SPECTRAL PROPERTIES OF A NON SELFADJOINT SYSTEM OF DIFFERENTIAL EQUATIONS WITH A SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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Abstract

In this paper we investigated the spectrum of the operator $L(\lambda)$ generated in Hilbert Space of vector-valued functions $L^2(\mathbf{R}_+, \mathbf{C}_2)$ by the system

$$(0.1)iy_1' + q_1(x)y_2 = \lambda y_1, \quad -iy_2' + q_2(x)y_1 = \lambda y_2, \quad x \in \mathbf{R}_+ := [0, \infty),$$

and the spectral parameter- dependent boundary condition

$$(a_1\lambda + b_1)y_2(0, \lambda) - (a_2\lambda + b_2)y_1(0, \lambda) = 0,$$

where λ is a complex parameter, q_i , $i = 1, 2$ are complex-valued functions

$a_i \neq 0$, $b_i \neq 0$, $i = 1, 2$ are complex constants. Under the condition

$$\sup_{x \in \mathbf{R}_+} \{\exp \varepsilon x |q_i(x)|\} < \infty, i = 1, 2, \varepsilon > 0,$$

we proved that $L(\lambda)$ has a finite number of eigenvalues and spectral singularities with finite multiplicities. Furthermore we show that the principal functions corresponding to eigenvalues of $L(\lambda)$ belong to the space $L^2(\mathbf{R}_+, \{\mathbf{C}_2\})$ and the principal functions corresponding to spectral singularities belong to a Hilbert space containing $L^2(\mathbf{R}_+, \mathbf{C}_2)$.

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1. Introduction

Let us consider the nonself-adjoint one dimensional Schrödinger operator L generated in $L^2(\mathbf{R}_+)$ by the differential expression

$$l(y) = -y'' + q(x)y, \quad x \in \mathbf{R}_+$$

and the boundary condition $y(0) = 0$ as $Ly = ly$, where q is a complex-valued function. The spectral analysis of L has been studied by Naimark [7]. Naimark has proved that there are some poles of resolvent's kernel which are not the eigenvalues of the operator L . (Schwartz [8] named these points as spectral singularities of L). Moreover Naimark has proved that spectral singularities are on the continuous spectrum, he has also shown that L has a finite number of eigenvalues and spectral singularities with finite multiplicities if the condition

$$\int_0^\infty e^{\varepsilon x} |q(x)| dx < \infty, \quad \varepsilon > 0$$

holds. Lyance has obtained the role of the spectral singularities in the spectral expansion of the operator L in terms of principal functions[6].

The properties of the eigenvalues and vector-valued eigenfunctions of a boundary value problem for a one-dimensional Dirac system with a spectral parameter in the boundary conditions has been investigated by Kerimov [4].

We now consider the operator $L(\lambda)$ generated in

$$L^2(\mathbf{R}_+, \mathbf{C}_2) := \left\{ f(x) : f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \int_0^\infty \left\{ |f_1(x)|^2 + |f_2(x)|^2 \right\} dx < \infty \right\}$$

by the system

$$(1.1) \quad iy_1' + q_1(x)y_2 = \lambda y_1,$$

$$-iy_2' + q_2(x)y_1 = \lambda y_2, \quad x \in \mathbf{R}_+$$

and the spectral parameter-dependent boundary condition

$$(1.2) \quad (a_1\lambda + b_1)y_2(0, \lambda) - (a_2\lambda + b_2)y_1(0, \lambda) = 0,$$

where q_i , $i = 1, 2$, are complex-valued functions, λ is the spectral parameter, a_i, b_i are complex constants, $b_i \neq 0$, $i = 1, 2$; moreover $|a_1|^2 + |a_2|^2 \neq 0$.

The spectrum of the operator generated by the system (1.1) with the boundary condition $y_2(0) - hy_1(0) = 0$, (which is the special case of (1.2) when $a_i = 0$, $b = 1$) here $h \neq 0$ is a complex constant, has been investigated in [5] and in [1].

In this paper, we discussed the spectrum of $L(\lambda)$ defined by (1.1) and (1.2) and proved that $L(\lambda)$ has a finite number of eigenvalues and spectral singularities with finite multiplicities under the conditions

$$|q_i(x)| \leq ce^{-\varepsilon x} < \infty, \quad i = 1, 2, \quad \varepsilon > 0, \quad c > 0$$

by using analytic continuation method ([7]). Finally we observe the properties of the principal functions corresponding to eigenvalues and spectral singularities.

In the rest of the paper, we use the following notations:

$$\begin{aligned} \mathbf{C}_+ &= \{\lambda : \lambda \in \mathbf{C}, \operatorname{Im} \lambda > 0\}, \quad \overline{\mathbf{C}}_- = \{\lambda : \lambda \in \mathbf{C}, \operatorname{Im} \lambda < 0\}, \\ \overline{\mathbf{C}}_+ &= \{\lambda : \lambda \in \mathbf{C}, \operatorname{Im} \lambda \geq 0\}, \quad \overline{\mathbf{C}}_- = \{\lambda : \lambda \in \mathbf{C}, \operatorname{Im} \lambda \leq 0\}, \end{aligned}$$

$\sigma_p(L(\lambda))$ denotes the eigenvalues and $\sigma_{ss}(L(\lambda))$ denotes the spectral singularities of $L(\lambda)$.

2. Preliminaries

Let us suppose that

$$(2.1) \quad |q_i(x)| \leq c(1+x)^{-(1+\varepsilon)}, \quad i = 1, 2, \quad x \in \mathbf{R}_+, \quad \varepsilon > 0$$

holds, where $c > 0$ is a constant.

The following results were given in [1] and in the first reference there in. Under the conditions (2.1), equation (1.1) has the following vector solutions

$$(2.2) \quad e^+(x, \lambda) = \begin{pmatrix} e_1^+(x, \lambda) \\ e_2^+(x, \lambda) \end{pmatrix} = \begin{pmatrix} \int_x^\infty H_{12}(x, t) e^{i\lambda t} dt \\ e^{i\lambda x} + \int_x^\infty H_{22}(x, t) e^{i\lambda t} dt \end{pmatrix}$$

for $\lambda \in \overline{\mathbf{C}}_+$ and

$$(2.3) \quad e^-(x, \lambda) = \begin{pmatrix} e_1^-(x, \lambda) \\ e_2^-(x, \lambda) \end{pmatrix} = \begin{pmatrix} e^{-i\lambda x} + \int_x^\infty H_{11}(x, t) e^{-i\lambda t} dt \\ \int_x^\infty H_{21}(x, t) e^{-i\lambda t} dt \end{pmatrix}$$

for $\lambda \in \overline{\mathbf{C}}_-$; moreover the kernels $H_{ij}(x, t)$, $i, j = 1, 2$, satisfy the inequalities

$$(2.4) \quad |H_{ij}(x, t)| \leq c \sum_{k=1}^2 \left| q_k \left(\frac{x+t}{2} \right) \right|,$$

where $c > 0$ is a constant. Therefore the functions $e_i^+(x, \lambda)$ and $e_i^-(x, \lambda)$, $i = 1, 2$, are analytic with respect to λ in $\mathbf{C}_+$, $\mathbf{C}_-$, and continuous on $\overline{\mathbf{C}}_+$ and $\overline{\mathbf{C}}_-$, respectively. Moreover e^+ and e^- satisfy the following asymptotic equalities ([1])

$$(2.5) \quad e^+(x, \lambda) = \begin{pmatrix} 0 \\ e^{i\lambda x} \end{pmatrix} [1 + o(1)], \quad \lambda \in \overline{\mathbf{C}}_+ \quad x \rightarrow \infty$$

and

$$(2.6) \quad e^-(x, \lambda) = \begin{pmatrix} e^{-i\lambda x} \\ 0 \end{pmatrix} [1 + o(1)], \quad \lambda \in \overline{\mathbf{C}}_-, \quad x \rightarrow \infty.$$

From (2.5) and (2.6) we have

$$(2.7) \quad W\{e^+, e^-\} = \lim_{x \rightarrow \infty} W\{e^+(x, \lambda), e^-(x, \lambda)\} = -1$$

for $\lambda \in \mathbf{R}$, where $W\{y^{(1)}, y^{(2)}\}$ is the wronskian of the solutions of $y^{(1)}$ and $y^{(2)}$ which is defined as $W\{y^{(1)}, y^{(2)}\} = y_1^{(1)}y_2^{(2)} - y_1^{(2)}y_2^{(1)}$, here $y^{(i)} = \begin{pmatrix} y_1^{(i)} \\ y_2^{(i)} \end{pmatrix}$, $i = 1, 2$. Therefore e^+ , e^- are the fundamental system of solutions of the system (1.1) for $\lambda \in \mathbf{R}$.

Let $\varphi(x, \lambda)$ be the solution of (1.1) satisfying the initial conditions

$$\varphi_1(0, \lambda) = a_1\lambda + b_1, \quad \varphi_2(0, \lambda) = a_2\lambda + b_2.$$

Clearly the solution $\varphi(x, \lambda)$ exists uniquely and is an entire function of λ .

3. Eigenvalues and spectral singularities

Let us define

$$a^+(\lambda) = (a_1\lambda + b_1)e_2^+(0, \lambda) - (a_2\lambda + b_2)e_1^+(0, \lambda) = 0$$

$$(3.1) \quad a^-(\lambda) = (a_1\lambda + b_1)e_2^-(0, \lambda) - (a_2\lambda + b_2)e_1^-(0, \lambda) = 0.$$

Let

$$(3.2) \quad R(x, t; \lambda) = \begin{cases} R^+(x, t; \lambda), & \text{Im } \lambda \geq 0 \\ R^-(x, t; \lambda), & \text{Im } \lambda \leq 0 \end{cases}$$

be Green's function of $L(\lambda)$ which is obtained by using classical methods, here

$$(3.3) \quad R^+(x, t; \lambda) = \frac{i}{a^+(\lambda)} \begin{cases} e^+(x, \lambda) \varphi^*(t, \lambda), & 0 \leq t \leq x \\ \varphi(x, \lambda) (e^+)^*(t, \lambda), & x < t \leq \infty \end{cases}$$

and

$$(3.4) \quad R^-(x, t; \lambda) = \frac{i}{a^-(\lambda)} \begin{cases} e^-(x, \lambda) \varphi^*(t, \lambda), & 0 \leq t \leq x \\ \varphi(x, \lambda) (e^-)^*(t, \lambda), & x < t \leq \infty \end{cases}$$

and $(e^\pm)^* := (e_2^\pm, e_1^\pm)$, $\varphi^* := (\varphi_2, \varphi_1)$. Moreover from (2.5) and (2.6) we have

$$(3.5) \quad e^+(x, \lambda) \in L^2(\mathbf{R}_+, \mathbf{C}_2)$$

for $\lambda \in \mathbf{C}_+$ and

$$(3.6) \quad e^-(x, \lambda) \in L^2(\mathbf{R}_+, \mathbf{C}_2)$$

for $\lambda \in \mathbf{C}_-$. In this case we state the following

Lemma 3.1.

- a) $\sigma_p(L(\lambda)) = \{\lambda : \lambda \in \mathbf{C}_+, a^+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbf{C}_-, a^-(\lambda) = 0\}$,
- b) $\sigma_{ss}(L(\lambda)) = \{\lambda : \lambda \in \mathbf{R} \setminus \{0\}, a^+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbf{R} \setminus \{0\}, a^-(\lambda) = 0\}$.

Proof. a) It is clear that

$$\{\lambda : \lambda \in \mathbf{C}_+, a^+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbf{C}_-, a^-(\lambda) = 0\} \subset \sigma_p(L(\lambda)).$$

Now let us suppose that $\lambda_0 \in \sigma_p(L(\lambda))$. If $\lambda_0 \in \mathbf{C}_+$ then (1.1) has a nontrivial solution $y(x, \lambda_0)$ in $L^2(\mathbf{R}_+, \mathbf{C}_2)$ for $\lambda = \lambda_0$ satisfying (1.2).

Since $W\{y(x, \lambda_0), \varphi(x, \lambda_0)\} = 0$ then there exists a constant $c \neq 0$ such that $y(x, \lambda_0) = c\varphi(x, \lambda_0)$. Therefore

$$(3.7) \quad \begin{aligned} & W\{y(x, \lambda_0), e^+(x, \lambda_0)\} \\ & = y_1(0, \lambda_0) e_2^+(0, \lambda_0) - y_2(0, \lambda_0) e_1^+(0, \lambda_0) = ca^+(\lambda_0). \end{aligned}$$

Moreover we find from (3.5) that

$$\begin{aligned}
(3.8) \quad & W \{y(x, \lambda_0), e^+(x, \lambda_0)\} \\
& = \lim_{x \rightarrow \infty} \left\{ y_1(x, \lambda_0) e_2^+(x, \lambda_0) - y_2(x, \lambda_0) e_1^+(x, \lambda_0) \right\} \\
& = 0
\end{aligned}$$

So we obtain from (3.7) and (3.8) that $a^+(\lambda_0) = 0$.

If $\lambda_0 \in \mathbf{C}_-$ then we prove that $a^-(\lambda_0) = 0$ similarly.

If $\lambda_0 \in \mathbf{R}$, then the general solution of (1.1) is

$$y(x, \lambda_0) = c_1 e^+(x, \lambda_0) + c_2 e^-(x, \lambda_0)$$

for $\lambda = \lambda_0$. From (2.5) and (2.6) we have

$$y(x, \lambda_0) = \begin{pmatrix} c_2 e^{-i\lambda_0 x} \\ c_1 e^{i\lambda_0 x} \end{pmatrix} (1 + o(1))$$

as $x \rightarrow \infty$. Therefore $y(x, \lambda_0) \notin L^2(\mathbf{R}_+, \mathbf{C}_2)$. Hence $\sigma_p(L(\lambda)) \cap \mathbf{R} = \emptyset$, so (a) follows.

(b) Spectral singularities which are not the eigenvalues of $L(\lambda)$, are the poles of the resolvent's kernel. From (3.1) – (3.4) and (a), we can say that the spectral singularities of $L(\lambda)$ are the real zeros of a^+ and a^- . So (b) follows.

Furthermore

$$W\{e^+(x, \lambda), e^-(x, \lambda)\} = e_1^+(0, \lambda) e_2^-(0, \lambda) - e_2^+(0, \lambda) e_1^-(0, \lambda) = -1$$

for $\lambda \in \mathbf{R}$. Therefore we have

$$(3.9) \quad \{\lambda : \lambda \in \mathbf{R}, a^+(\lambda) = 0\} \cap \{\lambda : \lambda \in \mathbf{R}, a^-(\lambda) = 0\} = \emptyset$$

Now as we see from Lemma 3.1 that to investigate the properties of the eigenvalues and the spectral singularities of $L(\lambda)$, we need to investigate the properties of the zeros of a^+ and a^- in $\overline{\mathbf{C}}_+$, $\overline{\mathbf{C}}_-$, respectively. For simplicity, we will consider only the zeros of a^+ in $\overline{\mathbf{C}}_+$. In this point of view let us define the sets $Z_+ = \{\lambda : \lambda \in \mathbf{C}_+, a^+(\lambda) = 0\}$, $Z = \{\lambda : \lambda \in \mathbf{R}, a^+(\lambda) = 0\}$.

Lemma 3.2. (a) The set Z_+ is bounded and has at most a countable number of elements, and its limit points can lie only in a bounded subinterval of the real axis.

(b) Z is a compact set.

Proof. From (2.2) we get that $a^+(\lambda)$ is analytic in $\mathbf{C}_+$ and satisfies

$$(3.10) \quad a^+(\lambda) = a_1\lambda + b_1 + \int_0^\infty \{(a_1\lambda + b_1)H_{22}(0, t) - (a_2\lambda + b_2)H_{12}(0, t)\} e^{i\lambda t} dt.$$

From (3.10) we get

$$(3.11) \quad a^+(\lambda) = \lambda \left(a_1 + \int_0^\infty \{a_1H_{22}(0, t) - a_2H_{12}(0, t)\} e^{i\lambda t} dt \right) + O(1)$$

for $\lambda \in \overline{\mathbf{C}}_+$, $|\lambda| \rightarrow \infty$. From (3.11) we find that the zeros of a^+ must lie in a bounded domain. Since a^+ is analytic in $\mathbf{C}_+$ then these zeros are at most countable numbers. From the uniqueness of analytic functions the limit points of Z_+ can lie only in a bounded subinterval of the real axis. So (a) follows. (b) is obtained from the uniqueness theorem of analytic functions [3]

From Lemma 3.1 and Lemma 3.2 we have

Theorem 3.3. If the conditions (2.1) hold, then the set of eigenvalues and spectral singularities of $L(\lambda)$ are bounded, countable and their limit points can lie only in a bounded subinterval of the real axis.

Definition 3.4. The multiplicity of a zero of a^+ (or a^-) in $\overline{\mathbf{C}}_+$ (or $\overline{\mathbf{C}}_-$) is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of $L(\lambda)$.

Let us suppose that

$$(3.12) \quad |q_i(x)| \leq ce^{-\varepsilon x}, \quad c > 0, \quad \varepsilon > 0, \quad i = 1, 2$$

hold. From (2.4) we obtain that

$$(3.13) \quad |H_{ij}(x, t)| \leq c \exp \left\{ \frac{-\varepsilon}{2} (x + t) \right\}.$$

From (3.10) and (3.13), a^+ has an analytic continuation from the real axis to the half plane $\text{Im } \lambda > -\frac{\varepsilon}{2}$. So the limit points of the sets Z_+ and Z cannot lie in $\mathbf{R}$ i.e. the sets Z_+ and Z have no limit points. Therefore the number of zeros of a^+ in $\overline{\mathbf{C}}_+$ are finite with finite multiplicities. Similarly we can

show that a^- has a finite number of zeros with finite multiplicities in $\overline{\mathbf{C}}_-$. So we have proved the following

Theorem 3.5. The operator $L(\lambda)$ has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity if the conditions (3.12) hold.

4. Principal functions

Assume that (3.12) holds. Let $\lambda_1^+, \dots, \lambda_j^+$ and $\lambda_1^-, \dots, \lambda_k^-$ denote the zeros of a^+ in $\mathbf{C}_+$ and a^- in $\mathbf{C}_-$ with multiplicities $m_1^+, \dots, m_j^+$ and $m_1^-, \dots, m_k^-$, respectively. Similarly, let $\lambda_1, \dots, \lambda_p$ and $\lambda_{p+1}, \dots, \lambda_q$ denote the zeros of a^+ and a^- on the real axis with multiplicities $m_1, \dots, m_p$ and $m_{p+1}, \dots, m_q$, respectively. In this case we have

$$(4.1) \quad \left\{ \frac{\partial^n}{\partial \lambda^n} W[\varphi(x, \lambda), e^+(x, \lambda)] \right\}_{\lambda=\lambda_i^+} = \left\{ \frac{d^n}{d\lambda^n} a^+(\lambda) \right\}_{\lambda=\lambda_i^+} = 0$$

for $n = 0, 1, \dots, m_i^+ - 1$, $i = 1, 2, \dots, j$, and

$$(4.2) \quad \left\{ \frac{\partial^n}{\partial \lambda^n} W[\varphi(x, \lambda), e^-(x, \lambda)] \right\}_{\lambda=\lambda_l^-} = \left\{ \frac{d^n}{d\lambda^n} a^-(\lambda) \right\}_{\lambda=\lambda_l^-} = 0$$

for $n = 0, 1, \dots, m_l^- - 1$, $l = 1, 2, \dots, k$. Clearly we have

$$(4.3) \quad \varphi(x, \lambda_i^+) = c_0(\lambda_i^+) e^+(x, \lambda_i^+), \quad i = 1, 2, \dots, j,$$

$$(4.4) \quad \varphi(x, \lambda_l^-) = d_0(\lambda_l^-) e^-(x, \lambda_l^-), \quad l = 1, 2, \dots, k,$$

when $n = 0$. Therefore $c_0(\lambda_i^+) \neq 0$, $d_0(\lambda_l^-) \neq 0$. Therefore we can state the following lemma

Theorem 4.1. The following equalities

$$(4.5) \quad \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(x, \lambda) \right\}_{\lambda=\lambda_i^+} = \sum_{v=0}^n \binom{n}{v} c_{n-v}^+(\lambda_i^+) \left\{ \frac{\partial^v}{\partial \lambda^v} e^+(x, \lambda) \right\}_{\lambda=\lambda_i^+},$$

for $n = 0, 1, \dots, m_i^+ - 1$, $i = 1, 2, \dots, j$ and

$$(4.6) \quad \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(x, \lambda) \right\}_{\lambda=\lambda_l^-} = \sum_{v=0}^n \binom{n}{v} d_{n-v}^-(\lambda_l^-) \left\{ \frac{\partial^v}{\partial \lambda^v} e^-(x, \lambda) \right\}_{\lambda=\lambda_l^-}$$

for $n = 0, 1, \dots, m_l^- - 1$, $l = 1, 2, \dots, k$ hold, where the constants $c_0^+, c_1^+, \dots, c_n^+$ and $d_0^-, d_1^-, \dots, d_n^-$ depend on λ_i^+ and λ_l^- , respectively.

Proof. Using mathematical induction, we prove first (4.5). For $n = 0$, The proof is clear by (4.3). Now we suppose that (4.5) holds for $1 \leq n_0 \leq m_i^+ - 2$, i.e.

$$(4.7) \left\{ \frac{\partial^{n_0}}{\partial \lambda^{n_0}} \varphi(x, \lambda) \right\}_{\lambda=\lambda_i^+} = \sum_{v=0}^{n_0} \binom{n_0}{v} c_{n_0-v}^+(\lambda_i^+) \left\{ \frac{\partial^v}{\partial \lambda^v} e^+(x, \lambda) \right\}_{\lambda=\lambda_i^+}.$$

Now we will show that (4.5) also holds for $n_0 + 1$. If $U(x, \lambda)$ is a solution of the equation (1.1), then $\frac{\partial^n}{\partial \lambda^n} U(x, \lambda)$ satisfies the following equation:

$$(4.8) \quad \left\{ J \frac{d}{dx} + Q(x) - \lambda \right\} \frac{\partial^n}{\partial \lambda^n} U(x, \lambda) = n \frac{\partial^{n-1}}{\partial \lambda^{n-1}} U(x, \lambda),$$

where

$$J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad Q(x) = \begin{bmatrix} 0 & q_1(x) \\ q_2(x) & 0 \end{bmatrix}, \quad U(x, \lambda) = \begin{bmatrix} U_1(x, \lambda) \\ U_2(x, \lambda) \end{bmatrix}.$$

Writing (4.8) for $\varphi(x, \lambda_i^+)$ and $e^+(x, \lambda_i^+)$, then using (4.7), we get

$$\left\{ J \frac{d}{dx} + Q(x) - \lambda_i^+ \right\} f_{n_0+1}(x, \lambda_i^+) = 0,$$

where

$$\begin{aligned} f_{n_0+1}(x, \lambda_i^+) &= \left\{ \frac{\partial^{n_0+1}}{\partial \lambda^{n_0+1}} \varphi(x, \lambda) \right\}_{\lambda=\lambda_i^+} \\ &\quad - \sum_{v=1}^{n_0+1} \binom{n_0+1}{v} c_{n_0+1-v}^+(\lambda_i^+) \left\{ \frac{\partial^v}{\partial \lambda^v} e^+(x, \lambda) \right\}_{\lambda=\lambda_i^+}. \end{aligned}$$

therefore we have

$$W[f_{n_0+1}(x, \lambda_i^+), e^+(x, \lambda_i^+)] = \left\{ \frac{\partial^{n_0+1}}{\partial \lambda^{n_0+1}} W[\varphi(x, \lambda), e^+(x, \lambda)] \right\}_{\lambda=\lambda_i^+} = 0$$

by (4.1). Hence there exists a constant $c_{n_0+1}^+(\lambda_i^+)$ such that

$$f_{n_0+1}(x, \lambda_i^+) = c_{n_0+1}^+(\lambda_i^+) e^+(x, \lambda_i^+)$$

which proves the theorem. Similarly we can prove that (4.6) holds.

Now we introduce the principal functions corresponding to the eigenvalues as follows:

$$\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(x, \lambda) \right\}_{\lambda=\lambda_i^+}, \quad n = 0, 1, \dots, m_i^+ - 1, \quad i = 1, 2, \dots, j,$$

$$\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(x, \lambda) \right\}_{\lambda=\lambda_l^-}, \quad n = 0, 1, \dots, m_l^- - 1, \quad l = 1, 2, \dots, k$$

are called the principal functions corresponding to the eigenvalues $\lambda = \lambda_i^+$, $i = 1, 2, \dots, j$ and $\lambda = \lambda_l^-$, $l = 1, 2, \dots, k$ of $L(\lambda)$, respectively.

Therefore we arrive at the following result for the principal functions given above:

Theorem 4.2. The principal functions corresponding to the eigenvalues of $L(\lambda)$ are in $L^2(\mathbf{R}_+, \mathbf{C}_2)$, i.e.:

$$(4.9) \quad \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\}_{\lambda=\lambda_i^+} \in L^2(\mathbf{R}_+, \mathbf{C}_2),$$

$$n = 0, 1, \dots, m_i^+ - 1, \quad i = 1, 2, \dots, j,$$

$$(4.10) \quad \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\}_{\lambda=\lambda_l^-} \in L^2(\mathbf{R}_+, \mathbf{C}_2),$$

$$n = 0, 1, \dots, m_l^- - 1, \quad l = 1, 2, \dots, k.$$

Proof. From (3.13) and (2.2) we obtain that

$$\left| \left\{ \frac{\partial^n}{\partial \lambda^n} e_1^+(x, \lambda) \right\}_{\lambda=\lambda_i^+} \right| \leq c_1 e^{-\varepsilon x},$$

$$\left| \left\{ \frac{\partial^n}{\partial \lambda^n} e_2^+(x, \lambda) \right\}_{\lambda=\lambda_i^+} \right| \leq x^n e^{-x \operatorname{Im} \lambda_i^+} + c_2 e^{-\varepsilon x}$$

for $n = 0, 1, \dots, m_i^+ - 1$, $i = 1, 2, \dots, j$ which gives (4.9) by using (4.5). Equation (4.10) may be derived, by using (4.6), analogously.

Definition 4.3.

Obviously we also have

$$\left\{ \frac{\partial^n}{\partial \lambda^n} W[\varphi(x, \lambda), e^+(x, \lambda)] \right\}_{\lambda=\lambda_i} = \left\{ \frac{d^n}{d\lambda^n} a^+(\lambda) \right\}_{\lambda=\lambda_i} = 0$$

for $n = 0, 1, \dots, m_i - 1$, $i = 1, 2, \dots, p$ and

$$\left\{ \frac{\partial^n}{\partial \lambda^n} W[\varphi(x, \lambda), e^-(x, \lambda)] \right\}_{\lambda=\lambda_l} = \left\{ \frac{d^n}{d\lambda^n} a^-(\lambda) \right\}_{\lambda=\lambda_l} = 0$$

for $n = 0, 1, \dots, m_l - 1$, $l = p + 1, p + 2, \dots, q$. Using the last two formulas given above, in a similar way to Theorem 4.1 we get that

Remark 4.3. The formulas

$$(4.11) \quad \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(x, \lambda) \right\}_{\lambda=\lambda_i} = \sum_{v=0}^n \binom{n}{v} c_{n-v}(\lambda_i) \left\{ \frac{\partial^v}{\partial \lambda^v} e^+(x, \lambda) \right\}_{\lambda=\lambda_i},$$

for $n = 0, 1, \dots, m_i - 1$, $i = 1, 2, \dots, p$, and

$$(4.12) \quad \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(x, \lambda) \right\}_{\lambda=\lambda_l} = \sum_{v=0}^n \binom{n}{v} d_{n-v}(\lambda_l) \left\{ \frac{\partial^v}{\partial \lambda^v} e^-(x, \lambda) \right\}_{\lambda=\lambda_l},$$

for $n = 0, 1, \dots, m_l - 1$, $l = p + 1, p + 2, \dots, q$ hold, where the constants $c_0, c_1, \dots, c_n$

and $d_0, d_1, \dots, d_n$ depend on λ_i and λ_l , respectively.

Now we introduce the principal functions corresponding to the spectral singularities as follows:

$$\begin{aligned} & \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(x, \lambda) \right\}_{\lambda=\lambda_i}, \quad n = 0, 1, \dots, m_i - 1, \quad i = 1, 2, \dots, p, \\ & \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(x, \lambda) \right\}_{\lambda=\lambda_l}, \quad n = 0, 1, \dots, m_l - 1, \quad l = p + 1, p + 2, \dots, q \end{aligned}$$

are called the principal functions corresponding to the spectral singularities $\lambda = \lambda_i$ $i = 1, 2, \dots, p$ and $\lambda = \lambda_l$ $l = p + 1, p + 2, \dots, q$ of $L(\lambda)$, respectively. Therefore we arrive at the following

Lemma 4.4 The principal functions for the spectral singularities do not belong to the space $L^2(\mathbf{R}_+, \mathbf{C}_2)$, i.e: $\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\}_{\lambda=\lambda_i} \notin L^2(\mathbf{R}_+, \mathbf{C}_2)$, for $n = 0, 1, \dots, m_i - 1$, $i = 1, 2, \dots, p$, $\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\}_{\lambda=\lambda_l} \notin L^2(\mathbf{R}_+, \mathbf{C}_2)$, for $n = 0, 1, \dots, m_l - 1$, $l = p + 1, p + 2, \dots, q$.

The proof of the lemma is obtained from (2.2), (2.3), (4.11) and (4.12).

Now let us introduce the following Hilbert spaces [2]

$$\begin{aligned} H(\mathbf{R}_+, \mathbf{C}_2, m) : &= \left\{ f(x) : f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \right. \\ & \left. \int_0^\infty (1+x)^{2m} \left\{ |f_1(x)|^2 + |f_2(x)|^2 \right\} dx < \infty \right\}, \end{aligned}$$

$m = 0, 1, \dots$ with norm

$$\|f\|_{H(\mathbf{R}_+, \mathbf{C}_2, m)}^2 = \int_0^\infty (1+x)^{2m} \left\{ |f_1(x)|^2 + |f_2(x)|^2 \right\} dx$$

and

$$H(\mathbf{R}_+, \mathbf{C}_2, -m) : = \left\{ g(x) : g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}, \right. \\ \left. \int_0^\infty (1+x)^{-2m} \left\{ |g_1(x)|^2 + |g_2(x)|^2 \right\} dx < \infty \right\},$$

$m = 0, 1, \dots$ with norm

$$\|g\|_{H(\mathbf{R}_+, \mathbf{C}_2, -m)}^2 = \int_0^\infty (1+x)^{-2m} \left\{ |g_1(x)|^2 + |g_2(x)|^2 \right\} dx.$$

Clearly $H(\mathbf{R}_+, \mathbf{C}_2, 0) = L^2(\mathbf{R}_+, \mathbf{C}_2)$ and

$$H(\mathbf{R}_+, \mathbf{C}_2, m) L^2(\mathbf{R}_+, \mathbf{C}_2,) H(\mathbf{R}_+, \mathbf{C}_2, -m).$$

Therefore we reach to the following

Theorem 4.5.

$$(4.13) \quad \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\}_{\lambda=\lambda_i} \in H(\mathbf{R}_+, \mathbf{C}_2, -(n+1)),$$

for $n = 0, 1, \dots, m_i - 1$, $i = 1, 2, \dots, p$, and

$$(4.14) \quad \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\}_{\lambda=\lambda_l} \in H(\mathbf{R}_+, \mathbf{C}_2, -(n+1))$$

for $n = 0, 1, \dots, m_l - 1$, $l = p+1, p+2, \dots, q$.

Proof. From (2.2), we obtain that

$$(4.15) \quad \left| \left\{ \frac{\partial^n}{\partial \lambda^n} e_1^+(x, \lambda) \right\}_{\lambda=\lambda_i} \right| \leq \int_x^\infty t^n |H_{12}(x, t)| dt$$

and

$$(4.16) \quad \left| \left\{ \frac{\partial^n}{\partial \lambda^n} e_2^+(x, \lambda) \right\}_{\lambda=\lambda_i} \right| \leq x^n + \int_x^\infty t^n |H_{22}(x, t)| dt.$$

for $n = 0, 1, \dots, m_i - 1$, $i = 1, 2, \dots, p$. By the definition of

$H(\mathbf{R}_+, \mathbf{C}_2, -(n+1))$

and using (4.15) and (4.16) we arrive at (4.13). In a similar way, we can show that (4.14) also holds.

Now let us choose n_0 so that

$$n_0 = \max \{m_1, \dots, m_p, m_{p+1}, \dots, m_q\}.$$

Then

$$H(\mathbf{R}_+, \mathbf{C}_2, n_0) \subsetneq L^2(\mathbf{R}_+, \mathbf{C}_2) \subsetneq H(\mathbf{R}_+, \mathbf{C}_2, -n_0).$$

From Theorem 4.5, we finally reach to the following

Conclusion 4.6. The principal functions for the spectral singularities of the operator $L(\lambda)$ belong to the space $H(\mathbf{R}_+, \mathbf{C}_2, -n_0)$, i.e.:

$$\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\}_{\lambda=\lambda_i} \in H(\mathbf{R}_+, \mathbf{C}_2, -n_0),$$

for $n = 0, 1, \dots, m_i - 1$, $i = 1, 2, \dots, p$ and

$$\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\}_{\lambda=\lambda_l} \in H(\mathbf{R}_+, \mathbf{C}_2, -n_0)$$

for $n = 0, 1, \dots, m_l - 1$, $l = p + 1, p + 2, \dots, q$.

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