STABILITY RESULTS FOR THE SOLUTIONS OF CERTAIN NON-AUTONOMOUS DIFFERENTIAL EQUATIONS OF FIFTH - ORDER

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Abstract

The paper is concerned with the stability of solutions of a class of general type fifth order non-autonomous differential equations (1.3) and (1.4). It is shown that under some less restrictive conditions that all solutions of (1.3) and (1.4) tend to zero as $t \rightarrow \infty$. Our results improve that the results obtained by Sadek [9].

Keywords : *Stability, differential equation of fifth order.*

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1. Introduction

In a recent paper Sadek [9] constituted two results on the stability of solutions of non-autonomous differential equations

$$\begin{aligned} x^{(5)} + f(t, \dot{x}, \ddot{x}, \ddot{x})x^{(4)} + \phi(t, \ddot{x}, \ddot{x}) + \psi(t, \ddot{x}) \\ + g(t, \ddot{x}) + e(t)h(x) = 0 \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} x^{(5)} + a(t)f(\dot{x}, \ddot{x}, \ddot{x})x^{(4)} + b(t)\phi(\ddot{x}, \ddot{x}) + c(t)\psi(\ddot{x}) \\ + d(t)g(\dot{x}) + e(t)h(x) = 0, \end{aligned} \quad (1.2)$$

respectively. In this paper, we are interested in fifth order non-autonomous differential equations as follows

$$\begin{aligned} x^{(5)} + f(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}) + \phi(t, \ddot{x}, \ddot{x}) + \psi(t, \dot{x}, \ddot{x}) \\ + g(t, x, \dot{x}) + e(t)h(x) = 0 \end{aligned} \quad (1.3)$$

and

$$x^{(5)} + f(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}) + b(t)\phi(\ddot{x}, \ddot{x}) + c(t)\psi(\dot{x}, \ddot{x}) + d(t)g(x, \dot{x}) + e(t)h(x) = 0. \quad (1.4)$$

It is assumed that all functions that appear in (1.3) and (1.4), that is, $f, \phi, \psi, g, h, b, c, d$ and e are continuous functions for the arguments displayed explicitly, and the dots denote differentiation with respect to t and all solutions considered are assumed real valued.

In the relevant literature till now and after, in a sequence of results, the qualitative properties of solutions of fifth order ordinary differential equations of the form

$$\begin{aligned} x^{(5)} + A_1 x^{(4)} + A_2 \ddot{x} + A_3 \ddot{x} + A_4 \dot{x} + A_5 x \\ = p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}), \end{aligned} \quad (1.5)$$

in which A_1, A_2, A_3, A_4 and A_5 are not necessarily constants, have been the object of much study for several authors. In these works, the authors have investigated the qualitative properties of solutions of certain non-linear differential equations of fifth order. It should be noted that the qualitative properties of solutions of differential equations of the form (1.5) which have been discussed include the following: Stability and instability of solution in the case $p = 0$, boundedness of solutions, convergence of solutions, existence of periodic solutions and boundary value problems. According to the our observations in the relevant literature, some works related to the qualitative properties of solutions of differential equations of the fifth order, especially for stability, boundedness and convergence of solutions, can be summarized as follows: First, in 1971, Burganskaja [4] considered the fifth order nonlinear differential equation

$$x^{(5)} + f(x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}) = 0$$

and, by means of a special construction of Lyapunov functions, the author obtained sufficient conditions for stability in the large of the zero solution of this equation. Then, in 1975, Chukwu [5] investigated the equation

$$x^{(5)} + f_1(x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)})x^{(4)} + b \ddot{x} + f_3(\ddot{x}) + f_4(\dot{x}) + f_5(x) = p(t)$$

in the two cases: (i) $p \equiv 0$, (ii) $p(\neq 0)$ is bounded. For the case (i) the asymptotic stability (in the large) of the trivial solution $x = 0$ is established, and for the case (ii) a boundedness result is deduced with a bound dependent on the initial conditions. Next, in 1976 and 1977, respectively, the same author ([6], [7]) continued his study of fifth order differential equations and considers the equations

$$x^{(5)} + ax^{(4)} + f_2(\ddot{x}) + c\ddot{x} + f_4(\dot{x}) + f_5(x) = p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)})$$

and

$$\begin{aligned} x^{(5)} + f_1(x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)})x^{(4)} + f_2(\ddot{x}, \ddot{x})\ddot{x} + f_3(\dot{x}, \ddot{x}) + f_4(\dot{x}) + f_5(x) \\ = p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}), \end{aligned}$$

and obtained some similar results about stability and boundedness of the solutions of that equations for the cases $p \equiv 0$ and $p \neq 0$, respectively. Later, in 1990, Yu [16] took into consideration the equation of the form

$$\begin{aligned} x^{(5)} + \varphi(x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)})x^{(4)} + b\ddot{x} + h(\ddot{x}) + g(\dot{x}) + f(x) \\ = p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}), \end{aligned}$$

and he deduced a result on the asymptotic stability in the large of the solution $x = 0$ (for the case $p \equiv 0$) and a boundedness result for the case $p \neq 0$ related to the all solutions of the equation considered. Then, in 1995, Abou-El-Ela and Sadek [1] proved a theorem on the stability of the solutions of a nonlinear differential equation as follows:

$$x^{(5)} + f_1(\ddot{x})x^{(4)} + f_2(\ddot{x}) + f_3(\ddot{x}) + f_4(\dot{x}) + f_5(x) = p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}).$$

Afterward, in 1999 and 2000, Abou-El-Ela&Sadek ([2], [3]) and, in 2002, Sadek [8] established sufficient conditions for the uniform boundedness, and the tending to zero, of all solutions of the equations

$$\begin{aligned} x^{(5)} + a(t)f_1(\ddot{x}, \ddot{x})x^{(4)} + b(t)f_2(\ddot{x}, \ddot{x}) + c(t)f_3(\ddot{x}) + d(t)f_4(\dot{x}) + e(t)f_5(x) \\ = p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}) \end{aligned}$$

and

$$x^{(5)} + f(t, \dot{x}, \ddot{x}, \ddot{x})x^{(4)} + \phi(t, \ddot{x}, \ddot{x}) + \psi(t, \ddot{x}) + g(t, \dot{x}) + e(t)h(x)$$

$$= p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}).$$

After that, in 1995, 1996, 2002 respectively, the author in ([10], [11], [12]) discussed the stability and boundedness of the solutions of the differential equations as follows:

$$\begin{aligned} x^{(5)} + \varphi(x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)})x^{(4)} + \psi(\ddot{x}, \ddot{x}) + h(\ddot{x}) + g(\dot{x}) + f(x) \\ = p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}), \end{aligned}$$

and

$$\begin{aligned} x^{(5)} + \varphi(x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)})x^{(4)} + b \ddot{x} + h(\ddot{x}) + g(x, \dot{x}) + f(x) \\ = p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}), \end{aligned}$$

and

$$\begin{aligned} x^{(5)} + \varphi(x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)})x^{(4)} + \psi(x, \dot{x}, \ddot{x}, \ddot{x}) \\ + h(x, \dot{x}, \ddot{x}) + g(x, \dot{x}) + f(x) \\ = p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}). \end{aligned}$$

Besides these works, recently, the same author ([13], [14], [15]) obtained sufficient conditions which ensure that all solutions of the equations of the form

$$\begin{aligned} x^{(5)} + f_1(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)})x^{(4)} + \\ b(t)f_2(\ddot{x}, \ddot{x}) + c(t)f_3(\dot{x}, \ddot{x}) + d(t)f_4(\dot{x}) + e(t)f_5(x) \\ = p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}), \end{aligned}$$

$$\begin{aligned} x^{(5)} + f(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)})x^{(4)} + \\ \phi(t, \ddot{x}, \ddot{x}) + \psi(t, \dot{x}, \ddot{x}) + g(t, x, \dot{x}) + e(t)h(x) \\ = p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}) \end{aligned}$$

and

$$\begin{aligned} x^{(5)} + f_1(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)})x^{(4)} + b(t)f_2(x, \dot{x}, \ddot{x}, \ddot{x}) \\ + c(t)f_3(x, \dot{x}, \ddot{x}) + d(t)f_4(x, \dot{x}) + e(t)f_5(x) = p(t, x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}) \end{aligned}$$

are uniformly bounded and tend to zero as $t \rightarrow \infty$. The motivation for the present work has been inspired basically by the paper of Sadek [9] and the references cited in that paper.

In what follows it will be convenient to use the equivalent differential systems:

$$\begin{aligned}
 \dot{x} &= y \\
 \dot{y} &= z \\
 \dot{z} &= w \\
 \dot{w} &= u \\
 \dot{u} &= -f(t, x, y, z, w, u) - \phi(t, z, w) - \psi(t, y, z) \\
 &\quad -g(t, x, y) - e(t)h(x),
 \end{aligned}
 \tag{1.6}$$

and

$$\begin{aligned}
 \dot{x} &= y \\
 \dot{y} &= z \\
 \dot{z} &= w \\
 \dot{w} &= u \\
 \dot{u} &= -f(t, x, y, z, w, u) - b(t)\phi(z, w) - c(t)\psi(y, z) \\
 &\quad -d(t)g(x, y) - e(t)h(x),
 \end{aligned}
 \tag{1.7}$$

which were obtained from the equations (1.3) and (1.4) by setting $\dot{x} = y, \ddot{x} = z, \ddot{\ddot{x}} = w$ and $x^{(4)} = u$, respectively.

2. ASSUMPTIONS AND MAIN RESULTS

The following assumptions will be accepted on the functions that appeared in (1.3).

Assumptions:

(1) $h(x)$ is a continuously differentiable function in $\mathbb{R}^1$, and $e(t)$ is a continuously differentiable function in $\mathbb{R}^+ = [0, \infty)$.

(2) The function $g(t, x, y)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^2$, and for the function $g(t, x, y)$ there exist non-negative functions $d(t)$, $g_0(x, y)$ and $g_1(x, y)$ which satisfy the inequalities

$$d(t)g_0(x, y) \leq g(t, x, y) \leq d(t)g_1(x, y)$$

for all $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^2$. The function $d(t)$ is continuously differentiable for all $t \in \mathbb{R}^+$. Let

$$\tilde{g}(x, y) \equiv \frac{1}{2} [g_0(x, y) + g_1(x, y)],$$

and $\tilde{g}(x, y)$, $\frac{\partial}{\partial x}\tilde{g}(x, y)$ and $\frac{\partial}{\partial y}\tilde{g}(x, y)$ are continuous for all $(x, y) \in \mathbb{R}^2$.

(3) The function $\psi(t, y, z)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^2$. For the function $\psi(t, y, z)$ there exist non-negative functions $c(t)$, $\psi_0(y, z)$ and $\psi_1(y, z)$ which satisfy the inequalities

$$c(t)\psi_0(y, z) \leq \psi(t, y, z) \leq c(t)\psi_1(y, z)$$

for all $(t, y, z) \in \mathbb{R}^+ \times \mathbb{R}^2$. The function $c(t)$ is continuously differentiable for all $t \in \mathbb{R}^+$. Let

$$\tilde{\psi}(y, z) \equiv \frac{1}{2} [\psi_0(y, z) + \psi_1(y, z)],$$

$\tilde{\psi}(y, z)$ and $\frac{\partial}{\partial y}\tilde{\psi}(y, z)$ are continuous for all $(y, z) \in \mathbb{R}^2$.

(4) The function $\phi(t, z, w)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^2$. For the function $\phi(t, z, w)$ there exist non-negative functions $b(t)$, $\phi_0(z, w)$ and $\phi_1(z, w)$ which satisfy the inequalities

$$b(t)\phi_0(z, w) \leq \phi(t, z, w) \leq b(t)\phi_1(z, w)$$

for all $(t, z, w) \in \mathbb{R}^+ \times \mathbb{R}^2$. The function $b(t)$ is continuously differentiable for all $t \in \mathbb{R}^+$. Let

$$\tilde{\phi}(z, w) \equiv \frac{1}{2} [\phi_0(z, w) + \phi_1(z, w)],$$

and $\tilde{\phi}(z, w)$ and $\frac{\partial}{\partial z}\tilde{\phi}(z, w)$ are continuous for all $(z, w) \in \mathbb{R}^2$.

Remark 1. The assumptions just established above are less restrictive than those constituted in Sadek [9]. Because the result discussed in [9] can be proved here without the assumptions

$$a(t)f_0(y, z, w) \leq f(t, y, z, w) \leq a(t)f_1(y, z, w)$$

and

$$\tilde{f}(y, z, w) \equiv \frac{1}{2} [f_0(y, z, w) + f_1(y, z, w)]$$

set up there.

The first main result is the following theorem.

Theorem 1. Let all the assumptions (1)-(4) be satisfied, and in addition, we also assume that the existence of arbitrary positive constants $\alpha_1, \dots, \alpha_5$ and of sufficiently small positive constants $\varepsilon, \varepsilon_0, \varepsilon_1, \dots, \varepsilon_5$ such that the following conditions are satisfied:

(i) $B, C, D, E, b_0, c_0, d_0$ and e_0 are some constants satisfying the inequalities $B \geq b(t) \geq b_0 \geq 1, C \geq c(t) \geq c_0 \geq 1, D \geq d(t) \geq d_0 \geq 1, E \geq e(t) \geq e_0 \geq 1$ for all $t \in \mathbb{R}^+$.

(ii)

$$(2.1) \quad \alpha_1 > 0, \alpha_1\alpha_2 - \alpha_3 > 0, (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0,$$

$$(2.2) \quad \delta_0 := (\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0, \alpha_5 > 0$$

$$\Delta_1 := \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{(\alpha_1\alpha_4 - \alpha_5)} - \left[\alpha_1 d(t) \frac{\partial}{\partial y} \tilde{g}(x, y) - \alpha_5 \right]$$

$$(2.3) \quad \geq 2\varepsilon\alpha_2$$

for all $t \in \mathbb{R}^+$ and for all x, y ;

$$(2.4) \quad \Delta_2 := \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{(\alpha_1\alpha_4 - \alpha_5) \cdot \gamma \cdot d(t)}{\alpha_4(\alpha_1\alpha_2 - \alpha_5)} - \frac{\varepsilon}{\alpha_1} > 0$$

for all $t \in \mathbb{R}^+$ and for all x, y , where

$$(2.5) \quad \gamma := \begin{cases} \frac{\tilde{g}(x, y)}{y}, y \neq 0 \\ \frac{\partial}{\partial y} \tilde{g}(x, 0), y = 0. \end{cases}$$

(iii) $\varepsilon_0 \leq \frac{f(t,x,y,z,w,u)}{u} - \alpha_1 \leq \varepsilon_1$ for all $t \in \mathbb{R}^+$ and for all $x, y, z, w, u \neq 0$.

(iv) $\phi(t, z, 0) = 0, 0 \leq \frac{\phi(t, z, w)}{w} - \alpha_2 \leq \varepsilon_2$ for all $t \in \mathbb{R}^+$ and for all $z, w \neq 0$,
and $\frac{\partial}{\partial z} \tilde{\phi}(z, w) \leq 0$ for all z, w .

(v) $\psi(t, y, 0) = 0, 0 \leq \frac{\psi(t, y, z)}{z} - \alpha_3 \leq \varepsilon_3$ for all $t \in \mathbb{R}^+$ and for all $y, z \neq 0$,
and $\frac{\partial}{\partial y} \tilde{\psi}(y, z) \leq 0$ for all y, z .

(vi)

$$(2.6) \quad \tilde{g}(x, 0) = 0, \frac{\tilde{g}(t, x, y)}{y} \geq \frac{E\alpha_4}{d_0} \text{ for all } t \in \mathbb{R}^+ \text{ and for all } x, y \neq 0,$$

$$(2.7) \quad \frac{\partial}{\partial y} \tilde{g}(x, y) - \frac{\tilde{g}(x, y)}{y} \leq \frac{\alpha_5 \delta_0}{D\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)} \text{ for all } x, y \neq 0,$$

$$(2.8) \quad \left| \alpha_4 - \frac{\partial}{\partial y} \tilde{g}(x, y) \right| \leq \varepsilon_4 \text{ for all } x, y,$$

$$(2.9) \quad \left[\frac{\partial}{\partial x} \tilde{g}(x, y) \right]^2 \leq \min \left[\frac{\varepsilon^2 \alpha_4^2 (\alpha_1\alpha_2 - \alpha_3)}{16D(\alpha_1\alpha_4 - \alpha_5)}, \frac{\varepsilon^2 \alpha_2 \alpha_4^2 (\alpha_1\alpha_2 - \alpha_3)}{16D\alpha_1^2 (\alpha_1\alpha_4 - \alpha_5)} \right] \text{ for all } x, y,$$

and

$$(2.10) \quad y \int_0^y \frac{\partial}{\partial x} \tilde{g}(x, \eta) d\eta \leq -\frac{\varepsilon\alpha_4}{2} y^2 \text{ for all } x, y \neq 0.$$

(vii)

$$h(0) = 0, h(x) \operatorname{sgn} x > 0 \ (x \neq 0), H(x) = \int_0^x h(\xi) d\xi \rightarrow \infty \text{ as } |x| \rightarrow \infty,$$

and

$$0 \leq \alpha_5 - h'(x) \leq \varepsilon_5 \text{ for all } x.$$

(viii)

$$\int_0^\infty \beta_0(t) dt < \infty, \quad e'(t) \rightarrow 0 \text{ as } t \rightarrow \infty,$$

where

$$\beta_0(t) := b'_+(t) + c'_+(t) + |d'(t)| + |e'(t)|,$$

$$b'_+(t) = \max \{b'(t), 0\}, c'_+(t) := \max \{c'(t), 0\}.$$

(ix)

$$|B(\phi_1 - \phi_0) + C(\psi_1 - \psi_0) + D(g_1 - g_0)| \leq \Delta(y^2 + z^2 + w^2 + u^2)^{\frac{1}{2}}.$$

Then every solution of (1.3) satisfy

$$x(t) \rightarrow 0, \dot{x}(t) \rightarrow 0, \ddot{x}(t) \rightarrow 0, \ddot{\ddot{x}}(t) \rightarrow 0, x^{(4)}(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Remark 2. It should be also expressed that the theorem just stated above includes the results in [9] and also improves the first result obtained in [9] because the result stated here can be proved without the restriction $A \geq a(t) \geq a_0 \geq 1$ in [9], and the assumption (ix) of Theorem 1, that is,

$$|B(\phi_1 - \phi_0) + C(\psi_1 - \psi_0) + D(g_1 - g_0)| \leq \Delta(y^2 + z^2 + w^2 + u^2)^{\frac{1}{2}}$$

is less restrictive than the assumption

$$|A(f_1 - f_0) + B(\phi_1 - \phi_0) + C(\psi_1 - \psi_0) + D(g_1 - g_0)| \leq \Delta(y^2 + z^2 + w^2 + u^2)^{\frac{1}{2}}$$

constituted in Sadek [9; Theorem 2].

Now, before stating our second main result we impose some additional acceptations on the coefficient functions that appeared in (1.7). Namely, it is assumed that the functions $b, \dots, e$ are positive definite and differentiable in $\mathbb{R}^+ = [0, \infty)$ and the derivatives $\frac{\partial}{\partial z}\phi(z, w)$, $\frac{\partial}{\partial y}\psi(y, z)$, $\frac{\partial}{\partial x}g(x, y)$, $\frac{\partial}{\partial y}g(x, y)$ and $h'(x)$ exist and are continuous for all x, y, z and w . It can be taken the function $g(x, y)$ instead of the functions $g_0(x, y)$ and $g_1(x, y)$; the function $\psi(y, z)$ instead of the functions $\psi_0(y, z)$ and $\psi_1(y, z)$; the function $\phi(z, w)$ instead of the functions $\phi_0(z, w)$ and $\phi_1(z, w)$ in the assumptions (2)-(4). It should also be noted that in this case the functions $\tilde{g}(x, y)$, $\psi(y, z)$ and $\phi(z, w)$ coincide with the functions $g(x, y)$, $\psi(y, z)$ and $\phi(z, w)$ appeared in (1.7), respectively.

Next, the second main result concerning to the equation (1.4) is the following theorem.

Theorem 2. Let all the basic assumptions on the functions $b, c, d, e, f, \phi, \psi, g, h$ and p that appeared in (1.7) be satisfied, and in addition, it is assumed that the assumptions (i), (iii), (vii) and (viii) of Theorem 1 and following conditions are fulfilled (for some arbitrary positive constants $\alpha_1, \dots, \alpha_5$ and sufficiently small positive constants $\varepsilon, \varepsilon_0$ and ε_1) :

(i)'

$$\alpha_1 > 0, \alpha_1\alpha_2 - \alpha_3 > 0, (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0,$$

$$\delta_0 := (\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0, \alpha_5 > 0;$$

$$\Delta_1 := \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \left[\alpha_1 d(t) \frac{\partial}{\partial y} g(x, y) - \alpha_5 \right] \geq 2\varepsilon\alpha_2$$

for all $t \in \mathbb{R}^+$ and for all x, y ;

$$\Delta_2 := \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{(\alpha_1\alpha_4 - \alpha_5) \cdot \gamma \cdot d(t)}{\alpha_4(\alpha_1\alpha_2 - \alpha_5)} - \frac{\varepsilon}{\alpha_1} > 0$$

for all $t \in \mathbb{R}^+$ and for all x, y ; where

$$\gamma := \begin{cases} \frac{g(x, y)}{y}, y \neq 0 \\ \frac{\partial}{\partial y} g(x, 0), y = 0. \end{cases}$$

(ii)' $\phi(z, 0) = 0, 0 \leq \frac{\phi(z, w)}{w} - \alpha_2 \leq \varepsilon_2$ for all z and $w \neq 0$ and $\frac{\partial}{\partial z} \phi(z, w) \leq 0$ for all z, w .

(iii)' $\psi(y, 0) = 0, 0 \leq \frac{\psi(y, z)}{z} - \alpha_3 \leq \varepsilon_3$ for all $y, z \neq 0$, and $\frac{\partial}{\partial y} \psi(y, z) \leq 0$ for all y, z .

(iv)'

$$g(x, 0) = 0, \frac{g(x, y)}{y} \geq \frac{E\alpha_4}{d_0} \text{ for all } x, y \neq 0,$$

$$\left| \alpha_4 - \frac{\partial}{\partial y} g(x, y) \right| \leq \varepsilon_4 \text{ for all } x, y,$$

$$\frac{\partial}{\partial y} g(x, y) - \frac{g(x, y)}{y} \leq \frac{\alpha_5 \delta_0}{D\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)} \text{ for all } x, y \neq 0,$$

$$\left[\frac{\partial}{\partial x} g(x, y) \right]^2 \leq \min \left[\frac{\varepsilon^2 \alpha_4^2 (\alpha_1 \alpha_2 - \alpha_3)}{16D(\alpha_1 \alpha_4 - \alpha_5)}, \frac{\Delta_1 \varepsilon \alpha_4^2 (\alpha_1 \alpha_2 - \alpha_3)}{16D\alpha_1^2 (\alpha_1 \alpha_4 - \alpha_5)} \right]$$

and

$$y \int_0^y \frac{\partial}{\partial x} g(x, \eta) d\eta \leq -\frac{\varepsilon \alpha_4}{2} y^2 \quad \text{for all } x, y.$$

Then all solutions of (1.4) satisfy

$$x(t) \rightarrow 0, \dot{x}(t) \rightarrow 0, \ddot{x}(t) \rightarrow 0, \dddot{x}(t) \rightarrow 0, x^{(4)}(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Remark 3. It should be clarified Theorem 2 improves the second result of Sadek [9; Theorem 2] and also contains less restrictive conditions than those established for Theorem 2 by Sadek [9].

Remark 4. Since the proof of Theorem 2 just stated above is similar to that of our first theorem, Theorem 1, we will not give the proof here.

3. THE LYAPUNOV FUNCTION $V_0(t, x, y, z, w, u)$

Our main tool in the proof of Theorem 1 is the continuous differentiable function $V_0 = V_0(t, x, y, z, w, u)$ defined by:

$$\begin{aligned}
2V_0 = & u^2 + 2\alpha_1 uw + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} uz + 2\delta uy + 2b(t) \int_0^w \tilde{\phi}(z, \rho) d\rho \\
& + \left[\alpha_1^2 - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \right] w^2 + 2 \left[\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta \right] wz + 2\alpha_1 \delta wy \\
& + 2d(t)w\tilde{g}(x, y) + 2e(t)wh(x) + 2\alpha_1 c(t) \int_0^z \tilde{\psi}(y, \zeta) d\zeta \\
& + \left[\frac{\alpha_2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_4 - \alpha_1\delta \right] z^2 + 2\delta\alpha_2 yz + 2\alpha_1 d(t)z\tilde{g}(x, y) \\
& - 2\alpha_5 zy + 2\alpha_1 e(t)zh(x) + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} d(t) \int_0^y \tilde{g}(x, \eta) d\eta \\
& + (\delta\alpha_3 - \alpha_1\alpha_5)y^2 + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} e(t)yh(x) + 2\delta e(t) \int_0^x h(\xi) d\xi + k,
\end{aligned}
\tag{3.1}$$

where δ is a positive constant satisfying

$$\delta := \frac{\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} + \varepsilon,
\tag{3.2}$$

and k is a positive constant to be determined later in the proof.

Now, we shall some standard results needed in the proof our results.

Lemma 1: Assume that the assumptions (i) - (vii) of the Theorem 1 are satisfied. Then, there exist positive constants D_7 and D_8 such that

$$D_7 \left[H(x) + y^2 + z^2 + w^2 + u^2 + k \right] \leq V_0 \leq D_8 \left[H(x) + y^2 + z^2 + w^2 + u^2 + k \right].
\tag{3.3}$$

Proof. Since the arguments required here are essentially similar as those used for the proof of Theorem 1 in Sadek [9], therefore we omit the detailed proof of the lemma.

Lemma 2. Assume that all the conditions of the Theorem 1 hold. Then there exist positive constants

D_i ($i = 11, 12, 13$) such that

$$\begin{aligned} \dot{V}_0 \leq & -D_{13} [y^2 + z^2 + w^2 + u^2] + 2D_{12}(y^2 + z^2 + w^2 + u^2)^{\frac{1}{2}} [p_1(t) + p_2(t)] \\ & + 2D_{12}p_2(t) [H(x) + y^2 + z^2 + w^2 + u^2] + D_{11}\beta_0 V_0. \end{aligned} \quad (3.4)$$

Proof. On differentiating (3.1) and using (1.6), we obtain that (for $y, z, w \neq 0$)

$$\begin{aligned} \dot{V}_0 = & -u^2 \left[\frac{f(t, x, y, z, w, u)}{u} - \alpha_1 \right] - w^2 \left[\alpha_1 \frac{\phi(t, z, w)}{w} - \left\{ \alpha_3 + \frac{\alpha_1 \alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \delta \right\} \right] \\ & - z^2 \left[\frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \frac{\psi(t, y, z)}{z} - \left\{ \delta \alpha_2 + \alpha_1 d(t) \frac{\partial}{\partial y} \tilde{g}(x, y) - \alpha_5 \right\} \right] \\ & - y^2 \left[\delta \frac{g(t, x, y)}{y} - \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} e(t) h'(x) \right] - \alpha_1 u w \left[\frac{f(t, x, y, z, w, u)}{u} - \alpha_1 \right] \\ & - u z \left[\frac{\psi(t, y, z)}{z} - \alpha_3 \right] - w z [\alpha_4 - d(t) \frac{\partial}{\partial y} \tilde{g}(x, y)] + b(t) w \int_0^w \frac{\partial}{\partial z} \tilde{\phi}(z, \rho) d\rho \\ & - \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} u z \left[\frac{f(t, x, y, z, w, u)}{u} - \alpha_1 \right] - \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} w z \left[\frac{\phi(t, z, w)}{w} - \alpha_2 \right] \\ & - \delta u y \left[\frac{f(t, x, y, z, w, u)}{u} - \alpha_1 \right] - w y e(t) [\alpha_5 - h'(x)] - \delta w y \left[\frac{\phi(t, z, w)}{w} - \alpha_2 \right] \\ & - \alpha_1 e(t) z y [\alpha_5 - h'(x)] - \delta y z \left[\frac{\psi(t, y, z)}{z} - \alpha_3 \right] - \alpha_5 w y [1 - e(t)] \\ & - \alpha_1 \alpha_5 z y [1 - e(t)] + d(t) w y \frac{\partial}{\partial x} \tilde{g}(x, y) + \alpha_1 c(t) z \int_0^z \frac{\partial}{\partial y} \tilde{\psi}(y, \zeta) d\zeta \\ & + \alpha_1 d(t) z y \frac{\partial}{\partial x} \tilde{g}(x, y) + \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} d(t) y \int_0^y \frac{\partial}{\partial x} \tilde{g}(x, \eta) d\eta \\ & + \frac{1}{2} [c(t)(\psi_1 - \psi_0)] \alpha_1 w + \frac{1}{2} [d(t)(g_1 - g_0)] \\ & \cdot \left[u + \alpha_1 w + \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z \right] + \frac{1}{2} [b(t)(\phi_1 - \phi_0)] u + \frac{\partial V_0}{\partial t}, \end{aligned} \quad (3.5)$$

where

$$\begin{aligned}
 (3.6) \quad \frac{\partial V_0}{\partial t} = & e'(t) \left[wh(x) + \alpha_1 zh(x) + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} yh(x) + \delta \int_0^x h(\xi) d\xi \right] \\
 & + d'(t) \left[w\tilde{g}(x, y) + \alpha_1 z\tilde{g}(x, y) + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \int_0^y \tilde{g}(x, \eta) d\eta \right] \\
 & + \alpha_1 c'(t) \int_0^z \tilde{\psi}(y, \zeta) d\zeta + b'(t) \int_0^w \tilde{\phi}(z, \rho) d\rho.
 \end{aligned}$$

The method of the proof in [14] is applicable here. And therefore if ones follows the lines indicated as in [14], then it can be easily obtained, subject to the conditions of the Theorem 1, from (3.5) and (3.6) that

$$\begin{aligned}
 \dot{V}_0 \leq & -\varepsilon_0 u^2 - \left(\frac{15\varepsilon}{16} \right) w^2 - \left(\frac{15\varepsilon\alpha_2}{16} \right) z^2 - (\varepsilon\alpha_4 E) y^2 \\
 & + D_9(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5)(y^2 + z^2 + w^2 + u^2) \\
 & + D_{12} [B(\phi_1 - \phi_0) + C(\psi_1 - \psi_0) + D(g_1 - g_0)] \sqrt{y^2 + z^2 + w^2 + u^2} \\
 & + D_{10} [b'_+(t) + c'_+(t) + |d'(t)| + |e'(t)|] [H(x) + y^2 + z^2 + w^2 + u^2] \\
 \leq & -\frac{1}{2} \min \left\{ \varepsilon_0, \frac{15\varepsilon}{16}, \frac{15\varepsilon\alpha_2}{16}, \varepsilon\alpha_4 E \right\} (y^2 + z^2 + w^2 + u^2) \\
 & + D_{12} [B(\phi_1 - \phi_0) + C(\psi_1 - \psi_0) + D(g_1 - g_0)] \sqrt{y^2 + z^2 + w^2 + u^2} \\
 & + D_{11}\beta_0 V_0,
 \end{aligned}$$

provided that

$$(3.7) \quad D_9(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) \leq \frac{1}{2} \min \left\{ \varepsilon_0, \frac{15\varepsilon}{16}, \frac{15\varepsilon\alpha_2}{16}, \varepsilon\alpha_4 E \right\},$$

where D_9 , D_{10} , D_{11} and D_{12} are positive constants and $D_{11} = D_{10} D_7^{-1}$, and it is also assumed that D_9 and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_5$ are so small that (3.7) holds.

The case $y, z, w = 0$ is clear. The rest of the proof will then follow as in [14] and hence it is omitted.

4. Completion of the proof of theorem 1

Taking into account the results of Lemma 1, Lemma 2 just proved above and Theorem 10.2, and Theorem 14.2 in Yoshizawa [17], if one follows the lines indicated in [14], except for some minor modifications, it can be easily performed the completion of the proof. Hence, we omit the details of the proof.

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