SPECTRAL PROPERTIES OF A NON SELFADJOINT SYSTEM OF DIFFERENTIAL EQUATIONS WITH A SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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Abstract

In this paper we investigated the spectrum of the operator $L(\lambda)$ generated in Hilbert Space of vector-valued functions $L^2(\mathbb{R}_+, \mathbb{C}_2)$ by the system

\[(0.1)iy_1' + q_1(x)y_2 = \lambda y_1, \quad -iy_2' + q_2(x)y_1 = \lambda y_2, \quad x \in \mathbb{R}_+ := [0, \infty), \]

and the spectral parameter-dependent boundary condition

\[(a_1\lambda + b_1)y_2(0,\lambda) - (a_2\lambda + b_2)y_1(0,\lambda) = 0, \]

where $\lambda$ is a complex parameter, $q_i, i = 1, 2$ are complex-valued functions $a_i \neq 0, b_i \neq 0, i = 1, 2$ are complex constants. Under the condition $\sup_{x \in \mathbb{R}_+} \{\exp \varepsilon |q_i(x)|\} < \infty, i = 1, 2, \varepsilon > 0,$

we proved that $L(\lambda)$ has a finite number of eigenvalues and spectral singularities with finite multiplicities. Furthermore we show that the principal functions corresponding to eigenvalues of $L(\lambda)$ belong to the space $L^2(\mathbb{R}_+, \mathbb{C}_2)$ and the principal functions corresponding to spectral singularities belong to a Hilbert space containing $L^2(\mathbb{R}_+, \mathbb{C}_2)$.

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1. Introduction

Let us consider the nonself-adjoint one dimensional Schrödinger operator $L$ generated in $L^2(R_+)$ by the differential expression

$$l(y) = -y'' + q(x) y, \quad x \in R_+$$

and the boundary condition $y(0) = 0$ as $Ly = ly$, where $q$ is a complex-valued function. The spectral analysis of $L$ has been studied by Naimark [7]. Naimark has proved that there are some poles of resolvent’s kernel which are not the eigenvalues of the operator $L$. (Schwartz [8] named these points as spectral singularities of $L$). Moreover, Naimark has proved that spectral singularities are on the continuous spectrum, he has also shown that $L$ has a finite number of eigenvalues and spectral singularities with finite multiplicities if the condition

$$\int_0^\infty e^{\varepsilon x} |q(x)| \, dx < \infty, \quad \varepsilon > 0$$

holds. Lyance has obtained the role of the spectral singularities in the spectral expansion of the operator $L$ in terms of principal functions [6].

The properties of the eigenvalues and vector-valued eigenfunctions of a boundary value problem for a one-dimensional Dirac system with a spectral parameter in the boundary conditions has been investigated by Kerimov [4].

We now consider the operator $L(\lambda)$ generated in

$$L^2(R_+, C_2) = \left\{ f(x) : f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \int_0^\infty \left( |f_1(x)|^2 + |f_2(x)|^2 \right) \, dx < \infty \right\}$$

by the system

$$i y_1' + q_1(x) y_2 = \lambda y_1, \quad (1.1)$$

$$-i y_2' + q_2(x) y_1 = \lambda y_2, \quad x \in R_+$$

and the spectral parameter-dependent boundary condition

$$\begin{align*}
(a_1 \lambda + b_1) y_2(0, \lambda) - (a_2 \lambda + b_2) y_1(0, \lambda) &= 0,
\end{align*} \quad (1.2)$$

where $q_i$, $i = 1, 2$, are complex-valued functions, $\lambda$ is the spectral parameter, $a_i, b_i$ are complex constants, $b_i \neq 0$, $i = 1, 2$; moreover $|a_1|^2 + |a_2|^2 \neq 0$. 
The spectrum of the operator generated by the system (1.1) with the boundary condition \( y_2(0) - hy_1(0) = 0 \), (which is the special case of (1.2) when \( a_i = 0, \ b = 1 \)) here \( h \neq 0 \) is a complex constant, has been investigated in [5] and in [1].

In this paper, we discussed the spectrum of \( L(\lambda) \) defined by (1.1) and (1.2) and proved that \( L(\lambda) \) has a finite number of eigenvalues and spectral singularities with finite multiplicities under the conditions

\[
|q_i(x)| \leq c e^{-\epsilon x} < \infty, \quad i = 1, 2, \ \epsilon > 0, \ \ c > 0
\]

by using analytic continuation method ([7]). Finally we observe the properties of the principal functions corresponding to eigenvalues and spectral singularities.

In the rest of the paper, we use the following notations:

\[
\begin{align*}
\mathbb{C}_+ & = \{ \lambda : \lambda \in \mathbb{C}, \ \text{Im} \lambda > 0 \}, \quad \mathbb{C}_- = \{ \lambda : \lambda \in \mathbb{C}, \ \text{Im} \lambda < 0 \}, \\
\mathbb{C}_+ & = \{ \lambda : \lambda \in \mathbb{C}, \ \text{Im} \lambda \geq 0 \}, \quad \mathbb{C}_- = \{ \lambda : \lambda \in \mathbb{C}, \ \text{Im} \lambda \leq 0 \},
\end{align*}
\]

\( \sigma_p(L(\lambda)) \) denotes the eigenvalues and \( \sigma_{ss}(L(\lambda)) \) denotes the spectral singularities of \( L(\lambda) \).

### 2. Preliminaries

Let us suppose that

\[
|q_i(x)| \leq c(1 + x)^{-1+\epsilon}, \quad i = 1, 2, \ x \in \mathbb{R}_+, \ \epsilon > 0
\]

holds, where \( c > 0 \) is a constant.

The following results were given in [1] and in the first reference there in. Under the conditions (2.1), equation (1.1) has the following vector solutions

\[
e^+(x, \lambda) = \begin{pmatrix} e_1^+(x, \lambda) \\ e_2^+(x, \lambda) \end{pmatrix} = \begin{pmatrix} \int_x^\infty H_{12}(x, t) e^{i\lambda t} dt \\ e^{i\lambda x} + \int_x^\infty H_{22}(x, t) e^{i\lambda t} dt \end{pmatrix},
\]

for \( \lambda \in \mathbb{C}_+ \) and

\[
e^-(x, \lambda) = \begin{pmatrix} e_1^-(x, \lambda) \\ e_2^-(x, \lambda) \end{pmatrix} = \begin{pmatrix} e^{-i\lambda x} + \int_x^\infty H_{11}(x, t) e^{-i\lambda t} dt \\ \int_x^\infty H_{21}(x, t) e^{-i\lambda t} dt \end{pmatrix}
\]
for \( \lambda \in \mathbb{C}_- \); moreover the kernels \( H_{ij}(x,t), \ i,j = 1,2 \), satisfy the inequalities
\[
|H_{ij}(x,t)| \leq c \sum_{k=1}^{2} q_k \left( \frac{x+t}{2} \right),
\]
where \( c > 0 \) is a constant. Therefore the functions \( e_i^+(x,\lambda) \) and \( e_i^-(x,\lambda) \), \( i = 1,2 \), are analytic with respect to \( \lambda \) in \( \mathbb{C}_+, \mathbb{C}_- \), and continuous on \( \mathbb{C}_+ \) and \( \mathbb{C}_- \), respectively. Moreover \( e^+ \) and \( e^- \) satisfy the following asymptotic equalities ([1])
\[
e^+(x,\lambda) = \begin{pmatrix} 0 \\ e^{i\lambda x} \end{pmatrix} \left[ 1 + o(1) \right], \quad \lambda \in \mathbb{C}_+, \ x \to \infty
\]
and
\[
e^-(x,\lambda) = \begin{pmatrix} e^{-i\lambda x} \\ 0 \end{pmatrix} \left[ 1 + o(1) \right], \quad \lambda \in \mathbb{C}_-, \ x \to \infty.
\]
From (2.5) and (2.6) we have
\[
W \{ e^+, e^- \} = \lim_{x \to \infty} W \{ e^+(x,\lambda), e^-(x,\lambda) \} = -1
\]
for \( \lambda \in \mathbb{R} \), where \( W \{ y^{(1)}, y^{(2)} \} \) is the wronskian of the solutions of \( y^{(1)} \) and \( y^{(2)} \) which is defined as \( W \{ y^{(1)}, y^{(2)} \} = y^{(1)}_{1} y^{(2)}_{2} - y^{(1)}_{2} y^{(2)}_{1} \), here \( y^{(i)} = (y^{(i)}_1, y^{(i)}_2), \ i = 1,2 \). Therefore \( e^+ \), \( e^- \) are the fundamental system of solutions of the system (1.1) for \( \lambda \in \mathbb{R} \).

Let \( \varphi(x,\lambda) \) be the solution of (1.1) satisfying the initial conditions
\[
\varphi_1(0,\lambda) = a_1 \lambda + b_1, \quad \varphi_2(0,\lambda) = a_2 \lambda + b_2.
\]
Clearly the solution \( \varphi(x,\lambda) \) exists uniquely and is an entire function of \( \lambda \).

3. Eigenvalues and spectral singularities

Let us define
\[
a^+(\lambda) = (a_1 \lambda + b_1) e_2^+(0,\lambda) - (a_2 \lambda + b_2) e_1^+(0,\lambda) = 0
\]
and
\[
a^-(\lambda) = (a_1 \lambda + b_1) e_2^-(0,\lambda) - (a_2 \lambda + b_2) e_1^-(0,\lambda) = 0.
\]
Let
\begin{equation}
R(x, t; \lambda) = \begin{cases} R^+(x, t; \lambda), & \text{Im } \lambda \geq 0 \\ R^-(x, t; \lambda), & \text{Im } \lambda \leq 0 \end{cases}
\end{equation}
be Green’s function of \( L(\lambda) \) which is obtained by using classical methods, here
\begin{equation}
R^+(x, t; \lambda) = \frac{i}{a^+(\lambda)} \begin{cases} e^+(x, \lambda) \varphi^*(t, \lambda), & 0 \leq t \leq x \\ \varphi(x, \lambda) (e^+)^*(t, \lambda), & x < t \leq \infty \end{cases}
\end{equation}
and
\begin{equation}
R^-(x, t; \lambda) = \frac{i}{a^-(\lambda)} \begin{cases} e^-(x, \lambda) \varphi^*(t, \lambda), & 0 \leq t \leq x \\ \varphi(x, \lambda) (e^-)^*(t, \lambda), & x < t \leq \infty \end{cases}
\end{equation}
and \((e^\pm)^* := \begin{pmatrix} e^\pm_2, e^\pm_1 \end{pmatrix} \), \( \varphi^* := (\varphi_2, \varphi_1) \). Moreover from (2.5) and (2.6) we have
\begin{equation}
e^+(x, \lambda) \in L^2(\mathbb{R}_+, \mathbb{C}_2) \end{equation}
for \( \lambda \in \mathbb{C}_+ \) and
\begin{equation}
e^-(x, \lambda) \in L^2(\mathbb{R}_+, \mathbb{C}_2) \end{equation}
for \( \lambda \in \mathbb{C}_- \). In this case we state the following

**Lemma 3.1.**

\( a) \) \( \sigma_p(L(\lambda)) = \{ \lambda : \lambda \in \mathbb{C}_+, \ a^+(\lambda) = 0 \} \cup \{ \lambda : \lambda \in \mathbb{C}_-, \ a^- (\lambda) = 0 \} \),

\( b) \) \( \sigma_{ss}(L(\lambda)) = \{ \lambda : \lambda \in \mathbb{R} \setminus \{0\} , \ a^+(\lambda) = 0 \} \cup \{ \lambda : \lambda \in \mathbb{R} \setminus \{0\} , \ a^- (\lambda) = 0 \} \).

**Proof.** \( a) \) It is clear that
\[ \{ \lambda : \lambda \in \mathbb{C}_+, \ a^+(\lambda) = 0 \} \cup \{ \lambda : \lambda \in \mathbb{C}_-, \ a^- (\lambda) = 0 \} \subset \sigma_p(L(\lambda)). \]

Now let us suppose that \( \lambda_0 \in \sigma_p(L(\lambda)) \). If \( \lambda_0 \in \mathbb{C}_+ \) then (1.1) has a nontrivial solution \( y(x, \lambda_0) \) in \( L^2(\mathbb{R}_+, \mathbb{C}_2) \) for \( \lambda = \lambda_0 \) satisfying (1.2).

Since \( W\{y(x, \lambda_0), \varphi(x, \lambda_0)\} = 0 \) then there exists a constant \( c \neq 0 \) such that \( y(x, \lambda_0) = c\varphi(x, \lambda_0) \). Therefore
\begin{equation}
W\{y(x, \lambda_0), e^+(x, \lambda_0)\} = y_1 (0, \lambda_0) e^+_2 (0, \lambda_0) - y_2 (0, \lambda_0) e^+_1 (0, \lambda_0) = ca^+(\lambda_0).
\end{equation}

Moreover we find from (3.5) that
\[ W \{ y(x, \lambda_0), e^+(x, \lambda_0) \} = \lim_{x \to \infty} \left\{ y_1(x, \lambda_0) e_2^+(x, \lambda_0) - y_2(x, \lambda_0) e_1^+(x, \lambda_0) \right\} = 0 \] (3.8)

So we obtain from (3.7) and (3.8) that \( a^+(\lambda_0) = 0 \).

If \( \lambda_0 \in \mathbb{C}_- \) then we prove that \( a^- (\lambda_0) = 0 \) similarly.

If \( \lambda_0 \in \mathbb{R} \), then the general solution of (1.1) is

\[ y(x, \lambda_0) = c_1 e^+(x, \lambda_0) + c_2 e^-(x, \lambda_0) \]

for \( \lambda = \lambda_0 \). From (2.5) and (2.6) we have

\[ y(x, \lambda_0) = \left( c_2 e^{-\lambda_0 x} / c_1 e^{\lambda_0 x} \right)(1 + o(1)) \]

as \( x \to \infty \). Therefore \( y(x, \lambda_0) \notin L^2(\mathbb{R}_+, \mathbb{C}_2) \). Hence \( \sigma_p(L(\lambda)) \cap \mathbb{R} = \), so (a) follows.

\( b) \) Spectral singularities which are not the eigenvalues of \( L(\lambda) \), are the poles of the resolvent’s kernel. From (3.1) – (3.4) and (a), we can say that the spectral singularities of \( L(\lambda) \) are the real zeros of \( a^+ \) and \( a^- \). So (b) follows.

Furthermore

\[ W\{e^+(x, \lambda), e^-(x, \lambda)\} = e_1^+(0, \lambda) e_2^-(0, \lambda) - e_2^+(0, \lambda) e_1^-(0, \lambda) = -1 \]

for \( \lambda \in \mathbb{R} \). Therefore we have

(3.9) \( \{ \lambda : \lambda \in \mathbb{R}, a^+(\lambda) = 0 \} \cap \{ \lambda : \lambda \in \mathbb{R}, a^- (\lambda) = 0 \} = \phi \)

Now as we see from Lemma 3.1 that to investigate the properties of the eigenvalues and the spectral singularities of \( L(\lambda) \), we need to investigate the properties of the zeros of \( a^+ \) and \( a^- \) in \( \mathbb{C}_+, \mathbb{C}_- \), respectively. For simplicity, we will consider only the zeros of \( a^+ \) in \( \mathbb{C}_+ \). In this point of view let us define the sets \( Z_+ = \{ \lambda : \lambda \in \mathbb{C}_+, a^+(\lambda) = 0 \} \), \( Z = \{ \lambda : \lambda \in \mathbb{R}, a^+(\lambda) = 0 \} \).

**Lemma 3.2.** (a) The set \( Z_+ \) is bounded and has at most a countable number of elements, and its limit points can lie only in a bounded subinterval of the real axis.

(b) \( Z \) is a compact set.
Proof. From (2.2) we get that \( a^+ (\lambda) \) is analytic in \( \mathbb{C}_+ \) and satisfies

\[
(3.10) \quad a^+ (\lambda) = a_1 \lambda + b_1 + \int_0^\infty \{(a_1 \lambda + b_1) H_{22}(0, t) - (a_2 \lambda + b_2) H_{12}(0, t)\} e^{i\lambda t} dt.
\]

From (3.10) we get

\[
(3.11) \quad a^+ (\lambda) = \lambda \left( a_1 + \int_0^\infty \{a_1 H_{22}(0, t) - a_2 H_{12}(0, t)\} e^{i\lambda t} dt \right) + O(1)
\]

for \( \lambda \in \mathbb{C}_+, \ |\lambda| \to \infty \). From (3.11) we find that the zeros of \( a^+ \) must lie in a bounded domain. Since \( a^+ \) is analytic in \( \mathbb{C}_+ \) then these zeros are at most countable numbers. From the uniqueness of analytic functions the limit points of \( Z_+ \) can lie only in a bounded subinterval of the real axis. So (a) follows. (b) is obtained from the uniqueness theorem of analytic functions [3].

From Lemma 3.1 and Lemma 3.2 we have

**Theorem 3.3.** If the conditions (2.1) hold, then the set of eigenvalues and spectral singularities of \( L(\lambda) \) are bounded, countable and their limit points can lie only in a bounded subinterval of the real axis.

**Definition 3.4.** The multiplicity of a zero of \( a^+ \) (or \( a^- \)) in \( \mathbb{C}_+ \) (or \( \mathbb{C}_- \)) is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of \( L(\lambda) \).

Let us suppose that

\[
(3.12) \quad |q_i(x)| \leq ce^{-\varepsilon x}, \quad c > 0, \quad \varepsilon > 0, \quad i = 1, 2
\]

hold. From (2.4) we obtain that

\[
(3.13) \quad |H_{ij}(x, t)| \leq c \exp \left\{ \frac{-\varepsilon}{2} (x + t) \right\}.
\]

From (3.10) and (3.13), \( a^+ \) has an analytic continuation from the real axis to the half plane \( \text{Im} \lambda > -\frac{\varepsilon}{2} \). So the limit points of the sets \( Z_+ \) and \( Z \) cannot lie in \( \mathbb{R} \) i.e. the sets \( Z_+ \) and \( Z \) have no limit points. Therefore the number of zeros of \( a^+ \) in \( \mathbb{C}_+ \) are finite with finite multiplicities. Similarly we can
show that \(a^-\) has a finite number of zeros with finite multiplicities in \(\mathbb{C}_-\). So we have proved the following

**Theorem 3.5.** The operator \(L(\lambda)\) has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity if the conditions (3.12) hold.

### 4. Principal functions

Assume that (3.12) holds. Let \(\lambda_1^+, \ldots, \lambda_j^+\) and \(\lambda_1^-, \ldots, \lambda_k^-\) denote the zeros of \(a^+\) in \(\mathbb{C}_+\) and \(a^-\) in \(\mathbb{C}_-\) with multiplicities \(m_1^+\), \ldots, \(m_j^+\) and \(m_1^-\), \ldots, \(m_k^-\), respectively. Similarly, let \(\lambda_1, \ldots, \lambda_p\) and \(\lambda_{p+1}, \ldots, \lambda_q\) denote the zeros of \(a^+\) and \(a^-\) on the real axis with multiplicities \(m_1, \ldots, m_p\) and \(m_{p+1}, \ldots, m_q\), respectively. In this case we have

\[
\left\{ \frac{\partial^n}{\partial \lambda^n} W[\varphi(x, \lambda), e^+(x, \lambda)] \right\}_{\lambda=\lambda_i^+} = \left\{ \frac{d^n}{d \lambda^n} a_+^+(\lambda) \right\}_{\lambda=\lambda_i^+} = 0
\]

for \(n = 0, 1, \ldots, m_i^+-1, \ i = 1, 2, \ldots, j\), and

\[
\left\{ \frac{\partial^n}{\partial \lambda^n} W[\varphi(x, \lambda), e^-(x, \lambda)] \right\}_{\lambda=\lambda_i^-} = \left\{ \frac{d^n}{d \lambda^n} a_-(\lambda) \right\}_{\lambda=\lambda_i^-} = 0
\]

for \(n = 0, 1, \ldots, m_i^- - 1, \ l = 1, 2, \ldots, k\). Clearly we have

\[
\varphi(x, \lambda_i^+) = c_0 \left( \lambda_i^+ \right) e^+(x, \lambda_i^+), \ i = 1, 2, \ldots, j,
\]

\[
\varphi(x, \lambda_i^-) = d_0 \left( \lambda_i^- \right) e^-(x, \lambda_i^-), \ l = 1, 2, \ldots, k,
\]

when \(n = 0\). Therefore \(c_0 \left( \lambda_i^+ \right) \neq 0, d_0 \left( \lambda_i^- \right) \neq 0\). Therefore we can state the following lemma

**Theorem 4.1.** The following equalities

\[
\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(x, \lambda) \right\}_{\lambda=\lambda_i^+} = \sum_{v=0}^{n} \binom{n}{v} e^+(x, \lambda_i^+) \left\{ \frac{\partial^v}{\partial \lambda^v} e^+(x, \lambda) \right\}_{\lambda=\lambda_i^+},
\]

for \(n = 0, 1, \ldots, m_i^+-1, \ i = 1, 2, \ldots, j\) and

\[
\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(x, \lambda) \right\}_{\lambda=\lambda_i^-} = \sum_{v=0}^{n} \binom{n}{v} d^-(x, \lambda_i^-) \left\{ \frac{\partial^v}{\partial \lambda^v} d^-(x, \lambda) \right\}_{\lambda=\lambda_i^-}.
\]
for \( n = 0, 1, ..., m_i^−-1, \ l = 1, 2, ..., k \) hold, where the constants \( c_0^+, c_1^+, ..., c_n^+ \)
and \( d_0^−, d_1^−, ..., d_n^− \) depend on \( \lambda_i^+ \) and \( \lambda_i^− \), respectively.

**Proof.** Using mathematical induction, we prove first (4.5). For \( n = 0 \), The proof is clear by (4.3). Now we suppose that (4.5) holds for \( 1 \leq n_0 \leq m_i^+ - 2 \), i.e.

\[
(4.7) \left\{ \frac{\partial^{n_0}}{\partial \lambda^{n_0}} \phi(x, \lambda) \right\}_{\lambda = \lambda_i^+} = \sum_{v=0}^{n_0} \binom{n_0}{v} c_{n_0-v}^+ \left( \lambda_i^+ \right) \left\{ \frac{\partial^v}{\partial \lambda^v} e^+(x, \lambda) \right\}_{\lambda = \lambda_i^+}.
\]

Now we will show that (4.5) also holds for \( n_0 + 1 \). If \( U(x, \lambda) \) is a solution of the equation (1.1), then \( \frac{\partial^{n_1}}{\partial \lambda^{n_1}} U(x, \lambda) \) satisfies the following equation:

\[
(4.8) \left\{ J \frac{d}{dx} + Q(x) - \lambda \right\} \frac{\partial^{n_1}}{\partial \lambda^{n_1}} U(x, \lambda) = n \frac{\partial^{n_1-1}}{\partial \lambda^{n_1-1}} U(x, \lambda),
\]

where

\[
J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad Q(x) = \begin{bmatrix} 0 & q_1(x) \\ q_2(x) & 0 \end{bmatrix}, \quad U(x, \lambda) = [U_1(x, \lambda), U_2(x, \lambda)].
\]

Writing (4.8) for \( \phi \left( x, \lambda_i^+ \right) \) and \( e^+(x, \lambda_i^+) \), then using (4.7), we get

\[
\left\{ J \frac{d}{dx} + Q(x) - \lambda_i^+ \right\} f_{n_0+1} \left( x, \lambda_i^+ \right) = 0,
\]

where

\[
f_{n_0+1} \left( x, \lambda_i^+ \right) = \left\{ \frac{\partial^{n_0+1}}{\partial \lambda^{n_0+1}} \phi(x, \lambda) \right\}_{\lambda = \lambda_i^+}
- \sum_{v=1}^{n_0+1} \binom{n_0+1}{v} c_{n_0+1-v}^+ \left( \lambda_i^+ \right) \left\{ \frac{\partial^v}{\partial \lambda^v} e^+(x, \lambda) \right\}_{\lambda = \lambda_i^+}.
\]

therefore we have

\[
W \left[ f_{n_0+1} \left( x, \lambda_i^+ \right), e^+ \left( x, \lambda_i^+ \right) \right] = \left\{ \frac{\partial^{n_0+1}}{\partial \lambda^{n_0+1}} W[\phi(x, \lambda), e^+(x, \lambda)] \right\}_{\lambda = \lambda_i^+} = 0
\]

by (4.1). Hence there exists a constant \( c_{n_0+1}^+ \left( \lambda_i^+ \right) \) such that
\[ f_{n_0+1}(x, \lambda^+_i) = c_{n_0+1}^+ (\lambda^+_i) e^+(x, \lambda^+_i) \]

which proves the theorem. Similarly we can prove that (4.6) holds.

Now we introduce the principal functions corresponding to the eigenvalues as follows:

\[
\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi (x, \lambda) \right\}_{\lambda=\lambda^+_i} \quad n = 0, 1, ..., m^+_i-1, \ i = 1, 2, ..., j,
\]

\[
\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi (x, \lambda) \right\}_{\lambda=\lambda^-_i} \quad n = 0, 1, ..., m^-_i-1, \ l = 1, 2, ..., k
\]

are called the principal functions corresponding to the eigenvalues \( \lambda = \lambda^+_i, \ i = 1, 2, ..., j \) and \( \lambda = \lambda^-_i, \ l = 1, 2, ..., k \) of \( L(\lambda) \), respectively.

Therefore we arrive at the following result for the principal functions given above:

**Theorem 4.2.** The principal functions corresponding to the eigenvalues of \( L(\lambda) \) are in \( L^2(\mathbb{R}_+, \mathbb{C}_2) \), i.e.

\[
\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi (\cdot, \lambda) \right\}_{\lambda=\lambda^+_i} \in L^2(\mathbb{R}_+, \mathbb{C}_2),
\]

\[
\begin{align*}
&n = 0, 1, ..., m^+_i-1, \ i = 1, 2, ..., j, \\
&n = 0, 1, ..., m^-_l-1, \ l = 1, 2, ..., k.
\end{align*}
\]

**Proof.** From (3.13) and (2.2) we obtain that

\[
\begin{align*}
&\left\{ \frac{\partial^n}{\partial \lambda^n} e^1_1 (x, \lambda) \right\}_{\lambda=\lambda^+_i} \leq c_1 e^{-\varepsilon x}, \\
&\left\{ \frac{\partial^n}{\partial \lambda^n} e^2_1 (x, \lambda) \right\}_{\lambda=\lambda^+_i} \leq x^n e^{-x} \operatorname{Im} \lambda^+_i + c_2 e^{-\varepsilon x}
\end{align*}
\]

for \( n = 0, 1, ..., m^+_i-1, \ i = 1, 2, ..., j \) which gives (4.9) by using (4.5). Equation (4.10) may be derived, by using (4.6), analogously.

**Definition 4.3.**

Obviously we also have

\[
\left\{ \frac{\partial^n}{\partial \lambda^n} W [\varphi (x, \lambda), e^+ (x, \lambda)] \right\}_{\lambda=\lambda_i} = \left\{ \frac{d^n}{d \lambda^n} a^+ (\lambda) \right\}_{\lambda=\lambda_i} = 0
\]

for \( n = 0, 1, ..., m_i - 1, \ i = 1, 2, ..., p \) and
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\[
\left\{ \frac{\partial^n}{\partial x^n} W[\varphi(x,\lambda), e^{-}(x,\lambda)] \right\}_{\lambda=\lambda_i} = \left\{ \frac{\partial^n}{\partial x^n} a^{-} (\lambda) \right\}_{\lambda=\lambda_i} = 0
\]

for \( n = 0, 1, ..., m_i - 1, \ l = p + 1, p + 2, ..., q \). Using the last two formulas given above, in a similar way to Theorem 4.1 we get that

**Remark 4.3.** The formulas

\[(4.11) \quad \left\{ \frac{\partial^n}{\partial x^n} \varphi(x,\lambda) \right\}_{\lambda=\lambda_i} = \sum_{v=0}^{n} \binom{n}{v} c_{n-v} (\lambda_i) \left\{ \frac{\partial^v}{\partial x^v} e^{+}(x,\lambda) \right\}_{\lambda=\lambda_i},
\]

for \( n = 0, 1, ..., m_i - 1, \ i = 1, 2, ..., p \), and

\[(4.12) \quad \left\{ \frac{\partial^n}{\partial x^n} \varphi(x,\lambda) \right\}_{\lambda=\lambda_l} = \sum_{v=0}^{n} \binom{n}{v} d_{n-v} (\lambda_l) \left\{ \frac{\partial^v}{\partial x^v} e^{-}(x,\lambda) \right\}_{\lambda=\lambda_l},
\]

for \( n = 0, 1, ..., m_l - 1, \ l = p + 1, p + 2, ..., q \) hold, where the constants \( c_0, c_1, ..., c_n \) and \( d_0, d_1, ..., d_n \) depend on \( \lambda_i \) and \( \lambda_l \), respectively.

Now we introduce the principal functions corresponding to the spectral singularities as follows:

\[
\left\{ \frac{\partial^n}{\partial x^n} \varphi(x,\lambda) \right\}_{\lambda=\lambda_i}, \ n = 0, 1, ..., m_i - 1, \ i = 1, 2, ..., p,
\]

\[
\left\{ \frac{\partial^n}{\partial x^n} \varphi(x,\lambda) \right\}_{\lambda=\lambda_l}, \ n = 0, 1, ..., m_l - 1, \ l = p + 1, p + 2, ..., q
\]

are called the principal functions corresponding to the spectral singularities \( \lambda = \lambda_i \ i = 1, 2, ..., p \) and \( \lambda = \lambda_l \ l = p + 1, p + 2, ..., q \) of \( L(\lambda) \), respectively. Therefore we arrive at the following

**Lemma 4.4** The principal functions for the spectral singularities do not belong to the space \( L^2(\mathbb{R}_{+}, C^2) \), i.e:

\[
\left\{ \frac{\partial^n}{\partial x^n} \varphi(.,\lambda) \right\}_{\lambda=\lambda_i} \notin L^2(\mathbb{R}_{+}, C^2),
\]

for \( n = 0, 1, ..., m_i - 1, \ i = 1, 2, ..., p \),

\[
\left\{ \frac{\partial^n}{\partial x^n} \varphi(.,\lambda) \right\}_{\lambda=\lambda_l} \notin L^2(\mathbb{R}_{+}, C^2),
\]

for \( n = 0, 1, ..., m_l - 1, \ l = p + 1, p + 2, ..., q \).

The proof of the lemma is obtained from (2.2), (2.3), (4.11) and (4.12).

Now let us introduce the following Hilbert spaces [2]

\[
H(\mathbb{R}_{+}, C^2, m) = \left\{ f(x) : f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \right. \\
\left. \int_0^\infty (1 + x)^{2m} \left\{ |f_1(x)|^2 + |f_2(x)|^2 \right\} dx < \infty \right\},
\]

\[
\left\{ \frac{\partial^n}{\partial x^n} \varphi(x,\lambda) \right\}_{\lambda=\lambda_i} \notin L^2(\mathbb{R}_{+}, C^2),
\]

for \( n = 0, 1, ..., m_i - 1, \ i = 1, 2, ..., p \),

\[
\left\{ \frac{\partial^n}{\partial x^n} \varphi(x,\lambda) \right\}_{\lambda=\lambda_l} \notin L^2(\mathbb{R}_{+}, C^2),
\]

for \( n = 0, 1, ..., m_l - 1, \ l = p + 1, p + 2, ..., q \).
\[ m = 0, 1, ... \] with norm
\[
\| f \|_{H(R_+, C_2, m)}^2 = \int_0^\infty (1 + x)^{2m} \left\{ |f_1(x)|^2 + |f_2(x)|^2 \right\} dx
\]

and
\[
H(R_+, C_2, -m) := \left\{ g(x) : g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}, \int_0^\infty (1 + x)^{-2m} \left\{ |g_1(x)|^2 + |g_2(x)|^2 \right\} dx < \infty \right\}
\]

\[ m = 0, 1, ... \] with norm
\[
\| g \|_{H(R_+, C_2, -m)}^2 = \int_0^\infty (1 + x)^{-2m} \left\{ |g_1(x)|^2 + |g_2(x)|^2 \right\} dx.
\]

Clearly \( H(R_+, C_2, 0) = L^2(R_+, C_2) \) and
\[
H(R_+, C_2, m) L^2(R_+, C_2) H(R_+, C_2, -m).
\]

Therefore we reach to the following

**Theorem 4.5.**

\[
\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi (., \lambda) \right\}_{\lambda = \lambda_i} \in H(R_+, C_2, -(n + 1)),
\]

for \( n = 0, 1, ..., m_i - 1, i = 1, 2, ..., p \), and

\[
\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi (., \lambda) \right\}_{\lambda = \lambda_l} \in H(R_+, C_2, -(n + 1))
\]

for \( n = 0, 1, ..., m_l - 1, l = p + 1, p + 2, ..., q \).

**Proof.** From (2.2), we obtain that

\[
\left\| \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi^+ (x, \lambda) \right\}_{\lambda = \lambda_i} \right\|_{L^\infty} \leq \int_x^\infty \left| H_{12} (x, t) \right| dt
\]
and
\[ \left\lfloor \frac{\partial^n}{\partial \lambda^n} e_2^+(x, \lambda) \right\rfloor_{\lambda=\lambda_i} \leq x^n + \int_x^\infty t^n |H_{22}(x, t)| \, dt. \tag{4.16} \]

for \( n = 0, 1, ..., m_i - 1, \ i = 1, 2, ..., p \). By the definition of
\( H(R_+, C_2, -(n+1)) \)
and using (4.15) and (4.16) we arrive at (4.13). In a similar way, we can show that (4.14) also holds.

Now let us choose \( n_0 \) so that
\[ n_0 = \max \{ m_1, ..., m_p, m_{p+1}, ..., m_q \}. \]

Then
\[ H(R_+, C_2, n_0) \subset L^2(R_+, C_2) \subset H(R_+, C_2, -n_0). \]

From Theorem 4.5, we finally reach to the following

**Conclusion 4.6.** The principal functions for the spectral singularities of the operator \( L(\lambda) \) belong to the space \( H(R_+, C_2, -n_0) \), i.e.:\n\[ \left\lfloor \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\rfloor_{\lambda=\lambda_i} \in H(R_+, C_2, -n_0), \]
for \( n = 0, 1, ..., m_i - 1, \ i = 1, 2, ..., p \) and
\[ \left\lfloor \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\rfloor_{\lambda=\lambda_l} \in H(R_+, C_2, -n_0) \]
for \( n = 0, 1, ..., m_l - 1, \ l = p + 1, p + 2, ..., q \).

**References**


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