstrated that $N$ is constriction operator at large values of $\mu > 0$.

\[
NA = \sum_{i=1}^{5} N_j A
\]

\[
N_1 A = \int_{0}^{\infty} [r(x-\xi)g(x, \xi; \mu) [Q(x) - Q(\xi)]] A(\xi, \eta) d\xi
\]

\[
N_2 A = \int_{0}^{\infty} 4r' (x-\xi) g''(x, \xi; \mu) A(\xi, \eta) d\xi
\]

\[
N_3 A = \int_{0}^{\infty} 6r'' (x-\xi) g'''(x, \xi; \mu) A(\xi, \eta) d\xi
\]

\[
N_4 A = \int_{0}^{\infty} 4r''(x-\xi) g'(x, \xi; \mu) A(\xi, \eta) d\xi
\]

\[
N_5 A = \int_{0}^{\infty} r(IV) (x-\xi) g(x, \xi; \mu) A(\xi, \eta) d\xi
\]

\[
\|N\| \leq \sum_{i=1}^{5} \|N_i\|
\]

can be written from nature of norm. Let’s do the operations for operator $N_1 A$.

\[
g(x, s; \mu) = \sum_{i=1}^{6} g_i
\]

\[
N_1 A = \int_{0}^{\infty} [r(x-\xi)g(x, \xi; \mu) [Q(x) - Q(\xi)]] A(\xi, \eta) d\xi
\]

\[
= \int_{0}^{\infty} \left[ r(x-\xi) \sum_{i=1}^{6} g_i(x, \xi; \mu) [Q(x) - Q(\xi)] \right] A(\xi, \eta) d\xi
\]

\[
N_1 A = \sum_{i=1}^{6} N_{1i} A
\]

\[
\|N_1\| \leq \sum_{i=1}^{6} \|N_{1i}\|
\]

\[
N_{11} A = \int_{0}^{\infty} r(x-\xi) g_1(x, \xi; \mu) [Q(x) - Q(\xi)] A(\xi, \eta) d\xi
\]

\[
= \int_{|x-\xi| \leq 1} r(x-\xi) g_1(x, \xi; \mu) [Q(x) - Q(\xi)] A(\xi, \eta) d\xi
\]

\[
+ \int_{|x-\xi| > 1} r(x-\xi) g_1(x, \xi; \mu) [Q(x) - Q(\xi)] A(\xi, \eta) d\xi
\]

\[
= b_1 + b_2
\]

where $b_2 = 0$ according to $r(u) = 0$, $|u| > 1$.

\[
\|N_{11} A(x, \xi)\|_2 = \|b_1\|_2
\]

\[
(\|.\|_2 \text{ is a norm in space } X_2)
\]

\[
\|b_1\|_2^2 = \int_{|x-\xi| \leq 1} \left[ r(x-\xi) g_1(x, \xi; \mu) [Q(x) - Q(\xi)] A(\xi, \eta) d\xi \right]^2
\]

\[
\leq \int_{|x-\xi| \leq 1} \left[ r(x-\xi) g_1(x, \xi; \mu) [Q(x) - Q(\xi)] A(\xi, \eta) \right]_2 d\xi ^2
\]

\[
\|b_1\|_2^2 \leq c^2 \mu^2 q \int_{|x-\xi| \leq 1} |x-\xi|^{-\delta} \|A(\xi, \eta)\|_2 d\xi ^2
\]
is found. Hence
\[
\|b_1\|_2^2 \leq \int_0^\infty \int_0^\infty c\mu^{2q} \left[ \int_{|x-\xi| \leq 1} |x-\xi|^{-\varepsilon} \|A(\xi, \eta)\|_2 \, d\xi \right]^2 \, dx \, d\eta
\]
is obtained, or
\[
\|b_1\|_2 \leq c\mu^{2q} \|A(x, \eta)\|_2 < \infty.
\]

Therefore,
\[
\|N_{11}A\|_2^2 \leq c\mu^{2q} \|A(x, \eta)\|_2
\]
is derived. Here, \( q < 0 \) is constant. Thus, operator \( N_{11}A(x, \xi) \) is bounded with small enough norm of large \( \mu > 0 \). In a similar way, operators \( N_{i1}A \) \((i = 1, \ldots, 6)\) are bounded with small enough norm of large \( \mu > 0 \). Then it is obtained that operator \( NA \) is constriction operator in space \( X_2 \) for \( \mu >> 0 \). Thus Lemma 1 is proved.

If it can be shown that \( r(x-s)g(x, s; \mu) \) belongs to the space \( X_2 \), then, it is obtained that Eq.(6) has only one solution belonging to \( X_2 \) for \( \mu >> 0 \).
\[
\|g\|_2 \leq \sum_{i=1}^{6} \|g_i\|_2
\]
can be written.

Now let us show that \( rg \) belongs to space \( X_2 \) assuming that the condition 4-) of function \( Q(x) \) is fulfilled. Let us perform the operation for any term included by \( rg \), for example the term \( r(x-s)g_1(x, s; \mu) \). That is, let us show that \( rg_1 \in X_2 \). In a same manner, it is indicated that other terms also belong to \( X_2 \). Since \( r(x-s)g_1(x, s; \mu) \) is a function of normal operator valued function \( Q(x) \), using the spectral expansion formula for normal operators [17]:
\[
\|rg_1\|_2^2 = 
\sum_{j=1}^{\infty} \left( \frac{\sqrt{2}}{8} \right)^2 \left| r(x-s)(1+i)(\alpha_j(x) + \mu)^{-3/4} e^{-(\alpha_j(x)+\mu)^{1/4} \sqrt{2}(1+i)|x-s|} \right|^2
\]
\[
\|rg_1\|_2^2 \leq (1/16) \sum_{j=1}^{\infty} |\alpha_j(x) + \mu|^{-3/2} e^{-\sqrt{2}\delta|x-s|} \text{Re}(\alpha_j(x)+\mu)^{1/4} \delta
\]
\((\delta = \text{const} > 0)\)
is implied. From the fourth property of \( Q(x) \)
(c = const > 0) \[ \int_0^\infty \int_0^\infty \|rg_1\|^2_2 dsdx = c \int_0^\infty \sum_{j=1}^\infty \frac{dx}{|\alpha_j(x) + \mu|^{7/4}} < \infty, \]

is obtained. Thus it is denoted that \( rg_1 \in X_2 \). Therefore the following theorem has been proved.

**THEOREM 1**: If the conditions 4-) and 6-) of operator \( Q(x) \) are satisfied, then, for \( \mu >> 0 \), there exists a solution in the space \( X_2 \) for Eq.(6) and it is unique. This solution can be found by successive approximation method.

The following lemma can be proved.

**LEMMA 2**: If operator function \( Q(x) \) satisfies the conditions in Lemma 1 then for \( \mu >> 0 \), operator \( N \) is a constriction operator in every spaces \( X_2, X_3^{(1)}, X_4^{(-1/4)} \) and \( X_5 \). At the same time in addition to the conditions 1-) and 6-), if operator function \( Q(x) \) satisfies the condition \[ \left\| Q^{1/4}(x)Q^{-1/4}(s) \right\| \leq c, \ c = constant, \] then \( g \in X_4^{(-1/4)} \).

3. DERIVATIONS OF GREEN’S FUNCTION

Let us try to show that operator function \( G(x, s; \mu) \) has the derivatives \( \frac{\partial^j G(x, s; \mu)}{\partial s^j} \) \( (j = 1, 2, 3) \). If the derivatives of both sides of Eq.(6) is calculated according to \( s \)

\[
\frac{\partial^j G(x, s; \mu)}{\partial s^j} = \frac{\partial^j [r(x-s)g(x, s; \mu)]}{\partial s^j} - \int_0^\infty \{r^{(IV)}(x-\xi)g(x, \xi; \mu) \\
+4r''(x-\xi)g'(x, \xi; \mu) + 6r''(x-\xi)g''(x, \xi; \mu) + 4r'(x-\xi)g'''(x, \xi; \mu) \\
+r(x-\xi)[Q(\xi) - Q(x)]\} \frac{\partial^j G(\xi, s; \mu)}{\partial s^j} d\xi \tag{9}
\]

(10) \[ K_j(x, s; \mu) = \frac{\partial^j [r(x-s)g(x, s; \mu)]}{\partial s^j} - NK_j(\xi, s; \mu) \quad (j = 1, 2, 3) \]

can be written. Let us investigate integral Eq.(10) in Banach space \( X_3^{(1)} \).

In Lemma 2, \( N \) was denoted is a constriction operator in the space \( X_3^{(1)} \) for \( \mu >> 0 \). If it is implied that operator function \( \frac{\partial^j [r(x-s)g(x, s; \mu)]}{\partial s^j} \) \( (j = 1, 2, 3) \) belongs to \( X_3^{(1)} \), it is shown that there exists a solution for Eq.(10) in \( X_3^{(1)} \).
for $\mu >> 0$. It is seen that $\frac{\partial^j r g}{\partial s^j} \in X_3^{(1)}$, that is,

$$\sup_{0 \leq x < \infty} \int_0^\infty \left\| \frac{\partial^j (rg)}{\partial s^j} \right\|_H ds < \infty$$

from clear expression of operator function $g(x,s;\mu)$. It is demonstrated that

$$\frac{\partial^j G(x,s;\mu)}{\partial s^j} - \frac{\partial^j [r(x-s)g(x,s;\mu)]}{\partial s^j}$$

(j=1,2,3) is a continuous function for $s$, ($s \neq x$), performing the similar operations as in [10] and [4]. On the other hand, since $\frac{\partial^j [r(x-s)g(x,s;\mu)]}{\partial s^j}$ ($j = 1, 2$) is continuous, function $\frac{\partial^j G(x,s;\mu)}{\partial s^j}$ is also continuous according to $s$. From $\frac{\partial^j [r(x-s)g(x,s;\mu)]}{\partial s^j}$, it is denoted that this function satisfies the condition

$$\frac{\partial^3 [r g(x,s+0;\mu)]}{\partial s^3} - \frac{\partial^3 [r g(x,s-0;\mu)]}{\partial s^3} = I$$

at the point $s = x$. This results in that operator function $\frac{\partial^j G}{\partial s^j}$ fulfills the condition 4-) from the continuity of $\frac{\partial^j G}{\partial s^j} - \frac{\partial^j r g}{\partial s^j}$.

The following Lemma can be proved.

**LEMMA 3:** Assume that operator function $Q(x)$ satisfies the conditions 1-) and 3-) and

$$\sup_{0 \leq x < \infty} \int_0^\infty \left\| \frac{\partial^j [r(x-s)g(x,s;\mu)]}{\partial s^j} \right\| Q^{-1/4}(s) ds < \infty$$

while $|x - s| \leq 1$. In this case,

$$\frac{\partial^4 [r(x-s)g(x,s;\mu)]}{\partial s^4} \in X_3^{(-1/4)}$$

that is

$$\sup_{0 \leq x < \infty} \int_0^\infty \left\| \frac{\partial^j [r(x-s)g(x,s;\mu)]}{\partial s^j} \right\| Q^{-1/4}(s) ds < \infty.$$

**4. THE FOURTH DERIVATIVE OF GREEN’S FUNCTION**

In previous part it has been shown that the derivative $\frac{\partial^j G}{\partial s^j}$ of Green’s function $G(x,s;\mu)$ belongs to the space $X_3$ and it satisfies the continuity ($x \neq s$) for the variable $s$ and the following expression

$$\frac{\partial^3 G(x,s;\mu)}{\partial s^3} = \frac{\partial^3 [r(x-s)g(x,s;\mu)]}{\partial s^3} - \int_0^\infty P(x,\xi;\mu) \frac{\partial^3 G(\xi,s;\mu)}{\partial s^3} d\xi$$

where

$$P(x,\xi;\mu) = r^{IV}(x-\xi)g(x,\xi;\mu) + 4r''(x-\xi)g'(x,\xi;\mu)$$
\[+6r''(x - \xi)g''(x, \xi; \mu) + 4r'(x - \xi)g'''(x, \xi; \mu) + r(x - \xi)g(x, \xi; \mu) [Q(\xi) - Q(x)].\]

Let us write Eq. (9) as follows

\[(13) \quad L(x, s; \mu) = l(x, s; \mu) - \int_{0}^{\infty} P(x, \xi; \mu) L(\xi, s; \mu) d\xi.\]

Here

\[L(x, s; \mu) = \frac{\partial^3 G}{\partial s^3} - \frac{\partial^3 r g}{\partial s^3}\]

and

\[l(x, s; \mu) = -\int_{0}^{\infty} P(x, \xi; \mu) \frac{\partial^3 r g}{\partial s^3} d\xi,\]

Let us derive the Eq. (13) according to \(s\) as formal. From this

\[\frac{\partial L(x, s; \mu)}{\partial s} = \frac{\partial l(x, s; \mu)}{\partial s} - \int_{0}^{\infty} P(x, \xi; \mu) \frac{\partial L(\xi, s; \mu)}{\partial s} d\xi\]

is obtained. If the expression

\[\frac{\partial^3 [r(x - (x + 0))g(x, x + 0; \mu)]}{\partial s^3} - \frac{\partial^3 [r(x - (x - 0))g(x, x - 0; \mu)]}{\partial s^3} = I\]

is used and if we write as

\[l(x, s; \mu) = -\left(\int_{0}^{s-0} P(x, \xi; \mu) \frac{\partial^3 r g}{\partial s^3} d\xi + \int_{s+0}^{\infty} P(x, \xi; \mu) \frac{\partial^3 r g}{\partial s^3} d\xi\right),\]

\[\frac{\partial l(x, s; \mu)}{\partial s} = -P(x, s; \mu) - \left(\int_{0}^{\infty} P(x, \xi; \mu) \frac{\partial^3 r g}{\partial s^3} d\xi\right)\]

is found. Let us say that

\[\frac{\partial l(x, s; \mu)}{\partial s} = l_1(x, s; \mu).\]

If it can be shown that element \(l_1\) belongs to \(X_4^{(-1/4)}\), according to Lemma 2. It is obtained that there exists a derivative of the function \(\frac{\partial^3 G}{\partial s^3} - \frac{\partial^3 r g}{\partial s^3}\) according to \(s\) and \(\frac{\partial}{\partial s} \left(\frac{\partial^3 G}{\partial s^3} - \frac{\partial^3 r g}{\partial s^3}\right) \in X_4^{(-1/4)}\). From this point according to Lemma 3, \(\frac{\partial^4 G}{\partial s^4} \in X_4^{(-1/4)}\) is obtained. It is found that the element \(l_1\) belongs to \(X_4^{(-1/4)}\) by the studies [11], [4].

5. SATISFYING DIFFERENTIAL EQUATION OF GREEN'S FUNCTION

Let us show that Green’s function \(G(x, s, \mu)\) for \(x \neq s\) satisfies the equation

\[\frac{\partial^4 G}{\partial s^4} + G(x, s, \mu) [Q(s) + \mu I] = 0.\]

Let \(f \in D\). Then,
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\[ \frac{\partial^4 G}{\partial s^4} (f) + rg [Q(x) + \mu I] (f) = -rg [Q(s) - Q(x)] (f) - \int_0^\infty P(x, \xi; \mu) \frac{\partial^4 G(\xi, s; \mu)}{\partial s^4} (f) d\xi \]

or

\[ \frac{\partial^4 G}{\partial s^4} (f) = -rg [Q(s) + \mu I] (f) - \int_0^\infty P(x, \xi; \mu) \frac{\partial^4 G(\xi, s; \mu)}{\partial s^4} (f) d\xi \]

is obtained. Let \([Q(s) + \mu I] f = \varphi\). From this, Eq.(14) becomes as follows,

\[ \frac{\partial^4 G}{\partial s^4} [Q(s) + \mu I]^{-1} \varphi = -rg \varphi - \int_0^\infty P(x, \xi; \mu) \frac{\partial^4 G(\xi, s; \mu)}{\partial s^4} [Q(s) + \mu I]^{-1} \varphi d\xi. \]

Comparing this equation with Eq.(6),

\[ \frac{\partial^4 G}{\partial s^4} \left\{ -[Q(s) + \mu I]^{-1} \varphi \right\} = G(x, s; \mu) \varphi \]

is found. From the last expression, fourth property is obtained as elements' set of \(\varphi\) for every constant \(s \geq 0\) is dense everywhere in \(H\).

6. SATISFACTION OF BOUNDARY CONDITIONS

Let us show that \(G(x, s; \mu)\) satisfies the conditions

\[ \frac{\partial G(x, s; \mu)}{\partial s} \bigg|_{s=0} - h_1 G(x, s; \mu) \bigg|_{s=0} = 0 \]

\[ \frac{\partial^2 G(x, s; \mu)}{\partial s^2} \bigg|_{s=0} - h_2 \frac{\partial^2 G(x, s; \mu)}{\partial s^2} \bigg|_{s=0} = 0 \]

that is Green's function fulfills the condition 5-).

\[ G(x, s; \mu) = r(x - s)g(x, s; \mu) - \int_0^\infty P(x, \xi; \mu)G(\xi, s; \mu) d\xi \]

(15)

\[ \frac{\partial G(x, s; \mu)}{\partial s} = \frac{\partial [r(x - s)g(x, s; \mu)]}{\partial s} - \int_0^\infty P(x, \xi; \mu) \frac{\partial G(\xi, s; \mu)}{\partial s} d\xi \]

(16)

From the Eq.15 and 16;

\[ \frac{\partial [r(x - s)g(x, s; \mu)]}{\partial s} \bigg|_{s=0} - \int_0^\infty P(x, \xi; \mu) \frac{\partial G(\xi, s; \mu)}{\partial s} d\xi \bigg|_{s=0} = 0 \]

\[ -h_1 [r(x - s)g(x, s; \mu) - \int_0^\infty P(x, \xi; \mu)G(\xi, s; \mu) d\xi] \bigg|_{s=0} = 0 \]

(17)
is obtained. Considering that
\[
\frac{\partial (rg)}{\partial s} |_{s=0} - h_1 rg |_{s=0} = 0
\]
from Eq.(17);

\[
\int_0^\infty P(x, \xi; \mu) \left[ \frac{\partial G(\xi, s; \mu)}{\partial s} - h_1 G(\xi, s; \mu) \right] s=0 d\xi = 0
\]
can be written. Homogen Eq.(18) can be written as below,
\[
(N + I) \left[ \frac{\partial G}{\partial s} - h_1 G \right] s=0 = 0.
\]

Since operator \(N\) is constriction operator for \(\mu >> 0\), then
\[
\frac{\partial G(\xi, s; \mu)}{\partial s} |_{s=0} - h_1 G(\xi, s; \mu) |_{s=0} = 0
\]
is obtained. Thus the first boundary condition of (5-) is satisfied.

Now let us calculate the second and third derivation of \(G(x, s; \mu)\) according to \(s\).
\[
\frac{\partial G}{\partial s} |_{s=0} = \frac{\partial (rg)}{\partial s} |_{s=0} - \int_0^\infty r(x - \xi) g(x, \xi; \mu) [Q(\xi) - Q(x)] \frac{\partial G(\xi, s; \mu)}{\partial s} d\xi |_{s=0}
\]
\[= \frac{\partial^3 G}{\partial s^3} |_{s=0} - \int_0^\infty P(x, \xi; \mu) \frac{\partial^3 G(\xi, s; \mu)}{\partial s^3} d\xi |_{s=0} - h_2 \left[ \frac{\partial^2 (rg)}{\partial s^2} |_{s=0} - \int_0^\infty P(x, \xi; \mu) \frac{\partial^2 G(\xi, s; \mu)}{\partial s^2} d\xi |_{s=0} \right]
\]
\[
= \frac{\partial^3 G}{\partial s^3} |_{s=0} - h_2 \frac{\partial^2 G}{\partial s^2} |_{s=0}
\]

From the expression of \(g(x, s; \mu)\), considering that
\[
\frac{\partial^3 (rg)}{\partial s^3} |_{s=0} - h_2 \frac{\partial^2 (rg)}{\partial s^2} |_{s=0} = 0
\]
from the Eq.19
\[- \int_0^\infty P(x, \xi; \mu) \left[ \frac{\partial G(\xi, s; \mu)}{\partial s} - h_2 \frac{\partial^2 G(\xi, s; \mu)}{\partial s^2} \right] s=0 d\xi = \frac{\partial^3 G(\xi, s; \mu)}{\partial s^3} - h_2 \frac{\partial^2 G(\xi, s; \mu)}{\partial s^2}
\]
is found. This homogen equation can be expressed by
\[
(N + I) \left[ \frac{\partial^3 G}{\partial s^3} - h_2 \frac{\partial^2 G}{\partial s^2} \right] s=0 = 0.
\]

Since \(N\) is constriction operator for \(\mu >> 0\) in \(X_3^{(1)}\)
\[
\frac{\partial^3 G(x, s; \mu)}{\partial s^3} - h_2 \frac{\partial^2 G(x, s; \mu)}{\partial s^2} |_{s=0} = 0
\]
Green's Function of Differential Equation with Fourth order and ...

... is obtained. Thus the second condition of 5-) is also fulfilled.

Consequently, it is shown that operator function \( G(x, s; \mu) \) satisfies all properties of Green's function.

If integral operator
\[
Af = \int_0^\infty G(x, s; \mu) f(s) ds,
\]
\( \mu > 0 \)
is formed in \( H_1 \) by using Green's function obtained, it is seen that \( A \) is a Hilbert-Schmidt (H-S) type operator from the property proved
\[
\int_0^\infty \int_0^\infty \| G(x, s; \mu) \|^2 dx ds < \infty.
\]

If \( Q(x) = Q^*(x) \), \( h_1 = h_2 \), are real numbers then \( G^*(x, s; \mu) = G(s, x; \mu) \) can be proved.

Appendix:

Example. Let \( \Omega \subset \mathbb{R}^m (m \geq 1) \) be any finite region with uniformly smooth boundary and \( R^+ = [0, \infty) \).

Let us consider the boundary value problem of
\[
\begin{align*}
\frac{\partial^4 u}{\partial x^4} + q(x)(-\nabla^2_y)^s u &= \lambda u \quad \text{(a)} \\
\frac{\partial u(x, y)}{\partial x} \big|_{x=0} + h_1 u(0, y) &= 0 \quad \text{(b)} \\
\frac{\partial^3 u(x, y)}{\partial x^3} \big|_{x=0} + h_2 u(0, y) &= 0 \quad \text{(c)} \\
u \big|_{\partial \Omega} &= \frac{\partial u}{\partial \nu} \big|_{\partial \Omega} = ... = \frac{\partial^{s-1} u}{\partial \nu^{s-1}} \big|_{\partial \Omega} = 0 \quad \text{(d)}
\end{align*}
\]
in space \( L_2(R^+ \times \Omega) \). Here \( y = (y_1, y_2, ..., y_m) \),
\(-\nabla_y = -\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} - ... - \frac{\partial^2}{\partial y_m^2}, s \) is any integer \( > \frac{2m}{4} \), \( \partial \Omega \) is the boundary of \( \Omega \) region, \( \gamma \) is the normal of \( \partial \Omega \) and \( q(x) \) is a complex valued function with values in \( C \setminus S_\varepsilon \) satisfying the conditions
\[
c_1(1 + x^\alpha) \leq |q(x)| \leq c_2(1 + x^\alpha)
\]
where \( \alpha > \frac{4}{7}, c_1, c_2 \) are positive constants and \( h_1, h_2 \) are arbitrary complex constants.

Let us define self-adjoint \( A \) operator (like in [14]) in space \( H = L_2(\Omega) \) by \((-\nabla^2_y)^s \) with boundary conditions (d).

Therefore the problem (a)-(d) in the space \( H_1 = L_2(R^+ \times \Omega) = L_2(R^+, H) \) can be writed as a boundary value problem with operator coefficient as follows:
\[
\begin{align*}
\frac{\partial^4 u}{\partial x^4} + Q(x) u - \lambda u &= 0 \\
u'(0) - h_1 u(0) &= 0 \\
u''(0) - h_2 u'(0) &= 0
\end{align*}
\]
where \( u(x) = u(x, .), Q(x) = q(x)A \).

Resolvent set of operator function \( Q(x) \) defined like this consists of region \( S_\varepsilon \) and it can be shown that conditions 1-) - 6-) are satisfied (See...
also [9], [5]). Applying the founded results in the theoretical part, Green function of the problem (a)-(d) can be examined.

7. REFERENCES


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**Kevser Ozden Koklu**
Mathematical Enginering Department
Yıldız Technical University
P. O. Box
Davutpasa Campus, Istanbul, Turkey
email:ozkoklu@yildiz.edu.tr
M. Bayramoglu
Mathematical Engineering Department
Yildiz Technical University
P. O. Box
Davutpasa Campus, Istanbul, Turkey
email: mbayram@yildiz.edu.tr