Proyecciones
Vol. 23, N° 1, pp. 51-60, May 2004.
Universidad Católica del Norte
Antofagasta - Chile

A NOTE ON QUASI n -MAPS

IVÁN CORREA ^{a *} and ALICIA LABRA ^{b †}

^a *Universidad de La Serena, Chile.*

^b *Universidad de Chile, Chile.*

Received May 2003. Accepted January 2004.

Abstract

Using a factorization of quasi n -maps we find a relationship between the module formed by the n -maps and the module formed by the quasi n -maps. In particular, we characterize the quasi cubic forms using a relation called the parallelepiped law. Moreover we give necessary and sufficient conditions for the equality of the modules of quasi cubic forms and cubic forms for any module M .

AMS classification : 13 C 99.

Keywords : n -maps, quasi n -maps, quasi cubic form, parallelepiped law.

---

*Supported by Fondecyt 1010196.

†Supported by Fondecyt 1030919.

1. Introduction

In this paper, every ring is commutative and has a unity element. Every module is unitary.

Let R be a ring and M, N be R -modules. Let $n \geq 1$ be an integer. We denote by M^n the direct sum of n copies of M . A map $f : M \rightarrow N$ is called a n -map if it satisfies

$$(1) \quad f(\lambda x) = \lambda^n f(x) \text{ for every } x \in M, \lambda \in R,$$

and the map $\phi_f : M^n \rightarrow N$ defined by

$$(2) \quad \phi_f(x_1, \dots, x_n) = \sum_{p=1}^n (-1)^{n-p} \sum_{1 \leq i_1 < \dots < i_p \leq n} f(x_{i_1} + \dots + x_{i_p})$$

is R -multilinear.

A map $f : M \rightarrow N$ is called a *quasi n -map* if it satisfies (1) and ϕ_f is n -additive.

The following examples show that both conditions are necessary to define a quasi n -map.

Example 1. Let $\mathbf{Q}(x)$ be a simple transcendental extension of the rational numbers $\mathbf{Q}$ and D a non-zero derivation on $\mathbf{Q}(x)$. We define a map $f : \mathbf{Q}(x)(\sqrt{2}) \rightarrow \mathbf{Q}(x)$ by $f(a+b\sqrt{2}) = b^2 D(a) - a^2 D(b)$. Then $f(\lambda x) \neq \lambda^3 f(x)$ for every $x = a + b\sqrt{2}$ and $\lambda \in \mathbf{Q}(x)$ such that $a \neq 0, b \neq 0, \lambda \neq 0, a \neq b$. Moreover, the map $\phi_f : (\mathbf{Q}(x)(\sqrt{2}))^3 \rightarrow \mathbf{Q}(x)$ associated to f is given by

$$\phi_f(a + b\sqrt{2}, c + d\sqrt{2}, e + g\sqrt{2}) =$$

$$2dgD(a) + 2gbD(c) + 2bdD(e) - 2ceD(b) - 2eaD(d) - 2acD(g)$$

and it is 3-additive. Hence, f is not a quasi 3-map.

Example 2. Let $\mathbf{R}$ be the real numbers, $M = \mathbf{R}^2$ and $N = \mathbf{R}$. Let $f : M \rightarrow N$ be the map defined by $f(x, y) = \frac{x^5}{y^2}$ for every $y \neq 0$ and $f(x, 0) = 0$. Then $f(\lambda x) = \lambda^3 f(x)$ for all $x \in \mathbf{R}^2, \lambda \in \mathbf{R}$. But if $x = y = (1, 0)$ and $x' = z = (0, 1)$ then $\phi_f(x, y, z) + \phi_f(x', y, z) = 30 + \frac{-7}{4} = \frac{113}{4}$ while $\phi_f(x + x', y, z) = \frac{-97}{4}$. Hence, f is not a quasi 3-map.

As it is clear, any n -map is a quasi n -map. In general the converse is not true as we see with the following example:

Example 3. Let $\mathbf{Q}(x)$ be a simple transcendental extension of $\mathbf{Q}$ and $D : \mathbf{Q}(x) \rightarrow \mathbf{Q}(x)$ a nonzero derivation. As it is clear, the map $f : \mathbf{Q}(x)(\sqrt{2}) \rightarrow \mathbf{Q}(x)$ defined by $f(a + b\sqrt{2}) = bD(a) - aD(b)$ satisfies $f(\lambda(a + b\sqrt{2})) = \lambda^2 f(a + b\sqrt{2})$. The map ϕ_f associated to f is given by

$$\phi_f(a + b\sqrt{2}, c + d\sqrt{2}) = bD(c) + dD(a) - cD(b) - aD(d)$$

and it is 2-additive but is not bilinear. Thus f is a quasi 2-map but it is not a 2-map.

In what follows we denote by $F_n(M, N)$ the R -module formed by the n -maps and by $QF_n(M, N)$ the R -module formed by the quasi n -maps, from M to N .

2. Linear factorization of quasi n -maps

Ferrero and Micali [1] proved that, for every R -module M , there exists a R -module $\Gamma_n(M)$ and a n -map $\gamma_n : M \rightarrow \Gamma_n(M)$ such that for every n -map $f : M \rightarrow N$ there is a unique R -linear map $\tilde{f} : \Gamma_n(M) \rightarrow N$ such that $\tilde{f} \circ \gamma_n = f$.

If we denote by $R^{(M)}$ the free R -module with basis M and by $M^{\otimes n}$ the tensor product of n copies of M , then $\Gamma_n(M) = R^{(M)} \times M^{\otimes n} / R_n(M)$. Here $R_n(M)$ is the submodule of $R^{(M)} \times M^{\otimes n}$ generated by the elements

$$(e_{\alpha x} - \alpha^n e_x, 0), \quad \left(\sum_{1 \leq i_1 < \dots < i_p \leq n} (-1)^{n-p} e_{x_{i_1} + \dots + x_{i_p}}, -x_1 \otimes \dots \otimes x_n \right),$$

where $\{e_x\}_{x \in M}$ is the canonical basis of $R^{(M)}$, $\alpha \in R$, $x, x_1, \dots, x_n \in M$. The n -map γ_n is the composition of $x \in M \rightarrow (e_x, 0) \in R^{(M)} \times M^{\otimes n}$ and the canonical mapping $R^{(M)} \times M^{\otimes n} \rightarrow \Gamma_n(M)$. The R -module $\Gamma_n(M)$ is generated by the family $\{\gamma_n(x)\}_{x \in M}$.

As a consequence of these results, we have

$$(3) \quad F_n(M, N) \cong \text{Hom}_R(\Gamma_n(M), N),$$

where $\text{Hom}_R(\Gamma_n(M), N)$ is the R -module of all R -linear maps from $\Gamma_n(M)$ to N .

Proposition 1. *Given an R -module M , there exists an R -module $H_n(M)$ and a quasi n -map $\mu_n : M \rightarrow H_n(M)$ with the following property: for every $f \in QF_n(M, N)$ there is a unique R -linear map $\tilde{f} : H_n(M) \rightarrow N$ such that $\tilde{f} \circ \mu_n = f$.*

Proof. Let $H_n(M) = R \otimes_Z \Gamma_n(M) / L_n(M)$ where $L_n(M)$ is the submodule of $R \otimes_Z \Gamma_n(M)$ generated by the elements $1 \otimes_Z \lambda^n \gamma_n(x) - \lambda^n \otimes_Z \gamma_n(x)$ with $\lambda \in R$ and $x \in M$. We make $R \otimes_Z \Gamma_n(M)$ an R -module by defining $\alpha(\lambda \otimes_Z \gamma_n(x)) = \alpha\lambda \otimes_Z \gamma_n(x)$. Then $H_n(M)$ is also an R -module. We define the map $\mu_n : M \rightarrow H_n(M)$ as follows: $\mu_n(x) = \pi(1 \otimes_Z \gamma_n(x))$, where $\pi : R \otimes_Z \Gamma_n(M) \rightarrow H_n(M)$ is the canonical mapping. As it is clear $\mu_n(\lambda x) = \lambda^n \mu_n(x)$. Also since $\phi_{\mu_n} = \pi \circ (1 \otimes_Z \phi_{\gamma_n})$ it follows that ϕ_{μ_n} is n -additive. Thus μ_n is a quasi n -map.

If $f \in QF_n(M, N)$ then ϕ_f is n -additive. Thus ϕ_f is $\mathbf{Z}$ -multilinear and consequently f is an n -map over $\mathbf{Z}$. Hence, there exists a unique $\mathbf{Z}$ -linear map $\tilde{f} : \Gamma_n(M) \rightarrow N$ such that $\tilde{f} \circ \gamma_n = f$. It follows that the map $\sigma : R \times \Gamma_n(M) \rightarrow N$ defined by $\sigma(\lambda, \gamma_n(x)) = \lambda \tilde{f}(\gamma_n(x)) = \lambda f(x)$ is $\mathbf{Z}$ -bilinear. Therefore, there is a unique $\mathbf{Z}$ -linear map $\bar{\sigma} : R \otimes_{\Gamma_n} (M) \rightarrow N$ such that $\bar{\sigma}(\lambda \otimes_{\gamma_n}(x)) = \sigma(\lambda, \gamma_n(x))$. Moreover, $\bar{\sigma}$ is R -linear. Since $L_n(M) \subseteq \text{Ker}(\bar{\sigma})$, there exists a unique R -linear map $\tilde{f} : H_n(M) \rightarrow N$ such that $\tilde{f} \circ \pi = \bar{\sigma}$. Thus $\tilde{f} \circ \mu_n = f$.

Finally, since the R -module $H_n(M)$ is generated by the set $\{\mu_n(x)\}_{x \in M} = \{\pi(1 \otimes_{\gamma_n}(x))\}_{x \in M}$, the uniqueness of $\tilde{f}$ follows from $\tilde{f} \circ \mu_n = f$.

Corollary 1. *For any R -modules M and N we have*

$$QF_n(M, N) \cong \text{Hom}_R(H_n(M), N).$$

Proposition 2. *For every R -module M there exists a surjective R -linear map $\tau_n : H_n(M) \rightarrow \Gamma_n(M)$. Furthermore, for any R -module N*

$$QF_n(M, N) \cong F_n(M, N) \oplus \text{Hom}_R(\Gamma'_n(M), N),$$

where $\Gamma'_n(M)$ denotes the kernel of τ_n .

Proof. The map $g : R \otimes_{\Gamma_n} (M) \rightarrow \Gamma_n(M)$ defined by $g(\sum \alpha_i \otimes_{\gamma_n}(x_i)) = \sum \alpha_i \gamma_n(x_i)$ is a surjective R -linear map and $L_n(M) \subseteq \text{Ker}(g)$. Therefore, there exists a unique R -linear map $\tau_n : H_n(M) \rightarrow \Gamma_n(M)$ such that $\tau_n \circ \pi = g$. Since g is surjective, τ_n is also surjective.

Let $\tau'_n : \Gamma_n(M) \rightarrow H_n(M)$ be the R -linear map defined by $\tau'_n(\gamma_n(x)) = \pi(1 \otimes_Z \gamma_n(x))$. Then $\tau_n \circ \tau'_n = I_{\Gamma_n(M)}$. Hence, the exact sequences

$$0 \rightarrow \Gamma'_n(M) \hookrightarrow H_n(M) \longrightarrow \Gamma_n(M) \rightarrow 0$$

and

$$0 \rightarrow \text{Hom}_R(\Gamma_n(M), N) \rightarrow \text{Hom}_R(H_n(M), N) \rightarrow \text{Hom}_R(\Gamma'_n(M), N) \rightarrow 0$$

split. The result follows now from (3) and Corollary 1.

3. Quasi cubic forms

We call the n -maps $f : M \rightarrow R$ n -forms.

Let M be an R -module. Given a map $f : M \rightarrow R$, we introduce the following identity:

$$(4) \quad \sum \phi_f(x_1, \pm x_2, \dots, \pm x_n) = 0,$$

where the $\sum$ is taken over all possible combinations of $+$ and $-$. For $n = 2$ we have the well-known *parallelogram law*

$$f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) - 2f(x_2) = 0.$$

Thus we call (4) *the generalized parallelogram law*. If ϕ_f is n -additive then f satisfies (4) since

$$\sum \phi_f(x_1, \pm x_2, \dots, \pm x_n) = \phi_f(x_1, x_2 - x_2, \dots, x_n - x_n) = 0.$$

In particular, any quasi n -form satisfies (4).

When $n = 3$ we call (4) *the parallelepiped law*, in this case f satisfies $f(x + y + z) + f(x + y - z) + f(x - y + z) + f(x - y - z) = 2[f(x + y) + f(x - y) + f(x + z) + f(x - z)] - 4f(x)$. Geometrically, this law gives a relation between the inner diagonals of the parallelepiped, those from the two contiguous faces and the common edge.

As proved by J. von Neumann and P. Jordan in 1935 for a more particular situation, if R is a ring such that 2 is not a zero divisor, then a map $f : M \rightarrow R$ satisfying (1) for $n = 2$ is a quasi quadratic form (i.e., a quasi 2-form) if and only if it verifies the parallelogram law. See Gleason [2] but notice that here we are using a different terminology and considering a more general situation.

Following the ideas given by Gleason, we give now a characterization of quasi cubic forms (i.e., quasi 3-forms) in terms of the parallelepiped law.

Theorem 1. *Let R be a ring. Assume that 2 is not a zero divisor in R . Let M be an R -module and $f : M \rightarrow R$ be a map. Then f is a quasi cubic*

form if and only if f satisfies the parallelepiped law and $f(\lambda x) = \lambda^3 f(x)$ for any $\lambda \in R$, $x \in M$.

Proof. As mentioned above any quasi n -form satisfies (4). In particular, any quasi cubic form satisfies the parallelepiped law.

Assume now that f satisfies the parallelepiped law and $f(\lambda x) = \lambda^3 f(x)$ for any $\lambda \in R$, $x \in M$. To conclude that f is a quasi cubic form we have to prove that ϕ_f is 3-additive. Since ϕ_f is symmetric in each variable, it suffices to prove the additivity in the first variable.

Using the fact that $f(-x) = -f(x)$ for each $x \in M$, by adding the identities

$$\sum \phi_f(x, \pm y, \pm z) = 0, \quad \sum \phi_f(y, \pm x, \pm z) = 0, \quad \sum \phi_f(z, \pm x, \pm y) = 0,$$

we obtain the identity

$$(5) \quad 4\phi_f(x, y, z) = f(x+y+z) - f(x+y-z) - f(x-y+z) + f(x-y-z).$$

The identity $\sum \phi_f(x, \pm z, \pm z) = 0$ yields

$$f(x+2z) + f(x-2z) = 4\{f(x+z) + f(x-z)\} - 6f(x).$$

This identity together with (5) implies

$$\begin{aligned} \phi_f(x+x', 2y, 2z) &= f(x+x'+y+z) - f(x+x'-y+z) \\ &\quad - f(x+x'+y-z) + f(x+x'-y-z). \end{aligned}$$

It follows from this by using the sum of the identities

$$\begin{aligned} \sum \phi_f(x, \pm(y+z), \pm x') &= 0, \quad \sum \phi_f(x', \pm(y+z), \pm x) = 0, \\ -\sum \phi_f(x, \pm(-y+z), \pm x') &= 0, \quad -\sum \phi_f(x', \pm(-y+z), \pm x) = 0, \end{aligned}$$

and (5) that

$$\phi_f(x+x', 2y, 2z) = 4\{\phi_f(x, y, z) + \phi_f(x', y, z)\}.$$

Letting $x' = 0$ we obtain $\phi_f(x, 2y, 2z) = 4\phi_f(x, y, z)$. Hence

$$4\phi_f(x+x', y, z) = \phi_f(x+x', 2y, 2z) = 4\{\phi_f(x, y, z) + \phi_f(x', y, z)\}$$

and the result follows.

We remark that the condition 2 is not a zero divisor is necessary as the following example shows:

Example 4. Let F_2 be the field with 2 elements and $f : F_2[x] \rightarrow F_2$ the map defined by $f(0) = 0$, $f(p(x)) = 1$ for every $p(x) \neq 0$. Then it is clear that f satisfies the parallelepiped law since 2 is a zero divisor. Moreover, $f(\lambda p(x)) = \lambda^3 f(p(x))$ for every $\lambda \in F_2$, $p(x) \in F_2[x]$, but $\phi_f(x^3 + 1, x^2, x^5) = 1$ while $\phi_f(x^3, x^2, x^5) + \phi_f(1, x^2, x^5) = 0$. Hence f is not a quasi cubic form.

Remark 1. Let $f : M \rightarrow R$ be a form satisfying $f(\lambda x) = \lambda^3 f(x)$ for any $\lambda \in R$, $x \in M$. Then $\phi_f(x, x, x) = 6f(x)$ for any $x \in M$.

Let M be an R -module. A form $h : M^3 \rightarrow R$ satisfying $h(x, x, x) = 0$, $\forall x \in M$, is called a *nearly alternating form*. A 3-additive form $h : M^3 \rightarrow R$ satisfying $h(\lambda x, \lambda y, \lambda z) = \lambda^3 h(x, y, z)$, $\forall \lambda \in R$, $\forall (x, y, z) \in M^3$, is called a *quasi trilinear form*.

There exist quasi trilinear forms that are not nearly alternating, as it is shown in the following example.

Example 5. Let R and M be $\mathbf{Z} / 25\mathbf{Z}$ and $\mathbf{Z} / 5\mathbf{Z}$ respectively. Let $f : M \rightarrow R$ be a function defined by $f(a + 5\mathbf{Z}) = 5a^3 + 25\mathbf{Z} \ \forall a \in \mathbf{Z}$. Then f satisfies the parallelepiped law and $f(\lambda x) = \lambda^3 f(x)$ for any $\lambda \in R$, $x \in M$. Theorem 1 implies that f is a quasi cubic form so that, its associated form ϕ_f is 3-additive. Moreover ϕ_f is a quasi trilinear form and for any $x \in M$, $\phi_f(x, x, x) = 6f(x) = 6f(a + 5\mathbf{Z}) = 6 \cdot 5a^3 + 25\mathbf{Z} = 30a^3 + 25\mathbf{Z} = 5a^3 + 25\mathbf{Z} \neq 0$. Therefore ϕ_f is not nearly alternating.

Using Theorem 1 and following the ideas of Davison[3], we improve the result of Proposition 2 in case of the R -module, $M = R^3$.

Denote $D(R^3, R)$ the R -module of nearly alternating and quasi trilinear form of R^3 to R .

Remark 2. $D(R^3, R) \neq \emptyset$. Let $f : \mathbf{Z}/9\mathbf{Z} \rightarrow \mathbf{Z}/9\mathbf{Z}$ be a function defined by $f(a + 9\mathbf{Z}) = 3a^3 + 9\mathbf{Z} \ \forall a \in \mathbf{Z}$. Then f satisfies $f(\lambda x) = \lambda^3 f(x)$ for any $\lambda, x \in \mathbf{Z}/9\mathbf{Z}$, and by Remark 1, $\phi_f(x, x, x) = 6f(x) = 6f(a + 9\mathbf{Z}) =$

$6 \cdot 3a^3 + 9\mathbf{Z} = 18a^3 + 9\mathbf{Z} = 9\mathbf{Z}$. Therefore ϕ_f is nearly alternating. Moreover $\phi_f(x, y, z) = 9\mathbf{Z} \forall x, y, z \in \mathbf{Z}/9\mathbf{Z}$. Then ϕ_f is a quasi trilinear form.

Theorem 2. *Let R be a ring. Assume that 2 is not a zero divisor in R . Then $QF_3(R^3, R) = F_3(R^3, R) \oplus D(R^3, R)$.*

Proof. Let $f \in QF_3(R^3, R)$, then $f(\lambda, \lambda, \lambda) = \lambda^3 f(1, 1, 1)$ by Theorem 1. Define the form $g : R^3 \rightarrow R$, by $g(\alpha, \beta, \gamma) = \alpha\beta\gamma f(1, 1, 1)$ then g is a cubic form. If $\Delta = f - g$, then Δ is 3-additive and $\Delta(\lambda, \lambda, \lambda) = f(\lambda, \lambda, \lambda) - g(\lambda, \lambda, \lambda) = \lambda^3 f(1, 1, 1) - \lambda\lambda\lambda f(1, 1, 1) = 0$. Moreover, let $\lambda, \alpha, \beta, \gamma$ be elements in R then

$$\begin{aligned} \Delta(\lambda\alpha, \lambda\beta, \lambda\gamma) &= f(\lambda\alpha, \lambda\beta, \lambda\gamma) - g(\lambda\alpha, \lambda\beta, \lambda\gamma) \\ &= \lambda^3 f(\alpha, \beta, \gamma) - (\lambda\alpha\lambda\beta\lambda\gamma)f(1, 1, 1) = \lambda^3 f(\alpha, \beta, \gamma) - \lambda^3 \alpha\beta\gamma f(1, 1, 1) \\ &= \lambda^3 [f(\alpha, \beta, \gamma) - g(\alpha, \beta, \gamma)] = \lambda^3 \Delta(\alpha, \beta, \gamma). \end{aligned}$$

Therefore $\Delta \in D(R^3, R)$ and $f = g + \Delta$.

Let g be an element in $F_3(R^3, R) \cap D(R^3, R)$. Since g is 3-additive we have that

$$g(1, 1, 0) = g(1, 1 + 0, 0) = g(1, 1, 0) + g(1, 0, 0),$$

$$g(0, 1, 1) = g(0, 1, 1 + 0) = g(0, 1, 1) + g(0, 1, 0)$$

and

$$g(1, 0, 1) = g(1 + 0, 0, 1) = g(1, 0, 1) + g(0, 0, 1).$$

Therefore $g(1, 0, 0) = g(0, 1, 0) = g(0, 0, 1) = 0$.

Moreover, since g is a cubic form we consider the trilinear form $\phi_g(x, y, z) = g(x + y + z) - g(x + y) - g(x + z) - g(y + z) + g(x) + g(y) + g(z)$ associated to g . If $x = (\alpha, 0, 0)$, $y = (0, \beta, 0)$ and $z = (0, 0, \gamma)$ then the 3-additivity of g implies that $g(\alpha, \beta, 0) = g(\alpha, 0, \gamma) = g(0, \beta, \gamma) = 0$. On the other hand, since g is a quasi trilinear form we have that $g(\alpha, 0, 0) = \alpha^3 g(1, 0, 0) = \alpha 0 = 0$. Similarly we prove that $g(0, \beta, 0) = g(0, 0, \gamma) = 0$. That is for every $\alpha, \beta, \gamma \in R$ we have proved that

$$\phi_g((\alpha, 0, 0), (0, \beta, 0), (0, 0, \gamma)) = g(\alpha, \beta, \gamma)$$

Finally, since g is a cubic form, ϕ_g is a trilinear form. Using that g is nearly alternating we obtain that $g(\alpha, \beta, \gamma) = \alpha\beta\gamma g(1, 1, 1) = \alpha\beta\gamma \cdot 0 = 0$. Therefore $g = 0$ and the sum is direct.

As a measuring how much a quasi trilinear form $g : M \times N \times P \rightarrow R$, falls short of being trilinear we define for every $(x, y, z) \in M \times N \times P$, the form $\Delta_{x,y,z}(g)$ from R^3 to R as follows

$$\Delta_{x,y,z}(g)(\alpha, \beta, \gamma) = g(\alpha x, \beta y, \gamma z) - \alpha\beta\gamma g(x, y, z).$$

We say that a 3-additive form $\Delta : R^3 \rightarrow R$ is a *measure of non trilinearity* of a quasi trilinear form g if there exist x, y, z such that $\Delta = \Delta_{x,y,z}(g)$.

Proposition 3. *A form $\Delta : R^3 \rightarrow R$ is a measure of non trilinearity of a quasi trilinear form $g : M \times N \times P \rightarrow R$ for some R -modules if and only if $\Delta \in D(R^3, R)$.*

Proof. For any quasi trilinear form $g : M \times N \times P \rightarrow R$, the form $\Delta_{x,y,z}(g)$ is a nearly alternating quasi trilinear form. Conversely, if $M = N = P = R^3$, $x = (1, 0, 0)$, $y = (0, 1, 0)$, $z = (0, 0, 1)$ then defining g by $g((\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2), (\alpha_3, \beta_3, \gamma_3)) = \Delta(\alpha_1, \beta_2, \gamma_3) + \Delta(\beta_1, \alpha_2, \beta_3) + \Delta(\gamma_1, \gamma_2, \alpha_3)$ for every $\alpha_i, \beta_i, \gamma_i \in R$, $i = 1, 2, 3$ we have that g is a quasi trilinear form and $\Delta = \Delta_{x,y,z}(g)$.

The following result gives necessary and sufficient conditions for the equality of the modules of quasi cubic forms and cubic forms for any R -module M .

Corollary 2. The followings conditions are equivalents.

- i) All quasi trilinear forms from $M \times N \times P \rightarrow R$ are trilinear, for every R -modules M, N, P .
- ii) $QF_3(M^3, R) = F_3(M^3, R)$ for any R -module M .
- iii) $QF_3(R^3, R) = F_3(R^3, R)$
- iv) $D(R^3, R) = \{0\}$.

Proof. It is clear that i) $\Rightarrow$ ii) $\Rightarrow$ iii) $\Rightarrow$ iv). Now we prove that iv) $\Rightarrow$ i). Let g be a quasi trilinear form from $M \times N \times P$ to R . Using the proof of Proposition 3 we have that for every $(x, y, z) \in M \times N \times P$, the map $\Delta_{x,y,z}(g) \in D(R^3, R)$. Since $D(R^3, R) = \{0\}$, the 3-linearity of g follows from the definition of $\Delta_{x,y,z}(g)$.

Remark 3. The authors think that in the case of the R module $M = R^n$, the techniques of this section can be generalized to quasi n -forms

using the generalized parallelogram law. Moreover they also think that the results concerning to the universal n -map, γ_n given in [1], can be brought to the context of quasi n -maps with μ_n playing the role of γ_n .

Acknowledgement The authors thank the referee for comments that improve the presentation of the paper. They also thank Professor Luiz A. Peresi for his comments about this paper.

REFERENCES

- [1] M. Ferrero and A. Micali, Sur les n -applications, *Bull. Soc. Math. France Mém.* **59**, pp. 33-53, (1979).
- [2] A. M. Gleason, The definition of a quadratic form, *Amer. Math. Monthly* **73**, pp. 1049-1056, (1966).
- [3] T. M. K. Davison, Jordan derivations and quasi-bilinear forms, *Comm. in Alg.* **12** (1), pp. 23-32, (1984).

Iván Correa

Departamento de Matemáticas
Universidad de La Serena
Avda. Cisternas 1200
La Serena
Chile
e-mail: icorrea@userena.cl

Alicia Labra

Departamento de Matemáticas
Facultad de Ciencias
Universidad de Chile
Casilla 653
Santiago
Chile
e-mail: alimat@uchile.cl