A NOTE ON POLYNOMIAL CHARACTERIZATIONS OF ASPLUND SPACES

GERALDO BOTELHO *

Universidade Federal de Uberlândia, Brasil.
and

DANIEL M. PELLEGRINO

Universidade Federal de Campina Grande, Brasil

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Abstract

In this note we obtain several characterizations of Asplund spaces by means of ideals of Pietsch integral and nuclear polynomials, extending previous results of R. Alencar and R. Cilia-J. Gutiérrez.


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Introduction

A Banach space $E$ is an Asplund space if every separable subspace of $E$ has a separable dual. Let $\mathcal{P}_{PI}(^{n}E;F)$ (resp. $\mathcal{P}_{N}(^{n}E;F)$) denote the space of Pietsch integral (resp. nuclear) $n$-homogeneous polynomials from $E$ to $F$ (see definitions below). For linear operators ($n=1$) we write $PI(E;F)$ and $N(E;F)$. The inclusion $\mathcal{P}_{N}(^{n}E;F) \subseteq \mathcal{P}_{PI}(^{n}E;F)$ holds true for every $E$, $F$ and $n$. The following results are due to R. Alencar:

**Theorem 1.** [1, Theorem 1.3] A Banach space $E$ is an Asplund space if and only if $PI(E;F) = N(E;F)$ for every Banach space $F$.

**Theorem 2.** [2, Proposition 1] Let $E$ be a Banach space and $n \in \mathbb{N}$. If $E$ is an Asplund space, then $\mathcal{P}_{PI}(^{n}E;F) = \mathcal{P}_{N}(^{n}E;F)$ for every Banach space $F$.

Improvements of Theorem 2 were proved by C. Boyd-R. Ryan [4] and D. Carando-V. Dimant [5]. Recently, R. Cilia-J. Gutiérrez [6, Theorem 6] proved the converse of Theorem 2. This note has a twofold purpose: to give a simpler non-tensorial proof of this result of [6] and to extend this characterization of Asplund spaces to other ideals of polynomials which are related to Pietsch integral and nuclear operators.

**Preliminaries**

Throughout this note $E, E_1, \ldots, E_n$ and $F$ are real or complex Banach spaces, $B_E$ denotes the closed unit ball of $E$ and $\mathbb{N}$ denotes the set of natural numbers. The Banach spaces of continuous $n$-linear mappings from $E_1 \times \cdots \times E_n$ into $F$ and of continuous $n$-homogeneous polynomials from $E$ into $F$ with the sup norm will be denoted by $\mathcal{L}(E_1, \ldots, E_n;F)$ ($\mathcal{L}(^{n}E;F)$ if $E_1 = \cdots = E_n = E$) and $\mathcal{P}(^{n}E;F)$, respectively. If $A \in \mathcal{L}(^{n}E;F)$ and $P$ is the polynomial generated by $A$ we write $P = A$. Conversely, we write $\tilde{P}$ for the (unique) symmetric $n$-linear mapping associated to the polynomial $P$. For the general theory of multilinear mappings and homogeneous polynomials the reader is referred to S. Dineen [8].

A polynomial $P \in \mathcal{P}(^{n}E;F)$ is nuclear, resp. Pietsch integral, if it can be written as

$$P(x) = \sum_{i=1}^{\infty} \varphi_i(x)^n y_i \text{ for every } x \in E,$$
where \((\varphi_i) \subset E'\) and \((y_i) \subset F\) are such that \(\sum_{i=1}^{\infty} \|\varphi_i\| \|y_i\| < \infty\),

\[ \text{resp. } P(x) = \int_{B_{E'}} \varphi(x)^n d\mu(\varphi), \text{ for every } x \in E, \]

where \(\mu\) is an \(F\)-valued regular countably additive Borel measure of bounded variation on \(B_{E'}\) with the weak-star topology. The notation for the spaces of such polynomials was established in the introduction. \textit{Mutatis mutandis}, one defines Pietsch integral and nuclear \(n\)-linear mappings.

According to [3], an \(n\)-linear mapping \(A \in \mathcal{L}(E_1, \ldots, E_n; F)\) is said to be \textit{semi-integral} if there exist \(C \geq 0\) and a regular probability measure \(\nu\) on the Borel sets of \(B_{E_1'} \times \cdots \times B_{E_n'}\) endowed with the weak-star topologies \(\sigma(E_j', E_j)\), \(j = 1, \ldots, n\), such that

\[ \|A(x_1, \ldots, x_n)\| \leq C \left( \int_{B_{E_1'} \times \cdots \times B_{E_n'}} |\varphi_1(x_1) \cdots \varphi_n(x_n)| d\nu(\varphi_1, \ldots, \varphi_n) \right), \]

for every \(x_j \in E_j, j = 1, \ldots, n\).

Now we describe two methods, introduced by A. Pietsch [9], for the generation of ideals of polynomials from a given operator ideal. For \(i = 1, \ldots, n\), let \(\Psi_i^{(n)} : \mathcal{L}(E_1, \ldots, E_n; F) \rightarrow \mathcal{L}(E_i; \mathcal{L}(E_1, \ldots, E_n; F))\) represent the canonical isometric isomorphism defined by \(\Psi_i^{(n)}(A)(x_i)(x_1, \ldots, x_n) := A(x_1, \ldots, x_n)\), where the notation \([i]\) means that the \(i\)-th coordinate is not involved. Let \(\mathcal{I}\) be an arbitrary operator ideal.

- The factorization method: a polynomial \(P \in \mathcal{P}(E; F)\) is of \textit{type} \(\mathcal{P}_{\mathcal{L}(\mathcal{I})}\) if there exist a Banach space \(G\), a linear operator \(u \in \mathcal{I}(E; G)\) and a polynomial \(Q \in \mathcal{P}(G; F)\) such that \(P = Q \circ u\).

- The linearization method: a multilinear mapping \(A \in \mathcal{L}(E_1, \ldots, E_n; F)\) is of \textit{type} \([\mathcal{I}]\) if \(\Psi_i^{(n)}(A) \in \mathcal{I}(E_i; \mathcal{L}(E_1, \ldots, E_n; F))\) for every \(i = 1, \ldots, n\). A polynomial \(P \in \mathcal{P}(E; F)\) is of \textit{type} \([\mathcal{I}]\) - \(P \in \mathcal{P}_{[\mathcal{I}]}(E; F)\) - if \(\hat{P}\) is of type \([\mathcal{I}]\).

\textbf{Results}

First we give an alternative simpler proof of [6, Theorem 6].
Theorem 3. Let $E$ be a Banach space. If $\mathcal{P}_{PI}^{(n)E;F} = \mathcal{P}_{N}^{(n)E;F}$ for every Banach space $F$ and some $n \in \mathbb{N}$, then $E$ is an Asplund space.

Proof. In view of Theorem 1 it suffices to show that $PI(E;F) \subseteq N(E;F)$ for every $F$. Let $u \in PI(E;F)$. By [7, Theorem VI.3.11] there exist a regular Borel measure $\mu$ on $B_{E'}$ with the weak-star topology and a linear operator $v : L_{1}(\mu) \to F$ such that $u = v \circ j \circ i$, where $i : E \to C(B_{E'})$ is the canonical injection and $j : C(B_{E'}) \to L_{1}(\mu)$ is the formal inclusion. Fix $0 \neq a \in E$ and choose a linear functional $\varphi$ on $C(B_{E'})$ such that $\varphi(i(a)) = 1$. Define $R \in L^{(n)C(B_{E'});L_{1}(\mu)}$ by

$$R(f_{1}, \ldots, f_{n}) := \frac{1}{n} \sum_{k=1}^{n} \left( j(f_{k}) \prod_{m=1, m \neq k}^{n} \varphi(f_{m}) \right).$$

It is easy to see that $R$ is semi-integral (use, e.g., the fact that $j$ is absolutely summing). From a result due to R. Alencar-M. Matos [3, Theorem 5.6], it follows that $R$ is Pietsch integral. By [2, Proposition 2], $\hat{R}$ is Pietsch integral, and consequently nuclear, by hypothesis. Now define a polynomial $P := (v \circ \hat{R} \circ i) \in \mathcal{P}^{(n)E;F}$ and a linear operator $S : E \to F$ by $S(x) = \hat{P}(x, a, \ldots, a)$. Then $S$ is nuclear ($\hat{R}$ nuclear $\Rightarrow P$ nuclear $\Rightarrow S$ nuclear).

From

$$\hat{P}(x_{1}, \ldots, x_{n}) = \frac{1}{n} \sum_{k=1}^{n} \left( u(x_{k}) \prod_{m=1, m \neq k}^{n} \varphi(i(x_{m})) \right),$$

for every $x_{1}, \ldots, x_{n} \in E$, we obtain

$$S(x) = \frac{1}{n} u(x) + \frac{n-1}{n} (\varphi \circ i)(x)u(a),$$

for every $x \in E$. But $S$ is nuclear and $(\varphi \circ i) (\cdot)u(a)$ is a finite rank operator, so we conclude that $u$ is nuclear, what completes the proof. $\square$

Theorem 4. For a Banach space $E$ and operator ideals $\mathcal{I}_{1}$ and $\mathcal{I}_{2}$, the following assertions are equivalent:

(i) $\mathcal{I}_{1}(E;F) \subseteq \mathcal{I}_{2}(E;F)$ for every Banach space $F$.
(ii) $\mathcal{P}_{\mathcal{L}(\mathcal{I}_{1})}^{(n)E;F} \subseteq \mathcal{P}_{\mathcal{L}(\mathcal{I}_{2})}^{(n)E;F}$ for every $F$ and every $n \in \mathbb{N}$.
(iii) $\mathcal{P}_{\mathcal{L}(\mathcal{I}_{1})}^{(n)E;F} \subseteq \mathcal{P}_{\mathcal{L}(\mathcal{I}_{2})}^{(n)E;F}$ for every $F$ and some $n \in \mathbb{N}$.
(iv) $\mathcal{P}_{\mathcal{I}_{1}}^{(n)E;F} \subseteq \mathcal{P}_{\mathcal{I}_{2}}^{(n)E;F}$ for every $F$ and every $n \in \mathbb{N}$.
(v) $\mathcal{P}_{\mathcal{I}_{1}}^{(n)E;F} \subseteq \mathcal{P}_{\mathcal{I}_{2}}^{(n)E;F}$ for every $F$ and some $n \in \mathbb{N}$.
Proof. (ii) ⇒ (iii) and (iv) ⇒ (v) are obvious.

(i) ⇒ (ii) and (i) ⇒ (iv): Let $P \in \mathcal{P}_{L(I_1)}(nE; F)$ (resp. $P \in \mathcal{P}_{[I_1]}(nE; F)$).

Then $P = Q \circ u$ with $u \in I_1(E; G) \subseteq I_2(E; G)$ (resp. $\Psi_i^{(n)}(\hat{P}) \in I_1(E; L^{(n-1)}E; F)) \subseteq I_2(E; L^{(n-1)}E; F))$, hence $P \in \mathcal{P}_{L(I_2)}(nE; F)$ (resp. $P \in \mathcal{P}_{[I_2]}(nE; F)$).

(iii) ⇒ (i): Assume that $\mathcal{P}_{L(I_1)}(nE; F) \subseteq \mathcal{P}_{L(I_2)}(nE; F)$ for every $F$ and let $u \in I_1(E; F)$, $u \neq 0$. Choosing $\varphi \in F'$, $\varphi \neq 0$, $a \in E$ such that $u(a) \neq 0$ and $\varphi(u(a)) = 1$, and defining $P \in \mathcal{P}(nE; F)$, $Q \in \mathcal{P}(nF; F)$ by

$$P(x) := \varphi(u(x))^{n-1}u(x); \quad Q(y) := \varphi(y)^{n-1}y,$$

we have that $P = Q \circ u$. Therefore $P \in \mathcal{P}_{L(I_1)}(nE; F) \subseteq \mathcal{P}_{L(I_2)}(nE; F)$.

Thus there exist a Banach space $G$, $R \in \mathcal{P}(nG; F)$ and $v \in I_2(E; G)$ so that $P = R \circ v$. For every $x \in E$, $P(x, a, \ldots, a) = (R(\cdot, v(a), \ldots, v(a)) \circ v)(x)$, hence $\hat{P}(\cdot, a, \ldots, a) \in I_2(E; F)$. From $\hat{P} = Q \circ (u, \ldots, u)$ we have that

$$\hat{P}(\cdot, a, \ldots, a) = \frac{1}{n}u(\cdot) + \frac{n-1}{n}\varphi(u(\cdot))u(a).$$

Since $\hat{P}(\cdot, a, \ldots, a) \in I_2(E; F)$ and $\varphi(u(\cdot))u(a)$ is a finite rank operator, we conclude that $u \in I_2(E; F)$.

(v) ⇒ (i): Assume that $\mathcal{P}_{[I_1]}(nE; F) \subseteq \mathcal{P}_{[I_2]}(nE; F)$ for every $F$ and let $u \in I_1(E; F)$, $u \neq 0$. Fixing $0 \neq a \in E$, choosing $\varphi \in E'$ such that $\varphi(a) = 1$ and defining $P \in \mathcal{P}(nE; F)$ by $P(x) := \varphi(x)^{n-1}u(x)$ we have that

$$n\Psi_1^{(n)}(\hat{P})(x_1)(x_2, \ldots, x_n) = \varphi(x_2) \cdots \varphi(x_n)u(x_1) + \cdots + \varphi(x_1) \cdots \varphi(x_{n-1})u(x_n),$$

for every $x_1, \ldots, x_n \in E$. Defining $R : E \to L^{(n-1)}E; F)$, $S : F \to L^{(n-1)}E; F)$ by

$$R(x_1)(x_2, \ldots, x_n) := \frac{1}{n}\varphi(x_2) \cdots \varphi(x_n)u(x_1),$$

$$S(y_1)(x_2, \ldots, x_n) := \frac{1}{n}\varphi(x_2) \cdots \varphi(x_n)y_1,$$

and $T : E \to L^{(n-1)}E; F)$ by $T(x_1)(x_2, \ldots, x_n) :=

$$\frac{1}{n}\varphi(x_1)\varphi(x_3) \cdots \varphi(x_n)u(x_2) + \cdots + \frac{1}{n}\varphi(x_1)\varphi(x_2) \cdots \varphi(x_{n-1})u(x_n),$$

it follows that $R = S \circ u$, hence $R \in I_1(E; L^{(n-1)}E; F))$, and that $T$ is a finite rank operator. Since $\Psi_1^{(n)}(\hat{P}) = R + T$ we have that $\Psi_1^{(n)}(\hat{P})$ belongs
to $\mathcal{I}_1$. So, $P \in \mathcal{P}_{[\mathcal{I}_1]}^{(n)}(E;F) \subseteq \mathcal{P}_{[\mathcal{I}_2]}^{(n)}(E;F)$, and therefore $\Psi_1^{(n)}(P)$ belongs to $\mathcal{I}_2$. Now let us define $J : \mathcal{L}^{(n-1)}(E;F) \rightarrow F$ by $J(A) := A(a, \ldots, a)$ to obtain

$$
( J \circ \Psi_1^{(n)}(\tilde{P}) ) (x) = \frac{1}{n} u(x) + \frac{n-1}{n} \varphi(x) u(a) \text{ for every } x \in E.
$$

But $J \circ \Psi_1^{(n)}(\tilde{P}) \in \mathcal{I}_2(E;F)$ and $\varphi(\cdot)u(a)$ has finite rank, so $u \in \mathcal{I}_2(E;F)$.

Combining Theorems 1, 2, 3 and 4 we obtain the announced characterizations of Asplund spaces.

**Theorem 5.** For a Banach space $E$, the following assertions are equivalent:

(i) $E$ is an Asplund space.

(ii) For all $n \in \mathbb{N}$ and every $F$, we have $\mathcal{P}_{\mathcal{L}(PI)}^{(n)}(E;F) = \mathcal{P}_{\mathcal{L}(NI)}^{(n)}(E;F)$.

(iii) There is $n \in \mathbb{N}$ such that $\mathcal{P}_{\mathcal{L}(PI)}^{(n)}(E;F) = \mathcal{P}_{\mathcal{L}(NI)}^{(n)}(E;F)$ for every $F$.

(iv) For all $n \in \mathbb{N}$ and every $F$, we have $\mathcal{P}_{[PI]}^{(n)}(E;F) = \mathcal{P}_{[NI]}^{(n)}(E;F)$.

(v) There is $n \in \mathbb{N}$ such that $\mathcal{P}_{[PI]}^{(n)}(E;F) = \mathcal{P}_{[NI]}^{(n)}(E;F)$ for every $F$.

(vi) For all $n \in \mathbb{N}$ and every $F$, we have $\mathcal{P}_{PI}^{(n)}(E;F) = \mathcal{P}_{NI}^{(n)}(E;F)$.

(vii) There is $n \in \mathbb{N}$ such that $\mathcal{P}_{PI}^{(n)}(E;F) = \mathcal{P}_{NI}^{(n)}(E;F)$ for every $F$.

The same techniques can be used to prove the following additional characterizations:

**Theorem 6.** For a Banach space $E$, the following assertions are equivalent:

(i) $E$ is an Asplund space.

(ii) For all $n \in \mathbb{N}$ and every $F$, we have $\mathcal{P}_{\mathcal{L}(PI)}^{(n)}(E;F) \subseteq \mathcal{P}_{[NI]}^{(n)}(E;F)$.

(iii) There is $n \in \mathbb{N}$ such that $\mathcal{P}_{\mathcal{L}(PI)}^{(n)}(E;F) \subseteq \mathcal{P}_{[NI]}^{(n)}(E;F)$ for every $F$.

(iv) For all $n \in \mathbb{N}$ and every $F$, we have $\mathcal{P}_{\mathcal{L}(PI)}^{(n)}(E;F) \subseteq \mathcal{P}_{NI}^{(n)}(E;F)$.

(v) There is $n \in \mathbb{N}$ such that $\mathcal{P}_{\mathcal{L}(PI)}^{(n)}(E;F) \subseteq \mathcal{P}_{NI}^{(n)}(E;F)$ for every $F$.

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References


**Geraldo Botelho**
Faculdade de Matemática  
Univ. Federal de Uberlandia  
38.400-902 Uberlandia  
Brazil  
e-mail: botelho@ufu.br

and
Daniel M. Pellegrino
Departamento de Matemática e Estatística
Univ. Federal de Campina Grande
58.109-970 Campina Grande
Brazil
e-mail: dmp@dme.ufcg.edu.br