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THE MINIMAL PRIME IDEAL OF A VALUATION RING *

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Abstract

In this paper we consider a non-archimedean ordered field $(K, \leq)$ and a valuation v of K compatible with respect to $\leq$. We will give a relation between the existence of a minimal prime ideal of the valuation ring A_v and the rank of the valuation.

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1. Introduction

In the theory of ordered fields the “Tarski-Principle ” says that the class of real closed fields shares with $\mathbf{R}$ all algebraic and order-theoretical properties. On the other hand, the topological properties of a non-archimedean real closed field have peculiarities that make them quite different from the properties of the field $\mathbf{R}$. In this work we deal essentially with the set of points of a non-archimedean ordered field whose powers sequence converges to zero. The motivation to study this set came out when we were trying to generalize the theory of control sets for the action of semigroups of non-empty interior in real compact homogeneous varieties to the algebraic varieties over real closed fields.

To illustrate the problem, we will consider the following example in the field of rational functions $\mathbf{R}(X_1, X_2, \dots)$.

Let $\mathbf{Z}^\infty$ be the group of all sequences $\alpha = (a_i)_{i \in \mathbf{N}}$, where $a_i \in \mathbf{Z}$ for all $i \in \mathbf{N}$ and such that $a_i = 0$ for all i or there is $i_\alpha \in \mathbf{N}$ with $a_{i_\alpha} \neq 0$ and $a_i = 0$ for $i > i_\alpha$. The *anti-lexicographical order* in $\mathbf{Z}^\infty$ is defined as follows: an element α is positive if $0 < a_{i_\alpha}$. Consider the integral ring $A = \mathbf{R}[X_1, X_2, \dots]$ of polynomials on infinitely many variables $X_1, X_2, \dots$ and the field of rational functions $k = \mathbf{R}(X_1, X_2, \dots)$ of A . A monomial in A can be written as

$$X^\alpha = X_1^{a_1} X_2^{a_2} \dots X_n^{a_n} \text{ where } n \in \mathbf{N}, \alpha = (a_1, \dots, a_n, 0, 0, \dots) \in \mathbf{Z}^\infty \text{ and } a_i \geq 0.$$

Hence a non zero element $f \in \mathbf{R}[X_1, X_2, \dots]$ is of the form

$$f = \sum_{i=0}^n c_i X^{\alpha_i} \text{ where } n \in \mathbf{N}, c_i \in \mathbf{R}, \alpha_i \in \mathbf{Z}^\infty, \alpha_0 < \alpha_1 < \dots < \alpha_n \text{ and } c_i \neq 0, \forall i$$

An element f is positive ($f > 0$) if $c_0 > 0$. This defines an order in the ring A , which can be extended to the field k in the usual way. We are looking for the elements $\frac{f}{g} \in k$ such that $(\frac{f}{g})^n \rightarrow 0$ in this order. According to the above notation,

$$f = \sum_{i=0}^n c_i X^{\alpha_i}, \quad g = \sum_{i=0}^m b_i X^{\beta_i}$$

and we can assume, without loss of generality, that f e g are positive.

Let $l = \max\{i_{\alpha_0}, \dots, i_{\alpha_n}, j_{\beta_0}, \dots, j_{\beta_m}\}$ and $\gamma = (a_i)_{i \in \mathbf{N}}$, where $a_{l+1} = 1$ and $a_i = 0$ for $i \neq l+1$. Then $(\frac{f}{g})^k > X^\gamma$ for all $k \in \mathbf{N}$. Indeed, since

$f > 0$ then $f^k - X^\gamma g^k > 0$, for all $k \in \mathbf{N}$. In this example we notice that the geometric series $\sum_{k=0}^{\infty} x^k$ converges only if $x = 0$.

The example shows that the set of all elements whose powers sequence converges to zero can be trivial. From this we are motivated to consider an arbitrary ordered field k and to study the set $\mathcal{I} = \{x \in k : x^n \rightarrow 0\}$. For archimedean orders the situation is trivial. Hence we will focus only on non-archimedean orders. The main result in this paper is Theorem 7, which gives a characterization of the rank of a valuation compatible with the order in terms of the properties of $\mathcal{I}$.

2. Orders and Valuations

The study of non-archimedean orders is close related with the study of a special type of valuations, the real places. In this section we will expose briefly the context where the problem mentioned in the introduction take place. A subgroup H of an (totally) ordered group G is called *isolated* (or *convex*) if $a \in H$, $b \in G$, $0 < b < a$, then $b \in H$.

The trivial examples of isolated subgroups of G are $\{0\}$ and G itself. Any intersection of isolated subgroups of G is still an isolated subgroup of G . It is well known that the set of all isolated subgroups of G is totally ordered by inclusion. The order type of the set of all non-zero isolated subgroups of G is called the *rank* G . On the other hand, the *rank* of a valuation v of a field K (in the sense of Krull) is the *rank* of the valuation ring $A_v = \{x \in K : v(x) \geq 0\}$.

Let v be a valuation of K with valuation ring A and value group Γ . Denote by $\mathcal{B}$ the set of all subrings of K containing A . Given $B \in \mathcal{B}$ consider the canonical valuation

$$v_B : K \rightarrow \Gamma_B \cup \{\infty\}.$$

Denote by U_B the kernel of v_B and by U the kernel of $v : K \rightarrow \Gamma \cup \{\infty\}$. Clearly $U \subseteq U_B$, so there exists one and only one group homomorphism $g_B : \Gamma \rightarrow \Gamma_B$ such that $v_B = g_B.v$. Note that g_B is surjective, it has kernel $\Phi_B = v(U_B)$ and it is order-preserving.

Theorem 1. ([3], 7.4.) *The mapping $B \rightarrow \Phi_B$ is bijective from $\mathcal{B}$ onto the set of all isolated subgroups of Γ . It is inclusion-preserving and so is its inverse.*

An immediate consequence of this classical result is that the rank of the value group of a valuation v of K coincides with the rank of the valuation ring corresponding to v .

Remark 2. It can also be shown that there is a inclusion inverting 1-1 correspondence between the set $\mathcal{B}$ and the set $\mathfrak{p}$ of all prime ideals of A_v (see [3], 6.6).

Hence the rank of the valuation v is equal to the order type of the set $\mathfrak{p} - \{0\}$.

Let $\leq$ be an ordering of K and v a valuation on K . We say that $\leq$ is compatible with v if $0 < a \leq b$ implies $v(a) \geq v(b)$ for all $a, b \in K$. We can also use that $\leq$ and v are compatible if $0 < a$, $v(a) < v(b)$ implies $b < a$.

For v a valuation on K , $\mathfrak{m}_v = \{x \in K : v(x) > 0\}$ is the maximal ideal of A_v . In [5], 7.2, Prestel proved that the following statements are equivalent:

1. $\leq$ is compatible with v
2. A_v is an isolated subgroup of K
3. $\mathfrak{m}_v$ is an isolated subgroup of K
4. $0 \leq a \in \mathfrak{m}_v$ implies $a < 1$.

An essencial aspect in the example given in the introduction is that the valuation compatible with the anti-lexicographical order has infinite rank. When the valuations have finite rank, the situation is totally different.

Proposition 3. Let $(K, \leq)$ be an ordered field and v a valuation of K compatible with respect to $\leq$. If the rank of v is finite, then $\mathcal{I} = \{x \in K : x^n \rightarrow 0\} \neq \{0\}$.

Proof. Since v is compatible with the order $\leq$ of K , it is easy to see that

$$\{x \in K : x^n \rightarrow 0\} = \{x \in K : v(x^n) \rightarrow \infty\}.$$

Let Γ be the value group of v . For each $\alpha \in \Gamma$, $\alpha > 0$, define

$$\Phi_\alpha = \cup_{n \in \mathbf{N}} \{\beta \in \Gamma : -n\alpha \leq \beta \leq n\alpha\},$$

which is an isolated subgroup of Γ containing α . Take $\alpha_1 \in \Gamma$ with $\alpha_1 > 0$. If $\Phi_{\alpha_1} \neq \Gamma$ we can take $\alpha_2 \in \Gamma - \Phi_{\alpha_1}$. Since the set of all isolated subgroups of Γ is totally ordered we have $\Phi_{\alpha_1} \subsetneq \Phi_{\alpha_2}$. Again, if $\Phi_{\alpha_2} \neq \Gamma$ we can take

$\alpha_3 \in \Gamma - \Phi_{\alpha_2}$. We must have $\Phi_{\alpha_1} \not\subseteq \Phi_{\alpha_2} \not\subseteq \Phi_{\alpha_3}$. In this way we built a chain of isolated subgroups of Γ

$$\Phi_{\alpha_1} \not\subseteq \Phi_{\alpha_2} \not\subseteq \dots \Phi_{\alpha_n} \not\subseteq \dots \not\subseteq \Gamma.$$

It follows that $\Phi_\alpha = \Gamma$ for some $\alpha \in \Gamma$, $\alpha > 0$, because the rank of v is finite. Hence given $\beta \in \Gamma$ we have that $\beta \leq n_o \alpha$ for some $n_o \in \mathbf{N}$. Thus $\beta < n\alpha$ for all $n \geq n_o$, that is, $n\alpha \rightarrow \infty$ as $n \rightarrow \infty$. Finally, take $\xi \in K$ such that $v(\xi) = \alpha$, then $\xi \neq 0$. We have $v(\xi^n) = nv(\xi) = n\alpha \rightarrow \infty$ as $n \rightarrow \infty$. Therefore $\xi \in \mathcal{I}$ and $\mathcal{I} \neq \{0\}$. $\square$

A question that appears naturally is the converse of the above result. We will give a partial answer to this question on Proposition 6. First, we need to better describe the set $\mathcal{I}$.

Proposition 4. *Let $(K, \leq)$ be a non-archimedean ordered field and v a valuation of K compatible with respect to $\leq$. If $\mathcal{I} = \{x \in K : x^n \rightarrow 0\} \neq \{0\}$, then it is a minimal prime ideal of A_v .*

Proof. If $x \in \mathcal{I}$, then $v(x^n) \rightarrow \infty$ as $n \rightarrow \infty$, i.e., $nv(x) \rightarrow \infty$. So we must have $v(x) > 0$. This shows that $\mathcal{I} \subset A_v$. Given $x, y \in \mathcal{I}$ we have

$$v[(x+y)^n] = nv(x+y) \geq n \cdot \min(v(x), v(y)).$$

Since $nv(x) \rightarrow \infty$ and $nv(y) \rightarrow \infty$, we have $x+y \in \mathcal{I}$. We know that $v[(xy)^n] = nv(xy) = nv(x) + nv(y)$. Now, if $x \in \mathcal{I}$ and $y \in A_v$, then $v[(xy)^n] \rightarrow \infty$ as $n \rightarrow \infty$. On the other hand, if $v[(xy)^n] \rightarrow \infty$, then either $nv(x) \rightarrow \infty$ or $nv(y) \rightarrow \infty$. Hence $\mathcal{I}$ is a prime ideal of A_v . It remains to prove that $\mathcal{I}$ is minimal. Let Γ be the value group of v . Consider the fundamental system of neighborhood of zero $(V_\gamma(0))_{\gamma \in \Gamma}$, with

$$V_\gamma(0) = \{x \in K : v(x) > \gamma\}.$$

Let $\xi \in A_v$ and $y \in V_{v(\xi)}(0)$, so $v(y) > v(\xi) > 0$. Hence $y \in A_v$ and we have that $V_{v(\xi)}(0) \subset A_v$ for all $\xi \in A_v$. Now, take $\mathfrak{p}$ a nonzero prime ideal of A_v and $x \in \mathfrak{p}$. Clearly $xV_{v(\xi)}(0) = \{xy : y \in V_{v(\xi)}(0)\}$ is contained in $\mathfrak{p}$. Moreover,

$$v(xy) = v(x) + v(y) > v(x) + v(\xi) = v(x\xi) = v(\xi'),$$

with $\xi' \in A_v$. Thus $xV_{v(\xi)}(0) = V_{v(\xi')}(0)$ and we conclude that $\mathfrak{p}$ contains a neighborhood of zero. Now we can prove that $\mathcal{I} \subset \mathfrak{p}$. Indeed, given $a \in \mathcal{I}$, we have that $a^n \rightarrow 0$ as $n \rightarrow \infty$. So for N sufficiently large $a^N \in \mathfrak{p}$. But $\mathfrak{p}$ is a prime ideal, so $a \in \mathfrak{p}$. $\square$

Remark 5. If $\mathcal{I} \neq \{0\}$ then $\mathcal{I}$ is relatively large in the sense that it has non-empty interior. In fact, if $x \in \mathcal{I}$ and $x > 0$ then the closed interval $[0, x]$ lies in $\mathcal{I}$.

Our goal now is to show that the reverse of Proposition 4 is true if we assume that v is a discrete valuation. By a discrete valuation we mean a valuation whose value group is isomorphic to a subgroup of the group $\mathbf{Z}^\infty$ described in the introduction. In this way, a valuation is discrete of finite rank r if and only if its value group is isomorphic to a anti-lexicographical product of r groups isomorphic to $\mathbf{Z}$. Notice that, if the rank of a discrete valuation is not finite, then its value group does not contain nontrivial isolated subgroups. In fact, if $\Gamma \subset \mathbf{Z}^\infty$ is a nontrivial isolated subgroup take $\gamma \in \mathbf{Z}^\infty$ such that $\gamma \notin \Gamma$. Hence the isolated subgroup of $\mathbf{Z}^\infty$ generated by γ must contain Γ properly.

With these observations, we can prove the following proposition:

Proposition 6. Let $(K, \leq)$ be a non-archimedean ordered field and v a discrete valuation of K compatible with respect to $\leq$. If $\mathcal{I} = \{x \in K : x^n \rightarrow 0\} \neq \{0\}$, then the rank of v is finite.

Proof. If $\mathcal{I} \neq \{0\}$, from Proposition 4, $\mathcal{I}$ is a minimal prime ideal of A_v . According to Theorem 1 and Remark 2, the prime ideal $\mathcal{I}$ corresponds to a maximal isolated subgroup of the value group Γ . By the above observations, the rank of Γ is finite. $\square$

We can summarize Proposition 4 and Proposition 6 in the following theorem.

Theorem 7. Let $(K, \leq)$ be a non-archimedean ordered field and v a discrete valuation of K compatible with respect to $\leq$. The rank of v is finite if and only if $\mathcal{I} = \{x \in K : x^n \rightarrow 0\} \neq \{0\}$.

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