Proyecciones
Vol. 23, N° 1, pp. 61-72, May 2004.
Universidad Católica del Norte
Antofagasta - Chile

ORLICZ - PETTIS THEOREMS FOR MULTIPLIER CONVERGENT OPERATOR VALUED SERIES

CHARLES SWARTZ
New Mexico State University, USA

Received November 2003. Accepted March 2004.

Abstract

Let X, Y be locally convex spaces and $L(X, Y)$ the space of continuous linear operators from X into Y . We consider 2 types of multiplier convergent theorems for a series $\sum T_k$ in $L(X, Y)$. First, if λ is a scalar sequence space, we say that the series $\sum T_k$ is λ multiplier convergent for a locally convex topology τ on $L(X, Y)$ if the series $\sum t_k T_k$ is τ convergent for every $t = \{t_k\} \in \lambda$. We establish conditions on λ which guarantee that a λ multiplier convergent series in the weak or strong operator topology is λ multiplier convergent in the topology of uniform convergence on the bounded subsets of X . Second, we consider vector valued multipliers. If E is a sequence space of X valued sequences, the series $\sum T_k$ is E multiplier convergent in a locally convex topology η on Y if the series $\sum T_k x_k$ is η convergent for every $x = \{x_k\} \in E$. We consider a gliding hump property on E which guarantees that a series $\sum T_k$ which is E multiplier convergent for the weak topology of Y is E multiplier convergent for the strong topology of Y .

1. INTRODUCTION

. The original Orlicz-Pettis Theorem which asserts that a series $\sum x_k$ in a normed space X which is subseries convergent in the weak topology of X is actually subseries convergent in the norm topology of X can be interpreted as a theorem about multiplier convergent series. Let λ be a scalar sequence space which contains the space of all sequences which are eventually 0 and let (E, τ) be a Hausdorff topological vector space. The series $\sum x_k$ in E is λ multiplier convergent with respect to τ if the series $\sum t_k x_k$ is τ convergent for every $t = \{t_k\} \in \lambda$; the elements of λ are called multipliers. Multiplier convergent series where the multipliers come from some of the classical sequence spaces, such as l^p , have been considered by various authors ([B],[FP]); in particular, c_0 multiplier convergent series have been used to characterize Banach spaces which contain no copy of c_0 ([D]). Thus, a series $\sum x_k$ is subseries convergent if and only if $\sum x_k$ is m_0 multiplier convergent, where m_0 is the sequence space of all scalar sequences which have finite range. Several Orlicz-Pettis Theorems have been established for multiplier convergent series where the multipliers are from various classical sequence spaces ([LCC], [SS],[WL]). In this note we consider Orlicz-Pettis Theorems for multiplier convergent series of linear operators where the multipliers are both scalar and vector valued. In general, as Example 2.1 below shows, a series of operators which is m_0 multiplier convergent in the weak operator topology is not m_0 multiplier convergent in the topology of uniform convergence on bounded subsets; however, if the space of multipliers satisfies certain conditions an Orlicz-Pettis result of this type does hold. One of our results contains a gliding hump assumption on the multiplier space which is of independent interest and also yields an Orlicz-Pettis result for series in the strong topology of a general locally convex space. In the final section we use the gliding hump property introduced in section 2 to establish a result on the weak sequential completeness of β -duals similar to a result of Stuart ([S1],[S2]).

We begin by fixing the notation and terminology which we use. Let X, Y be real Hausdorff locally convex spaces and let $L(X, Y)$ be the space of continuous linear operators from X into Y . If $x \in X, y' \in Y'$, let $x \otimes y'$ be the linear functional on $L(X, Y)$ defined by $\langle x \otimes y', T \rangle = \langle y', Tx \rangle$ and let $X \otimes Y'$ be the linear subspace spanned by $\{x \otimes y' : x \in X, y' \in Y'\}$. The weak operator topology on $L(X, Y)$ is the weak topology from the duality between $L(X, Y)$ and $X \otimes Y'$ ([DS]VI.1). The strong operator topology on $L(X, Y)$ is the topology of pointwise convergence on X ([DS]VI.1). Let

$L_b(X, Y)$ be $L(X, Y)$ with the topology of uniform convergence on the bounded subsets of X ; the topology of $L_b(X, Y)$ is generated by the semi-norms $p_A(T) = \sup\{p(Tx) : x \in A\}$, where p is a continuous semi-norm on Y and A is a bounded subset of X ([DS], VI.1).

2. SCALAR MULTIPLIERS

. In this section we establish Orlicz-Pettis type theorems for multiplier convergent series of operators with respect to the weak operator topology and the topology of $L_b(X, Y)$. As the following example illustrates, an operator valued series which is subseries or m_0 multiplier convergent in the strong (weak) operator topology needn't be m_0 multiplier convergent in $L_b(X, Y)$.

Example 1. Define $T_k : l^1 \rightarrow l^1$ by $T_k t = \langle e^k, t \rangle = t_k e^k$, where e^k is the sequence with 1 in the k th coordinate and 0 in the other coordinates.. Since for every subsequence $\{n_k\}$ the series $\sum_{k=1}^{\infty} T_{n_k} t = \sum_{k=1}^{\infty} t_{n_k} e^{n_k}$ converges in l^1 for every $t \in l^1$, the series $\sum T_k$ is m_0 multiplier convergent in the strong operator topology. But, $\|T_k\| = 1$ so $\sum T_k$ is not m_0 multiplier convergent in the operator norm.

However, as the following two theorems show, if the space of multipliers λ satisfies some additional conditions, an operator valued series which is λ multiplier convergent in the weak(strong) operator topology can indeed be λ multiplier convergent in $L_b(X, Y)$.

The sequence space λ is an AK space if λ has a Hausdorff locally convex topology such that the coordinate maps $t = \{t_k\} \rightarrow t_k$ from λ to $\mathbf{R}$ are continuous and each t has a series expansion $t = \sum_{k=1}^{\infty} t_k e^k$ which converges in the topology of λ ([BL]).

Theorem 2. If λ is a barrelled AK space and $\sum T_k$ is λ multiplier convergent in the weak operator topology of $L(X, Y)$, then $\sum T_k$ is λ multiplier convergent in $L_b(X, Y)$.

Proof: From Corollary 2.4 of [SS] it follows that $\sum T_k$ is λ multiplier convergent in the strong topology $\beta(L(X, Y), X \otimes Y')$. Thus, it suffices to show that if a sequence(net) $\{S_k\}$ in $L(X, Y)$ converges in $\beta(L(X, Y), X \otimes Y')$, then $\{S_k\}$ converges in $L_b(X, Y)$; that is, the strong topology $\beta(L(X, Y), X \otimes Y')$ is stronger than the topology of $L_b(X, Y)$. Let $A \subset X$ be bounded, $B \subset Y'$ be equicontinuous and let $C = \{x \otimes y' : x \in A, y' \in B\}$. It is easily

checked that C is $\sigma(X \otimes Y', L(X, Y))$ bounded so $\sup\{|\langle x \otimes y', S_k \rangle| : x \in A, y' \in B\} \rightarrow 0$ which implies that $S_k x \rightarrow 0$ uniformly for $x \in A$ or $S_k \rightarrow 0$ in $L_b(X, Y)$.

We next consider an analogue of Theorem 2 for other spaces of multipliers which are described by purely algebraic conditions in contrast to the AK and barrelledness assumptions of Theorem 2.

Let E be a vector space of X valued sequences which contains the space of all X valued sequences which are eventually 0. If $t = \{t_j\}$ is a scalar sequence and $x = \{x_j\}$ is either a scalar or X valued sequence, $tx = \{t_j x_j\}$ will be the coordinatewise product of t and x . A sequence of interval $\{I_j\}$ in $\mathbf{N}$ is increasing if $\max I_j < \min I_{j+1}$ for all j ; if I is an interval, then C_I will be the characteristic function of I . The space E has the *infinite gliding hump property* (∞ -GHP) if whenever $x \in E$ and $\{I_k\}$ is an increasing sequence of intervals, there exist a subsequence $\{n_k\}$ and $a_{n_k} > 0, a_{n_k} \rightarrow \infty$ such that every subsequence of $\{n_k\}$ has a further subsequence $\{p_k\}$ such that $\sum_{k=1}^{\infty} a_{p_k} C_{I_{p_k}} x \in E$ (coordinate sum). [The term ∞ -GHP is used to suggest that the "humps" $C_I x$ are multiplied by a sequence which tends to ∞ ; there are other gliding hump properties where the "humps" are multiplied by elements of classical sequence spaces ([Sw3])].

We now give several examples of spaces with the ∞ -GHP. The space E is normal (solid in the scalar case) if $l^\infty E = E$ ([KG]2.1); E is *c_0 -invariant* if $x \in E$ implies that there exist $t \in c_0, y \in E$ such that $x = ty$ ([G]; the term *c_0 -factorable* might be more descriptive).

Example 3. If E is normal and is c_0 -invariant, then E has ∞ -GHP. Let $x \in E$ with $x = ty$ where $t \in c_0, y \in E$ and let $\{I_k\}$ be an increasing sequence of intervals. Pick an increasing sequence $\{n_k\}$ such that $\sup\{|t_j| : j \in I_{n_k}\} = b_{n_k} > 0$ (if this choice is impossible there is nothing to do). Note that $b_{n_k} \rightarrow 0$ so $a_{n_k} = 1/b_{n_k} \rightarrow \infty$. Define $v_j = t_j a_{n_k}$ if $j \in I_{n_k}$ and $v_j = 0$ otherwise; then $v \in l^\infty$ so $vy \in E$ since E is normal. We have $\sum_{j=1}^{\infty} (vy)_j e^j = \sum_{k=1}^{\infty} a_{n_k} C_{I_{n_k}} x \in E$. Since the same argument can be applied to any subsequence of $\{n_k\}$, E has ∞ -GHP.

We now give some examples of spaces which satisfy the conditions of Example 3.

Example 4. Let X be a normed space and let $c_0(X)$ be the space of all null sequences in X . Then $c_0(X)$ is normal and c_0 -invariant and so has ∞ -GHP..

Example 5. Let $0 < p < \infty$ and let X be a normed space. Let $l^p(X)$ be the space of all p^{th} power summable sequences in X . Let $x \in l^p(X)$ and pick an increasing sequence $\{n_k\}$ such that $\sum_{j=n_k+1}^{n_{k+1}} \|x_j\|^p < 1/2^{k(p+1)}$. Set $I_k = [n_k + 1, n_{k+1}]$, $t = \sum_{k=1}^{\infty} 2^{-k} C_{I_k}$, $y = \sum_{k=1}^{\infty} 2^k C_{I_k} x$ so $x = ty$ with $t \in c_0$ and $y \in l^p$. Hence, $l^p(X)$ is c_0 -invariant and is obviously normal.

Likewise, it is easily checked that the spaces $d = \{t : \sup_k |t_k|^{1/k} < \infty\}$ and $\delta = \{t : \lim |t_k|^{1/k} = 0\}$ (see [KG] p.48 and 68) are normal and c_0 -invariant and, hence, have ∞ -GHP.

There exist non-normal sequence spaces with ∞ -GHP.

Example 6. Let cs be the space of convergent series ([KG]). Let $t \in cs$ and $\{I_k\}$ be an increasing sequence of intervals. Choose a subsequence $\{n_k\}$ such that $|\sum_{j \in I_{n_k} \cap I} t_j| < 1/k 2^k$ for any interval I . Consider $s = \sum_{k=1}^{\infty} k C_{I_{n_k}} t$. If I is any interval contained in the interval $[\min I_{n_k}, \infty)$, then

$$\left| \sum_{j \in I} s_j \right| = \left| \sum_{i=k}^{\infty} \sum_{j \in I_{n_i} \cap I} i t_j \right| \leq \sum_{i=k}^{\infty} 1/2^i = 2^{-k+1}$$

so the partial sums of the series generated by s are Cauchy and $s \in cs$. Since the same argument applies to any subsequence of $\{n_k\}$, cs has ∞ -GHP.

Note that the argument above shows that any Banach AK space has ∞ -GHP; e.g., bv_0 ([KG]).

The spaces l^∞ , m_0 , bs and bv do not have ∞ -GHP.

We next consider a result analogous to Theorem 2 except that we use the strong operator topology.

Theorem 7. Let λ have ∞ -GHP. If $\sum T_j$ is λ multiplier convergent in the strong operator topology, then $\sum T_j$ is λ multiplier convergent in $L_b(X, Y)$.

Proof: If the conclusion fails, there exist $\varepsilon > 0$, $t \in \lambda$, $A \subset X$ bounded, a continuous semi-norm p on Y and subsequences $\{m_k\}, \{n_k\}$ such that $m_1 < n_1 < m_2 < \dots$ and $p_A(\sum_{l=m_k}^{n_k} t_l T_l) > \varepsilon$. For every k there exists $x_k \in A$ such that

$$(1) \quad p(\sum_{l=m_k}^{n_k} t_l T_l x_k) > \varepsilon.$$

Set $I_k = [m_k, n_k]$. Since λ has ∞ -GHP, there exist $\{p_k\}, a_{p_k} > 0, a_{p_k} \rightarrow \infty$ such that every subsequence of $\{p_k\}$ has a further subsequence $\{q_k\}$ such that $s = \sum_{k=1}^{\infty} a_{q_k} C_{I_{q_k}} t \in \lambda$. Let $M = [m_{ij}] = [\sum_{l=m_j}^{n_j} (t_l a_{p_j}) T_l(x_i/a_{p_i})]$. We use the Antosik-Mikusinski Matrix Theorem ([Sw2]2.2.2) to show that the diagonal of M converges to 0; this will contradict (1). First, the columns of M converge to 0 since $x_i/a_{p_i} \rightarrow 0$ and each T_l is continuous. Next, given a subsequence there is a further subsequence $\{q_k\}$ such that $s = \sum_{k=1}^{\infty} a_{q_k} C_{I_{q_k}} t \in \lambda$. The series $\sum_{l=1}^{\infty} s_l T_l$ converges in the strong operator topology to an operator $T \in L(X, Y)$. Hence, $\sum_{j=1}^{\infty} m_{iq_j} = \sum_{j=1}^{\infty} \sum_{l \in I_{q_j}} s_l T_l(x_i/a_{p_i}) = T(x_i/a_{p_i}) \rightarrow 0$. It follows that M is a K matrix. By the Antosik-Mikusinski Matrix Theorem the diagonal of M converges to 0 which contradicts (1).

Remark 8. If the multiplier space λ in Theorem 7 is normal, we may replace the assumption that $\sum T_j$ is λ multiplier convergent in the strong operator topology with the assumption that the series is λ multiplier convergent in the weak operator topology. For if λ is normal and $\sum t_j T_j$ is convergent in the weak operator topology for every $t \in \lambda$, then the series is subseries convergent in the weak operator topology and, therefore, convergent in the strong operator topology by the classical Orlicz-Pettis Theorem.

The proof of Theorem 7 also establishes the following version of the Orlicz-Pettis Theorem for multiplier convergent series which should be compared to the version given in Corollary 2.4 of [SS] where it is assumed that λ is a barrelled AK-space.

Theorem 9.(Orlicz-Pettis) If λ has ∞ -GHP and if $\sum_j x_j$ is λ multiplier convergent in the weak topology $\sigma(X, X')$, then $\sum_j x_j$ is λ multiplier convergent in the strong topology $\beta(X, X')$.

The spaces $d = \{t : \sup_k |t_k|^{1/k} < \infty\}$ and $\delta = \{t : \lim |t_k|^{1/k} = 0\}$ ([KG]) give examples of spaces to which Theorems 7 and 9 apply but to which Theorem 2 and its scalar counterpart, Corollary 2.4 of [SS], do not apply. (The natural metric on d does not give a vector topology ([KG]p. 68).)

The scalar versions of Theorems 2 and 7 are both of interest. That is, the case when the space Y is the scalar field.

Corollary 10. Assume that λ has $\infty - GHP$ or that λ is a barrelled AK space.

If $\sum_j x'_j$ is λ multiplier convergent in the weak* topology $\sigma(X', X)$ of X' , then $\sum_j x'_j$ is λ multiplier convergent in the strong topology $\beta(X', X)$.

Corollary 10 should be compared to the Diestel-Faires Theorem concerning subseries convergence in the weak* topology of the dual of a Banach space. In the Diestel-Faires result the emphasis is on conditions on the space while in Corollary 10 the conditions are on the space of multipliers.

3. VECTOR MULTIPLIERS

. We now consider vector valued multipliers for operator valued series. Let $\{T_k\} \subset L(X, Y)$ and let E be a vector space of X valued sequences containing the space of all sequences which are eventually 0. If τ is a locally convex Hausdorff topology on Y , we say that the series $\sum T_k$ is E multiplier convergent with respect to τ if the series $\sum T_k x_k$ is τ convergent for every $x = \{x_k\} \in E$. We establish Orlicz-Pettis Theorems for multiplier convergent series analogous to Theorems 2 and 7 for the weak and strong topologies of Y .

As in section 2 the following example shows that if the space of multipliers does not satisfy some condition an Orlicz-Pettis Theorem does not in general hold for the weak and strong topologies of Y .

Example 1. Let $l^\infty(X)$ be the space of all bounded X valued sequences. Assume that l^∞ has the weak topology $\sigma(l^\infty, l^1)$. Define $P_k : l^\infty \rightarrow l^\infty$ by $P_k x = x_k e^k$. Let $E = l^\infty(l^\infty)$. If $x = \{x^k\} \in E$, then $\sum P_k x^k = \sum x_k^k e^k$ is $\sigma(l^\infty, l^1)$ convergent, but if $x = \{e^k\} \in E$, then $\sum P_k e^k = \sum e^k$ is not $\beta(l^\infty, l^1) = |||$ convergent.

We begin with the analogue of Theorem 2.7 since it is straightforward to state and prove.

Theorem 2. Let E have ∞ -GHP and $\{T_k\} \subset L(X, Y)$. If $\sum T_k$ is E multiplier convergent with respect to the weak topology $\sigma(Y, Y')$, then $\sum T_k$ is E multiplier convergent with respect to the strong topology $\beta(Y, Y')$.

Proof: If the conclusion fails, there exist $x \in E$, $\{y'_k\} \subset \sigma(Y', Y)$ bounded, $\varepsilon > 0$ and subsequences $\{m_k\}, \{n_k\}$ with $m_1 < n_1 < m_2 < \dots$ and

$\left| \sum_{l=m_k}^{n_k} \langle y'_k, T_l x_l \rangle \right| > \varepsilon$ for all k . Set $I_k = [m_k, n_k]$. Since E has ∞ -GHP, there exist $\{p_k\}, a_{p_k} > 0, a_{p_k} \rightarrow \infty$ such that every subsequence of $\{p_k\}$ has a further subsequence $\{q_k\}$ such that $\sum a_{q_k} C_{I_{q_k}} x \in E$. Define an infinite matrix $M = [m_{ij}] = [\sum_{l \in I_{p_j}} \langle y'_{p_i}/a_{p_i}, T_l(a_{p_j} x_l) \rangle]$. We show that M is a K matrix so the diagonal of M converges to 0 by the Antosik-Mikusinski Matrix Theorem ([Sw2]2.2.2) and this will contradict the inequality above. First, the columns of M converge to 0 since $\{y'_i\}$ is $\sigma(Y', Y)$ bounded and $a_{p_i} \rightarrow \infty$. Next, given a subsequence there is a further subsequence $\{q_j\}$ such that $y = \sum a_{q_k} C_{I_{q_k}} x \in E$. Let $\sum_{l=1}^{\infty} T_l y_l = \sum_{j=1}^{\infty} \sum_{l \in I_{q_j}} T_l(a_{q_j} x_l) = z$ be the $\sigma(Y, Y')$ sum of this series. Then $\sum_{j=1}^{\infty} m_{iq_j} = \langle y'_{q_i}/a_{q_i}, z \rangle \rightarrow 0$ so M is a K matrix and the result follows.

We next establish the analogue of Theorem 2.2. If $u \in X$, $e^k \otimes u$ is the sequence with u in the k^{th} coordinate and 0 in the other coordinates. If τ is a Hausdorff locally convex topology on E , (E, τ) , or E if τ is understood, is an AK space if $x = \tau - \lim_n \sum_{k=1}^n e^k \otimes x_k = \sum_{k=1}^{\infty} e^k \otimes x_k$ for every $x \in E$ ([BL]).

Let $\sum T_k$ be E multiplier convergent with respect to $\sigma(Y, Y')$. Define a linear map $\hat{T} : E \rightarrow Y$ by setting $\hat{T}x = \sigma(Y, Y') - \lim_n \sum_{k=1}^n T_k x_k = \sum_{k=1}^{\infty} T_k x_k$. Note that $\hat{T}(e^k \otimes u) = T_k u$ for $u \in X, k \in \mathbf{N}$. Recall that the (scalar) β -dual of E is defined to be $E^\beta = \{\{y'_i\} : \sum_{i=1}^{\infty} \langle y'_i, x_i \rangle \text{ converges } \forall x \in E\}$ ([BL]). If $y' = \{y'_i\} \in E^\beta$ and $x = \{x_i\} \in E$, we set $y' \cdot x = \sum_{i=1}^{\infty} \langle y'_i, x_i \rangle$ and note that E and E^β are in duality with respect this bilinear pairing. If $y' \in Y', x \in E$, then $\langle y', \hat{T}x \rangle = \langle y', \sum T_k x_k \rangle = \sum \langle y', T_k x_k \rangle = \sum \langle T'_k y', x_k \rangle$ so $\{T'_k y'\} \in E^\beta$ and $\langle y', \hat{T}x \rangle = \{T'_k y'\} \cdot x$. Hence, $\hat{T}$ is $\sigma(E, E^\beta) - \sigma(Y, Y')$ continuous and, therefore, $\beta(E, E^\beta) - \beta(Y, Y')$ continuous ([W]11.2.3, [Sw1]26.15) so we have

Theorem 3. If $(E, \beta(E, E^\beta))$ is an AK space and $\sum T_k$ is E multiplier convergent with respect to $\sigma(Y, Y')$, then $\sum T_k$ is E multiplier convergent with respect to $\beta(Y, Y')$.

Proof: If $x \in E$, $\hat{T}x = \hat{T}(\sum e^k \otimes x_k) = \beta(Y, Y') - \lim_n \sum_{k=1}^n \hat{T}(e^k \otimes x_k) = \beta(Y, Y') - \lim_n \sum_{k=1}^n T_k x_k$ by the strong continuity of $\hat{T}$ established above.

If E is a barreled AK space, then the argument in Lemma 3.9 of [KG] shows that $E' = E^\beta$ and since E is barreled, the original topology of E

is just $\beta(E, E^\beta)$ so Theorem 3 is applicable. If X is a Banach space, the spaces $c_0(X)$ of null sequences with the sup-norm and $l^p(X)$ of absolutely p-summable sequences with the l^p -norm give examples of barrelled (B-spaces), AK spaces to which Theorem 3 applies.

4. WEAK SEQUENTIAL COMPLETENESS OF

β -DUALS. The infinite gliding hump property introduced in section 2 also has applications to the β -duals of vector valued sequence spaces. In particular, we use one of the results to establish the weak sequential completeness of β -duals which serves as a complement to a weak sequential completeness result of Stuart ([S1],[S2]).

The β -dual of E with respect to Y , $E^{\beta Y}$, is the space of all sequences $\{T_j\} \subset L(X, Y)$ such that the series $\sum T_j$ is E multiplier convergent ([BL]). Our main result in this section concerns uniform convergence of series for elements in β -duals.

Theorem 1. Assume that E has ∞ -GHP. If $\Gamma \subset E^{\beta Y}$ is pointwise bounded on E with respect to $\beta(Y, Y')$, then for every $x \in E$ the series $\sum_k T_k x_k$ are uniformly $\beta(Y, Y')$ convergent for $T \in \Gamma$.

Proof: If the conclusion fails, there exist $\epsilon > 0$, $\{T_k\} \subset \Gamma$, $\{y'_k\} \subset Y'$ $\sigma(Y', Y)$ bounded and subsequences $\{m_k\}, \{n_k\}$ with $m_1 < n_1 < m_2 < \dots$ and

$$(2) \quad \left| \left\langle y'_k, \sum_{l=m_k}^{n_k} T_l x_l \right\rangle \right| > \epsilon.$$

Set $I_k = [m_k, n_k]$. By ∞ -GHP, there exist $\{p_k\}, a_{p_k} > 0, a_{p_k} \rightarrow \infty$ such that every subsequence of $\{p_k\}$ has a further subsequence $\{q_k\}$ such that $\sum a_{q_k} C_{I_{q_k}} \in E$. Define an infinite matrix

$M = [m_{ij}] = \left[\left\langle y'_{p_i} / a_{p_i}, \sum_{l=m_{p_j}}^{n_{p_j}} T_l^{p_i}(a_{p_j} x_l) \right\rangle \right]$. Since $\{T^i\}$ is pointwise $\beta(Y, Y')$ bounded on E , $\{y'_i\}$ is $\sigma(Y', Y)$ bounded and $1/a_{p_i} \rightarrow 0$, the columns of M converge to 0. Next, given any subsequence there is a further subsequence $\{q_j\}$ such that $u = \sum_{j=1}^{\infty} a_{q_j} C_{I_{q_j}} x \in E$. Therefore, $\sum_{j=1}^{\infty} m_{iq_j} = \left\langle y'_{p_i} / a_{p_i}, \sum_{j=1}^{\infty} T^{p_i} u_j \right\rangle \rightarrow 0$ by the same argument as above. Hence, M is a K matrix so the diagonal of M converges to 0. This contradicts (2).

Theorem 1 can be used to establish a weak sequential completeness result for the β -dual $E^{\beta Y}$. The weak topology, $\sigma(E^{\beta Y}, E)$, on $E^{\beta Y}$ from

E is the weakest topology on $E^{\beta Y}$ such that all of the maps $x \rightarrow \sum_k T_k x_k$ from E into Y are continuous for every $T \in E^{\beta Y}$. The pair (X, Y) has the Banach-Steinhaus property if whenever $\{T_k\} \subset L(X, Y)$ is such that $\lim T_k x = Tx$ exists for every $x \in X$, then $T \in L(X, Y)$. For example, if X is barrelled, then (X, Y) has the Banach-Steinhaus property ([W]9.3.7, [Sw]24.12).

Lemma 2. If $\{T^k\} \subset E^{\beta Y}$ is $\sigma(E^{\beta Y}, E)$ Cauchy, for each $x \in E$ the series $\sum_j T_j^k x_j$ are uniformly convergent for $k \in \mathbf{N}$ and (X, Y) has the Banach-Steinhaus property, then there exists $T \in E^{\beta Y}$ such that $T^k \rightarrow T$ with respect to $\sigma(E^{\beta Y}, E)$.

Proof: Suppose that $\{T^k\}$ is $\sigma(E^{\beta Y}, E)$ Cauchy. For each j and $z \in X$, $\lim_k T^k(e^j \otimes z) = T_j(z)$ exists. By the Banach-Steinhaus property $T_j \in E^{\beta Y}$; put $T = \{T_j\}$.

We claim that $T \in L(X, Y)$ and $T^k \rightarrow T$ in $\sigma(E^{\beta Y}, E)$. Let $x \in E$. Put $u = \lim T^k x$. It suffices to show that $u = \sum_{j=1}^{\infty} T_j x_j$. Let U be a balanced neighborhood of 0 in Y and pick a balanced neighborhood of 0 V such that $V + V + V \subset U$. There exists p such that $\sum_{j=n}^{\infty} T_j^k x_j \in V$ for $n \geq p, k \in \mathbf{N}$. Fix $n \geq p$. Pick $k_n = k$ such that $\sum_{j=1}^{\infty} T_j^k x_j - u \in V$ and $\sum_{j=1}^n (T_j^k - T_j) x_j \in V$. Then $\sum_{j=1}^n T_j x_j - u = (\sum_{j=1}^{\infty} T_j^k x_j - u) + \sum_{j=1}^n (T_j - T_j^k) x_j - \sum_{j=1}^{\infty} T_j^k x_j \in V + V + V \subset U$ and the result follows.

Corollary 3. Assume that E has ∞ -GHP and Y is barrelled. If (X, Y) has the Banach-Steinhaus property, then

$(E^{\beta Y}, \sigma(E^{\beta Y}, E))$ is sequentially complete.

Proof: Since Y is barrelled, the original topology of Y is $\sigma(E^{\beta Y}, E)$. The result then follows from Theorem 1 and Lemma 2.

Stuart has established a weak sequential completeness result similar the Corollary 3 but involving a different gliding hump condition call the signed gliding hump property (signedWGHP) ([S1],[S2]). The space E has signedWGHP if whenever $x \in E$ and $\{I_k\}$ is an increasing sequence of intervals, there is a subsequence $\{n_k\}$ and a sequence of signs $\{s_k\}, s_k = \pm 1$ such that $\sum s_k C_{I_{n_k}} x \in E$ [coordinatewise sum]. The two gliding hump conditions, the ∞ -GHP and the signedWGHP, are independent; the space bs has signedWGHP but not ∞ -GHP ([S1],[S2]) while the space bv_0 has ∞ -GHP but not signedWGHP (remark following Example 2.6). Thus, Corollary 3 and Theorem 3.5 of [S2] cover different spaces of multipliers or different β -duals.

References

- [B] G. Bennett, Some inclusion theorems for sequence spaces, *Pacific J. Math.*, 46, pp. 17-30, (1973).
- [BL] J. Boos and T. Leiger, Some distinguished subspaces of domains of operator valued matrices, *Results Math.*, 16, pp. 199-211, (1989).
- [D] J. Diestel, *Sequences and Series in Banach Spaces*, Springer-Verlag, N. Y., (1984).
- [DF] J. Diestel and B. Faires, Vector Measures, *Trans. Amer. Math. Soc.*, 198, pp. 253-271, (1974).
- [DS] N. Dunford and J. Schwartz, *Linear Operators I*, Interscience, N. Y., (1958).
- [FP] M. Florencio and P. Paul, A note on λ multiplier convergent series, *Casopis Pro Pest. Mat.*, 113, pp. 421-428, (1988).
- [G] D. J. H. Garling, The β - and γ -duality of sequence spaces, *Proc. Cambridge Phil. Soc.*, 63, pp. 963-981, (1967).
- [KG] P. K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Marcel Dekker, N. Y., (1981).
- [LCC] Li Ronglu, Cui Changri and Min Hyung Cho, An invariant with respect to all admissible (X, X') -polar topologies, *Chinese Ann. Math.*, 3, pp. 289-294, (1998).
- [S1] C. Stuart, Weak Sequential Completeness in Sequence Spaces, Ph.D. Dissertation, New Mexico State University, (1993).
- [S2] C. Stuart, Weak Sequential Completeness of β -Duals, *Rocky Mountain Math. J.*, 26, pp. 1559-1568, (1996).
- [SS] C. Stuart and C. Swartz, Orlicz-Pettis Theorems for Multiplier Convergent Series, *Journal for Analysis and Appl.*, 17, pp. 805-811, (1998).
- [Sw1] C. Swartz, *An Introduction to Functional Analysis*, Marcel Dekker, N. Y., (1992).

- [Sw2] C. Swartz, Infinite Matrices and the Gliding Hump, World Sci.Publ., Singapore, (1996).
- [Sw3] C. Swartz, A multiplier gliding hump property for sequence spaces, *Proy. Revista Mat.*, 20, pp. 19-31, (2001).
- [W] A. Wilansky, Modern Methods in Topological Vector Spaces, McGraw-Hill, N. Y., (1978).
- [WL] Wu Junde and Li Ronglu, Basic properties of locally convex A-spaces, *Studia Sci. Math. Hungar.*, to appear.

Charles Swartz

Mathematics Department
New Mexico state University
Las Cruces, NM 88003
USA
e-mail : cswartz@nmsu.edu