Abstract

In this article, we look for invariance in commutative baric algebras \((A, \omega)\) satisfying \((x^2)^2 = \omega(x)x^3\) and in algebras satisfying \((x^2)^2 = \omega(x^3)x\), using subspaces of kernel of \(\omega\) that can be obtained by polynomial expressions of subspaces \(U_e \oplus V_e\) of Peirce decomposition \(A = Ke \oplus U_e \oplus V_e\) of \(A\), where \(e\) is an idempotent element. Such subspaces are called \(p\)-subspaces. Basically, we prove that for these algebras, the \(p\)-subspaces have invariant dimension, besides that, we find out necessary and sufficient conditions for the invariance of the \(p\)-subspaces.

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1. Introduction

Let $A$ be a commutative and not necessarily associative algebra with finite dimension over $K$, where $K$ is a field with char $(K) \neq 2, 3$. We consider in this paper two classes of baric algebras $(A, w)$ satisfying respectively

\[(x^2)^2 = \omega(x)x^3\]  \hspace{1cm} (1.1)

or

\[(x^2)^2 = \omega(x^3)x\]  \hspace{1cm} (1.2)

We present in the next two sections some well known results about these two classes of algebras. In particular we look for the idempotents in these classes.

1.1. Algebras satisfying $(x^2)^2 = \omega(x)x^3$

Let $(A, \omega)$ be a baric algebra satisfying $(x^2)^2 = \omega(x)x^3$ for all $x \in A$. From this identity, we have

\[(x^2)^2 = 0\]  \hspace{1cm} (1.3)

for all $x \in N = \ker \omega$. By linearization of (1.3), we deduce that

\[x_1^2(x_1, x_2) = 0\]  \hspace{1cm} (1.4)

\[x_1^2(x_2, x_3) + 2(x_1, x_2)(x_1, x_3) = 0\]  \hspace{1cm} (1.5)

for every $x_1, x_2, x_3, x_4 \in N$. It was proved in [1] that the set of idempotents of weight 1 in algebras satisfying (1.1) is given by $\text{Ip}(A) = \{z^3 ; \omega(z) = 1\}$.

Every $e \in \text{Ip}(A)$ determines a decomposition $A = Ke \oplus U_e \oplus V_e$ called the Peirce decomposition of $A$, where $N = U_e \oplus V_e$ and

\[U_e = \{u \in A ; eu = \frac{1}{2}u\}\]

\[V_e = \{v \in A ; ev = 0\}\]

and, moreover,

\[U_e V_e \subseteq U_e \quad U_e^2 \subseteq V_e \quad V_e^2 \subseteq V_e\]

The elements $u \in U_e$ and $v \in V_e$ satisfy the identities

\[u^3 = 0\]

\[v^3 = 0\]

\[uv^2 = 2(uv)v\]

\[u^2v = 2u(uv)\]  \hspace{1cm} (1.6)
and their linearizations

\begin{align*}
(1.7) \quad u_1^2 u_2 + 2u_1(u_1 u_2) &= 0 \\
(1.8) \quad u_1(u_2 u_3) + u_2(u_3 u_1) + u_3(u_1 u_2) &= 0 \\
(1.9) \quad v_1(v_2 v_3) + v_2(v_3 v_1) + v_3(v_1 v_2) &= 0 \\
(1.10) \quad u(v_1 v_2) &= (uv_1)v_2 + (uv_2)v_1 \\
(1.11) \quad (u_1 u_2)v &= u_1(u_2 v) + u_2(u_1 v)
\end{align*}

It was also proved in \cite{1} that \( \text{Ip}(A) = \{e + u + u^2 ; \ u \in U_e\} \) and if \( f = e + u_0 + u_0^2 \), for \( u_0 \in U_e \) then

\begin{align*}
(1.12) \quad U_f &= \{u + 2u_0u; \ u \in U_e\} \\
(1.13) \quad V_f &= \{v - 2u_0v; \ v \in V_e\}
\end{align*}

Using the previous identities, it is easy to prove that

\begin{equation}
(1.14) \quad x_1(x_1^2 x_2) = 0
\end{equation}

for every \( x_1, x_2 \in N \). Since \( A \) is commutative, the identities (1.4) and (1.14) imply that \( N \) is a Jordan algebra (in fact, of a very special kind: \( x_1^2(x_1 x_2) = x_1(x_1^2 x_2) = 0 \)). For more information see \cite{2}

**1.2. Algebras satisfying \((x^2)^2 = \omega(x^3)x\)**

Etherington showed in \cite{4} that if \((A, \omega)\) satisfies the train equation in principal powers (of degree 3)

\begin{equation}
(1.15) \quad x^3 - (1 + \gamma)\omega(x)x^2 + \gamma \omega(x^2)x = 0
\end{equation}

with \( \gamma \in K \), then \((A, \omega)\) also satisfies the train equation in plenary powers (of degree 4)

\begin{equation}
(1.16) \quad (x^2)^2 - (1 + 2\gamma)\omega(x^2)x^2 + 2\gamma \omega(x^3)x = 0
\end{equation}

In \cite{6}, Walcher proved that (1.15) and (1.16) are equivalent, excepting for \( \gamma = 0 \) and \( \gamma = -\frac{1}{2} \). If \( \gamma = 0 \) in (1.16), then \((x^2)^2 = \omega(x^2)x^2\) and, in this case, \((A, \omega)\) is a Bernstein algebra. Now, if \( \gamma = -\frac{1}{2} \), then \((A, \omega)\) satisfies

\( (x^2)^2 = \omega(x^3)x \)

These algebras are also studied in \cite{1}, \cite{5}, and \cite{6}. The following results are proved in \cite{5}. Every algebra \( A \) satisfying (1.2) has an idempotent given
by $e = (d^3)^3$ where $\omega(d) = 1$. Each idempotent $e$ of $A$ determines a Peirce decomposition $A = Ke \oplus U_e \oplus V_e$, where

$$U_e = \{ u \in \ker \omega ; eu = \frac{1}{2}u \}$$

$$V_e = \{ v \in \ker \omega ; ev = -\frac{1}{2}v \}$$

These subspaces satisfy the inclusions

$$U_e V_e \subseteq U_e, \quad U_e^2 \subseteq V_e, \quad V_e^2 \subseteq V_e$$

As in the preceding section, we have $N = \ker \omega$ and $N = U_e \oplus V_e$. From (1.2), we have, for every $x \in N$,

$$(x^2)^2 = 0$$

Likewise, $A$ also satisfies the identities (1.4) and (1.5). Another identity in algebras satisfying (1.2) is

(1.17) \[ x_1(x_2^2x_3) = 0 \]

From (1.4) and (1.17), $N$ is a Jordan algebra of a very special kind, as remarked at the end of 1.1. For every $u \in U$ and $v \in V$ we have

$$u^3 = 0$$

$$v^3 = 0$$

(1.18) \[ u^2v = 2u(uv) \]

$$uv^2 = 2v(vu)$$

By linearization of these identities we obtain

(1.19) \[ u_1(u_2u_3) + u_2(u_3u_1) + u_3(u_1u_2) = 0 \]

$$v_1(v_2v_3) + v_2(v_3v_1) + v_3(v_1v_2) = 0$$

$$u_1(u_2v) = u_1(u_2v) + u_2(u_1v)$$

$$u(v_1v_2) = (uv_1)v_2 + (uv_2)v_1$$

It is proved in [1] that the set of idempotents of weight 1 of $A$ is given by

$$\mathbf{I}p(A) = \{ e + u + \frac{1}{2}u^2 ; \ u \in U_e \}$$

and if $f = e + u_0 + \frac{1}{2}u_0^2, u_0 \in U_e$, is another idempotent, then

$$U_f = \{ u + u_0u ; \ u \in U_e \}$$

$$V_f = \{ v - u_0v ; \ v \in V_e \}$$
2. Invariance of $p$ -subspaces

We will use in this section the same terminology for $p$ -subspaces found in [3]. Let $A = Ke \oplus Ue \oplus Ve$ be a Peirce decomposition of an algebra satisfying (1.1) or (1.2). Subspaces of $A$ obtained by means of a monomial expression in $Ue$ and $Ve$ such as $Ue, V e, U^2 e, V^2 e, UeV e, Ue(UeV e)$, etc

are called $p$ -monomials. If $m$ denotes a $p$ -monomial, then $\partial m$ indicates the degree of $m$. The inclusions $UV \subseteq U, U^2 \subseteq V e V^2 \subseteq V$, valid in both cases (1.1) and (1.2), imply that there are two possibilities for a $p$ -monomial $m : m \subseteq U$ or $m \subseteq V$. A $p$ -subspace of $A$ is a sum of $p$ -monomials. For instance, $Ue, V e, Ue + V e, UeV e + V^2 e, U^2 e + V^3 e + (UeV e)V e$

are examples of $p$ -subspaces. A $p$ -monomial is, of course, a particular case of $p$ -subspace. In general, all $p$ -subspaces can be obtained from an ordinary polynomial $p(x, y)$ in two commutative and non associative variables upon the substitution of $x$ for $Ue$ and $y$ for $Ve$. We denote such $p$ -subspaces by $pe$ or simply by $p$. Given a $p$ -subspace $p$, there are two subspaces $g \subseteq U$ and $h \subseteq V$ of $A$ such that $p = g \oplus h$. Choosing another idempotent $f \in Ip(A)$ and proceeding as before for the same polynomial $p(x, y)$ we obtain a subspace $pf$. If $pe = pf$ for every $e, f \in Ip(A)$, we say that $p$ is invariant. If $\dim pe = \dim pf$ for every $e, f \in Ip(A)$, we say that $p$ has invariant dimension. In the next section, we prove that all $p$ -subspaces of algebras satisfying (1.1) or (1.2) have invariant dimension. We will also find a necessary and sufficient condition, of easy verification, for a $p$ -subspace being invariant. Such a condition is also necessary and sufficient for a $p$ -subspace being an ideal of $A$. These results allow us to introduce a large number of numerical invariants both in cases (1.1) and (1.2), namely the dimension of $p$ -subspaces.

2.1. Invariance in algebras satisfying $(x^2)^2 = \omega(x)x^3$

We suppose in this section that $(A, \omega)$ satisfies the baric equation in the title. Given $e, f \in Ip(A)$, let the functions $\sigma : Ue \rightarrow Uf$ and $\tau : Ve \rightarrow Vf$ be defined by

$$
\sigma(u) = u + 2u_0 u \\
\tau(v) = v - 2u_0 v
$$
where \( f = e + u_0 + u_0^2 \) with \( u_0 \in U_e \). From (1.12) and (1.13), \( \sigma \) and \( \tau \) are surjective. If \( u \in U_e \) and \( v \in V_e \) are such that \( \sigma(u) = \tau(v) = 0 \), then \( u = -2u_0u \in U_e \cap V_e \) and so \( u = 0 \). In the same way, \( v = 0 \) so that \( \sigma \) and \( \tau \) are injective. Therefore \( \sigma \) and \( \tau \) are isomorphisms of vector spaces. Consequently, \( U \) and \( V \) have invariant dimension. The isomorphism of vector spaces \( \varphi : A \to A \) defined by \( \varphi(\alpha e + u + v) = \alpha f + \sigma(u) + \tau(v) \) is called the Peirce transformation of \( A \) associated to \( e \) and \( f \). The linear operators \( \xi : U_e \to U_e \) and \( \zeta : V_e \to V_e \) defined by

\[
\xi(u) = u - 2u_0^2u \\
\zeta(v) = v + 2u_0^2v
\]

are also isomorphisms of vector spaces. In fact, if \( \xi(u) = 0 \), then \( u = 2u_0^2u \). Multiplying this equality by \( u_0 \) and using (1.14), we have that \( u_0u = 2u_0u(2u_0^2u) = 0 \). Likewise, from (1.7) we have \( u = 2u_0^2u = -4u_0(u_0u) = 0 \). Therefore, \( \xi \) is injective. Now, if \( \zeta(v) = 0 \), then \( v = -2u_0^2v \). In the same way, \( u_0v = 0 \). Thus, from (1.6) we have \( v = 0 \).

**Lemma 2.1.** The functions \( \sigma, \tau, \xi \) and \( \zeta \) satisfy the following identities, for \( u, u_1, u_2 \in U \) and \( v, v_1, v_2 \in V \):

1. \( \sigma(u_1)\sigma(u_2) = \tau(\xi(u_1)\xi(u_2)) \);
2. \( \sigma(u)\tau(v) = \sigma(\xi(u)\xi(v)) \);
3. \( \tau(v_1)\tau(v_2) = \tau(\zeta(v_1)\zeta(v_2)) \).

**Proof.** From (1.5) and (1.8) we have

\[
\sigma(u_1)\sigma(u_2) = (u_1)(u_2) - 2u_0(u_1)(u_2) - 2(u_0^2)((u_1)(u_2)).
\]

Using (1.3), (1.5) and (1.11) we have \( \xi(u_1)\xi(u_2) = (u_1)(u_2) - 2u_0^2((u_1)(u_2)) \). Hence, \( \sigma(u_1)\sigma(u_2) = \tau(\xi(u_1)\xi(u_2)) \) by (1.14). Now, from (1.5) and (1.11),

\[
\sigma(u)\tau(v) = uv + 2u_0(uv) + 2u_0^2(uv).
\]

It follows from (1.3), (1.5) and (1.10) that \( \xi(u)\xi(v) = uv + 2u_0^2(uv) \). From (1.14) we obtain \( \sigma(u)\tau(v) = \sigma(\xi(u)\xi(v)) \). Finally (1.5) and (1.10) imply that \( \tau(v_1)\tau(v_2) = v_1v_2 - 2u_0(v_1v_2) - 2u_0^2(v_1v_2) \). Using (1.3), (1.5) and (1.9) we prove that \( \zeta(v_1)\zeta(v_2) = v_1v_2 - 2u_0^2(v_1v_2) \). Therefore, \( \tau(v_1)\tau(v_2) = \tau(\zeta(v_1)\zeta(v_2)) \) by (1.14).

The next corollary follows immediately from the previous lemma.

**Corollary 2.2.** Let \( X, X_1, X_2 \subseteq U \) and \( W, W_1, W_2 \subseteq V \) be subspaces of \( A = Ke \oplus U \oplus V \). Then
(a) $\sigma(X_1)\sigma(X_2) = \tau(\xi(X_1)\xi(X_2))$;
(b) $\sigma(X)\tau(W) = \sigma(\xi(X)\zeta(W))$;
(c) $\tau(W_1)\tau(W_2) = \tau(\zeta(W_1)\zeta(W_2))$.

In the following proposition we show that $p$-subspaces of $A$ absorb products by $V$.

**Proposition 2.3.** Every $p$-subspace $p$ of $A$ satisfies $V p \subseteq p$.

**Proof.** It is enough to prove the statement for monomials. We have, when the degree of $m$ is 1,

$$
VU \subseteq U \\
VV \subseteq V
$$

If $\partial m \geq 2$ then one of the 3 next possibilities might occur:

$$
m = \mu\nu \\
m = \mu_1\mu_2 \\
m = \nu_1\nu_2
$$

where $\mu, \mu_1, \mu_2 \subseteq U$ and $\nu, \nu_1, \nu_2 \subseteq V$ are $p$-monomials with lower degree than $\partial m$. A generator of $V(\mu\nu)$ has the form $v(\mu\nu)$, where $v \in V$, $u \in \mu$ and $w \in \nu$. From (1.10),

$$
v(uw) = u(vw) - w(uv) \in \mu(V\nu) + \nu(V\mu)
$$

We have that $V(\mu_1\mu_2) = \langle v((u_1)(u_2)) ; v \in V, u_1 \in \mu_1, u_2 \in \mu_2 \rangle$. Using (1.11) we obtain

$$
v(u_1u_2) = u_1(u_2v) + u_2(u_1v) \in \mu_1(\mu_2V) + \mu_2(\mu_1V)
$$

Finally, $V(\nu_1\nu_2)$ is spanned by elements having the form $v(w_1w_2)$, where $v \in V$, $w_1 \in \nu_1$ and $w_2 \in \nu_2$.

From (1.9),

$$
v(w_1w_2) = -w_1(w_2v) - w_2(vw_1) \in \nu_1(\nu_2V) + \nu_2(V\nu_1)
$$

Now, by induction on $\partial m$, we obtain $V m \subseteq m$. 

\square
Corollary 2.4. For all \( p \)-subspaces \( g \subseteq U \) and \( h \subseteq V \) we have
\[
\xi(g) = g \\
\zeta(h) = h
\]

Proof. Let \( u \in g \). From the preceding proposition, \( \xi(u) = u - 2u_0^2u \in g + Vg = g \). Next, \( \xi(g) \subseteq g \) and, as \( \xi \) is an isomorphism of vector spaces, \( \xi(g) = g \). In the same way, it is shown that \( \zeta(h) = h \). \( \blacksquare \)

Subsequently, we have the main result referring to algebras satisfying (1.1).

Theorem 2.5. Let \( A \) be a baric algebra satisfying (1.1). Then:

(1) Every \( p \)-subspace \( p \) of \( A \) satisfies \( \varphi(p_e) = p_f \), where \( \varphi \) is the Peirce transformation of \( A \) associated to the idempotents \( e \) and \( f \). In particular, every \( p \)-subspace has invariant dimension.

(2) The following statements are equivalent relative to the \( p \)-subspace \( p \) of \( A \):

\begin{enumerate}
\item \( p \) is invariant;
\item \( Up \subseteq p \);  
\item \( p \) is an ideal of \( A \).
\end{enumerate}

Proof. It suffices to show the statement (1) for a \( p \)-monomial \( m \) and, for this, we use induction on \( \partial m \). Let \( e \) be an idempotent of weight 1 in \( A \) and, for all \( u \in U_e \), we consider \( f = e + u + u^2 \). Since \( \sigma \) and \( \tau \) are isomorphisms, we have
\[
U_f = \sigma(U_e) = \varphi(U_e) \\
V_f = \tau(V_e) = \varphi(V_e)
\]

Let us suppose that the result is true for all \( p \)-monomials with degree \( \leq k \) and let \( m \) be a \( p \)-monomial with \( \partial m = k + 1 \). There are 3 possibilities: \( m = \mu \nu, m = \mu_1 \mu_2, m = \nu_1 \nu_2 \), where \( \mu, \mu_1, \mu_2 \subseteq U \) and \( \nu, \nu_1, \nu_2 \subseteq V \) are \( p \)-monomials with degree \( \leq k \). If \( m = \mu \nu \) then,
\[
m_f = \mu \nu_f = \sigma(\mu_e)\tau(\nu_e)
\]
From corollaries 2.2 and 2.4 we have
\[ m_f = \sigma(\xi(\mu_e)\zeta(\nu_e)) = \sigma(\mu_e
\nu_e) = \sigma(m_e) = \varphi(m_e) \]

The other cases are similar. To prove part (2), we let \( p = g \oplus h \), where \( g \) and \( h \) are \( p \)-subspaces with \( g \subseteq U \) and \( h \subseteq V \). From part (1),
\[ p_f = g_f \oplus h_f = \{ \sigma(x) + \tau(w) ; x \in g_e, w \in h_e \}. \]
Therefore,
\[
(2.20) \quad p_f = \{ (x - 2uw) + (w + 2ux) ; x \in g_e, w \in h_e \}
\]

Let us suppose that \( p \) is invariant. Then, \( p_f = p_e \) for every \( e, f \in \text{Ip}(A) \). It follows from (2.20) that for \( x \in g_e \) and \( w \in h_e \), there are \( x' \in g_e \) and \( w' \in h_e \) such that \( x - 2uw = x' \) and \( w + 2ux = w' \). Then,
\[
uw = \frac{1}{2}(x - x') \\
ux = \frac{1}{2}(w' - w)
\]

It means that \( U_e g_e \subseteq h_e \) and \( U_e h_e \subseteq g_e \) and so \( U_e p_e \subseteq p_e \). Reciprocally, let us suppose that \( Up \subseteq p \). We can state (2.20) as
\[
p_f = \{ x + w + 2u(x - w) ; x \in g_e, w \in h_e \}
\]
and so we obtain \( p_f \subseteq p_e + U e p_e \). Now, using the hypothesis, we conclude that \( p_f \subseteq p_e \). Since this inclusion is valid for every pair of idempotents, then \( p_e = p_f \) and \( p \) is invariant. Finally, as we know that \( A = K e \oplus U \oplus V \), \( ep = g \subseteq p \) and \( V p \subseteq p \), and so \( p \) is an ideal if and only if \( Up \subseteq p \). \( \square \)

2.2. **Invariance in algebras satisfying** \((x^2)^2 = \omega(x^3)x\)

We suppose now that the baric algebra \((A, \omega)\) satisfies the baric equation in the title. The proof of next proposition is the same as that of Proposition 2.3 and will be omitted.

**Proposition 2.6.** If \( p \) is a \( p \)-subspace of any algebra satisfying (1.2) then \( V p \subseteq p \).

\( \square \)

For any \( u_0 \in U \) and \( \alpha, \beta \in K \) we consider the linear operator
\[ T_{(\alpha, \beta)} : N \to N \text{ defined by} \]
\[ T_{(\alpha, \beta)}(x) = x + \alpha u_0 x + \beta u_0^2 x \]

Such operators satisfy the following properties:
Lemma 2.7. For every $u_0 \in U$ and $\alpha, \beta \in K$, we have:

1. $T_{(\alpha, \beta)}$ is an automorphism of vector spaces;
2. $T_{(0, \beta)}(p) = p$ for every $p$-subspace $p$ of $A$.

Proof. Let $x = u + v \in N$ ($u \in U$ and $v \in V$) be such that $T_{(\alpha, \beta)}(x) = 0$. Then $u + v + \alpha u_0(u + v) + \beta u_0^2(u + v) = 0$, and so

$$
\begin{align*}
u + \alpha u_0 v + \beta u_0^2 u &= 0 \\
v + \alpha u_0 u + \beta u_0^2 v &= 0
\end{align*}
$$

Multiplying these identities by $u_0$ and using (1.17), (1.18) and (1.19), we obtain

$$
\begin{align*}
u_0 u + \frac{1}{2} \alpha u_0^2 v &= 0 \\
u_0 v - \frac{1}{2} \alpha u_0^2 u &= 0
\end{align*}
$$

Again, multiplying by $u_0$ the latest 2 equalities and using (1.17), (1.18) and (1.19), we have $u_0^2 u = u_0^2 v = 0$. Then $u_0 u = u_0 v = 0$. Therefore $u = v = 0$, $T_{(\alpha, \beta)}$ is injective and so it is an isomorphism. Let $x \in p$; we know that $u_0 \in V$, and so from Proposition 2.6, we have

$$
\begin{align*}
T_{(0, \beta)}(p) &= x + \beta u_0^2 x \in p + Vp = p
\end{align*}
$$

Hence $T_{(0, \beta)}(p) \subseteq p$ and, since $T_{(\alpha, \beta)}$ is injective, we have $T_{(0, \beta)}(p) = p$. \hfill \Box

Given $e, f \in Ip(A)$, where $f = e + u_0 + \frac{1}{2} u_0^2$, $u_0 \in Ue$, we use the following notations:

$$
\sigma = T_{(1,0)}|Ue, \quad \tau = T_{(-1,0)}|Ve, \quad \xi = T_{(0, -\frac{1}{2})}|Ue, \quad \zeta = T_{(0, \frac{1}{2})}|Ve.
$$

We have that

$$
\sigma : U_e \rightarrow U_f, \quad \tau : V_e \rightarrow V_f, \quad \xi : U_e \rightarrow U_e \quad \text{and} \quad \zeta : V_e \rightarrow V_e.
$$

The vector space isomorphism $\varphi : A \rightarrow A$ defined by $\varphi(\alpha e + u + v) = \alpha f + \sigma(u) + \tau(v)$ is the Peirce transformation of $A$ associated to $e$ and $f$. The next result is proved similarly to Lemma 2.1:

Lemma 2.8. The functions $\sigma, \tau, \xi$ and $\zeta$ satisfy the identities:

(a) $\sigma(u_1)\sigma(u_2) = \tau(\xi(u_1)\xi(u_2))$;
(b) $\sigma(u)\tau(v) = \sigma(\xi(u)\zeta(v))$;
(c) $\tau(v_1)\tau(v_2) = \tau(\zeta(v_1)\zeta(v_2))$;
for every $u, u_1, u_2 \in U_e$ and $v, v_1, v_2 \in V_e$.

Finally, as we have done before, we can prove the next theorem.

**Theorem 2.9.** Let $A$ be a baric algebra satisfying (1.2). Then

1. Every $p$-subspace $p$ of $A$ satisfies $\varphi(p_e) = p_f$. In particular, every $p$-subspace has invariant dimension.

2. The following statements about a $p$-subspace $p$ of $A$ are equivalent:
   
   - (2.1) $p$ is invariant;
   - (2.2) $U_p \subseteq p$;
   - (2.3) $p$ is an ideal.

**References**


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