SPN-COMPACTNESS IN L -TOPOLOGICAL SPACES *

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Abstract

In this paper, the notions of SPN-compactness, countable SPN-compactness and the SPN-Lindelöf property are introduced in L -topological spaces by means of strongly preclosed L -sets. In an L -space, an L -set having the SPN-Lindelöf property is SPN-compact if and only if it is countably SPN-compact. (Countable) SPN-compactness implies (countable) N -compactness, the SPN-Lindelöf property implies the N -Lindelöf property, but each inverse is not true. Every L -set with finite support is SPN-compact. The intersection of an (a countable) SPN-compact L -set and a strongly preclosed L -set is (countably) SPN-compact. The strong preirresolute image of an (a countable) SPN-compact L -set is (countably) SPN-compact. Moreover SPN-compactness can be characterized by nets.

Keywords: L -topological space; strongly preopen L -set; strongly preclosed L -set; SPN-compactness; countable SPN-compactness, the SPN-Lindelöf property

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1. Introduction and preliminaries

The concept of compactness is one of most important concepts in general topology. The concept of compactness in $[0, 1]$ -fuzzy set theory was first introduced by C.L. Chang in terms of open cover [1]. J.A. Goguen was the first to point out a deficiency in Chang's compactness theory by showing that the Tychonoff Theorem is false [5]. Since Chang's compactness has some limitations, subsequently all kinds of notion of fuzzy compactness were presented [3, 8, 10, 12, 13, 14, 20, 21].

In this paper, based on the lines of N-compactness we shall introduce a new kind of compactness in L -topological spaces by means of strongly preclosed L -sets, which is called SPN-compactness.

Throughout this paper $(L, \vee, \wedge, ')$ is a completely distributive de Morgan algebra, X is a nonempty set. L^X is the set of all L -fuzzy sets on X . The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$.

An element a in L is called prime if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. An element a in L is called co-prime if a' is a prime element [4]. The set of nonunit prime elements in L is denoted by $P(L)$. The set of nonzero co-prime elements in L is denoted by $M(L)$. The set of nonzero co-prime elements in L^X is denoted by $M(L^X)$. Each member in $M(L^X)$ is also called a point.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [2]. In a completely distributive de Morgan algebra L , each member b is a sup of $\{a \in L \mid a \prec b\}$. In the sense of [11, 19], $\{a \in L \mid a \prec b\}$ is the greatest minimal family of b , denoted by $\beta(b)$. Moreover for $b \in L$, define $\beta^*(b) = \beta(b) \cap M(L)$, $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For an L -set $A \in L^X$, $\beta(A)$ denotes the greatest minimal family of A and $\beta^*(A) = \beta(A) \cap M(L^X)$.

An L -topological space (or L -space for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\underline{0}, \underline{1}$ and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L -topology on X . Each member of $\mathcal{T}$ is called an open L -set and its quasi-complement is called a closed L -set.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamily of Φ .

Definition 1.1 ([11, 19]). A net $S = \{S(n) : n \in D\}$ is called an α -net ($\alpha \in M(L)$) if for each $\lambda \in \beta^*(\alpha)$, there is $n_0 \in D$ such that $V(S(n)) \geq \lambda$ whenever $n \geq n_0$, where $V(S(n))$ is the height of molecule $S(n)$. A net $\{S(n)\}$ is said to be constant α -net if the height of each $S(n)$ is a constant value α .

In [18], the concepts of preopen and preclosed sets were introduced in $[0,1]$ -fuzzy set theory. They can easily be extended to L -sets as follows:

Definition 1.2. Let $(X, \mathcal{T})$ be an L -space, $G \in L^X$. Then G is called preopen if $G \leq \text{int}(\text{cl}(G))$; G is called preclosed if G' is preopen.

Definition 1.3. Let $(X, \mathcal{T})$ be an L -space, $G \in L^X$. Then we define:

- (1) $\text{int}_p(G) = \bigvee \{B \in L^X : B \leq G \text{ and } B \text{ is preopen}\};$
- (2) $\text{cl}_p(G) = \bigwedge \{B \in L^X : B \geq G \text{ and } B \text{ is preclosed}\}.$

In [6] and [7], the concepts of strongly preopen, strongly preclosed sets and SP-irresolute mapping were introduced in $[0,1]$ -fuzzy set theory by Biljana Krateska. They can easily be extended to L -sets as follows:

Definition 1.4. Let $(X, \mathcal{T})$ be an L -space, $G \in L^X$. Then G is called strongly preopen if $G \leq \text{int}(\text{cl}_p(G))$; G is called strongly preclosed if G' is strongly preopen.

$\mathbf{SPO}(X)$ and $\mathbf{SPC}(X)$ will always denote the family of all strongly preopen L -sets and the family of all strongly preclosed L -sets in $(X, \mathcal{T})$ respectively.

Lemma 1.5 ([15]). Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L -spaces and $f : X \rightarrow Y$ be a mapping. If $f_L^\rightarrow : L^X \rightarrow L^Y$ is the extension of f , then for any $\mathcal{P} \subset L^Y$, we have that

$$\bigvee_{y \in Y} \left(f_L^\rightarrow(G)(y) \wedge \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} f_L^-(B)(x) \right).$$

Definition 1.6 ([7]). Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L -spaces and $f : X \rightarrow Y$ be a mapping. f is called *SP-irresolute* if $f_L^-(B)$ is strongly preopen in $(X, \mathcal{T}_1)$ for each strongly preopen L -set B in $(Y, \mathcal{T}_2)$.

Definition 1.7 ([17]). Let $(X, \mathcal{T})$ be an L -space, $G \in L^X$ and $\alpha \in M(L)$. $\Phi \subset L^X$ is called an α -remote family of G if for any $x \in X$, $G(x) \wedge \bigwedge_{B \in \Phi} B(x) \not\geq \alpha$. Φ is called a strong α -remote family of G if $\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \Phi} B(x) \right) \not\geq \alpha$. If Φ is an (a strong) α -remote family of G and $\Phi \subset \mathcal{T}'$, then it is called a closed (strong) α -remote family of G .

Remark 1.8. It is easy to see that a closed α -remote family is an α - R -neighborhood family in the sense of [19, 21] and a closed strong α -remote family is an α^- - R -neighborhood family in the sense of [19, 21].

Wang [19] introduced the notion of N -compactness in $[0, 1]$ -topological spaces by means of α -nets. Zhao generalized the notion of N -compactness to L -fuzzy set theory in terms of closed α -remote family and α^- -remote family as follows:

Definition 1.9 ([21]). Let $(X, \mathcal{T})$ be an L -space, $\alpha \in M(L)$. $G \in L^X$ is called *N -compact* if for every $\alpha \in M(L)$, every closed α -remote family of G has a finite subfamily which is a closed α^- -remote family of G .

Analogously we can define countable N -compactness and the N -Lindelöf property (see [14]).

2. SPN-compactness of L -subsets

Definition 2.1. Let $(X, \mathcal{T})$ be an L -space, $x_\lambda \in M(L^X)$ and $G \in \mathbf{SPC}(\mathbf{X})$. G is called an *SP-closed remote set* of x_λ if $x_\lambda \not\leq G$. The set of all SP-closed remote set of x_λ is denoted by $\eta_{sp}(x_\lambda)$.

Definition 2.2. Let $(X, \mathcal{T})$ be an L -space and $G \in L^X$, $\alpha \in M(L)$ and $\Phi \subset L^X$. If Φ is an (a strong) α -remote family of G and $\Phi \subset \mathbf{SPC}(\mathbf{X})$, then it is called an *SP-closed (strong) α -remote family* of G .

Definition 2.3. Let $(X, \mathcal{T})$ be an L -space and $G \in L^X$. G is called (countably) SPN-compact if $\forall \alpha \in M(L)$, every (countable) SP-closed α -remote family Φ of G has a finite subfamily Ψ which is a strong α -remote family of G . Specifically, when $G = \underline{1}$ is (countably) SPN-compact, we call $(X, \mathcal{T})$ an (a countable) SPN-compact space.

Definition 2.4. Let $(X, \mathcal{T})$ be an L -space and $G \in L^X$. G is said to have the SPN-Lindelöf property (or be an SPN-Lindelöf L -set) if $\forall \alpha \in M(L)$, every SP-closed α -remote family Φ of G has a countable subfamily Ψ which is a strong α -remote family of G . Specifically, when $G = \underline{1}$ has the SPN-Lindelöf property, we call that $(X, \mathcal{T})$ has the SPN-Lindelöf property.

It is easy to see that the next two theorems are true.

Theorem 2.5. In an L -space, every (countably) SPN-compact L -set is (countably) N -compact and every SPN-Lindelöf L -set is an N -Lindelöf L -set.

Theorem 2.6. An L -set having the SPN-Lindelöf property is SPN-compact if and only if it is countably SPN-compact.

In general, N -compact doesn't imply SPN-compact. This can be seen from the following example.

Example 2.7. Let $(X, \mathcal{T})$ be an L -space, where $L = [0, 1]$, $X = (0, 1]$, $\mathcal{T} = \{\underline{0}, \underline{1}, A, B, A \vee B, A \wedge B\}$, A and B are defined by

$$A(x) = \begin{cases} 0.5 + x, & x \in (0, 0.5], \\ 0.1, & x \in (0.5, 1]. \end{cases}$$

$$B(x) = \begin{cases} 0.25, & x \in (0, 0.5], \\ 0.3, & x \in (0.5, 1]. \end{cases}$$

Then every closed L -set is N -compact from Theorem 4.2 and Theorem 4.10 in [9]. Take $\alpha = 0.75$ and $\Phi = \{([0.75] \wedge \chi_{(0,t)}) \vee [0.7] \mid t \in (0, 0.5)\}$.

By simple computation we obtain that $\text{int}(A' \vee B') = B$ and $\text{cl}(B) = B'$. Therefore $\text{cl}(\text{int}(A' \vee B')) = B'$. Hence $\forall P \in \Phi$, we have that

$$\text{cl}(\text{int}(A' \vee B')) = B' \leq P \leq A' \vee B'.$$

So Φ is a family of strongly semiclosed L -sets. Hence it is a family of strongly preclosed L -sets, and it is an SP -closed α -remote family of $\underline{1}$. But no finite subfamily of Φ is an SP -closed α -remote family of $\underline{1}$. Therefore $(X, \mathcal{T})$ is not SPN -compact.

Theorem 2.8. Let $(X, \mathcal{T})$ be an L -space, $G \in L^X$. Then G is (countably) SPN -compact if and only if the following two conditions hold.

- (1) For every $\alpha \in M(L)$, every (countably) SP -closed α -remote family Φ of G has a finite subfamily Ψ which is an α -remote family of G ;
- (2) For every $\alpha \in M(L)$, every SP -closed α -remote family $\Phi = \{P\}$ of G is a strong α -remote family of G .

Proof. We only prove the theorem for SPN -compactness.

The necessity is obvious since an SP -closed strong α -remote family of G must be an SP -closed α -remote family of G .

The sufficiency can be proved from the following fact. Φ is an SP -closed strong α -remote family of G if and only if $\{\bigwedge\{B \mid B \in \Phi\}\}$ is an SP -closed strong α -remote family of G . $\square$

Analogously we have the following result.

Theorem 2.9. Let $(X, \mathcal{T})$ be an L -space, $G \in L^X$. Then G has the SPN -Lindelöf property if and only if the following two conditions hold.

- (1) For every $\alpha \in M(L)$, every SP -closed α -remote family Φ of G has a countable subfamily Ψ which is an α -remote family of G ;
- (2) For every $\alpha \in M(L)$, every SP -closed α -remote family $\Phi = \{P\}$ of G is a strong α -remote family of G .

Definition 2.10. Let $(X, \mathcal{T})$ be an L -space, $a \in P(L)$ and $G \in L^X$. A family $\mathcal{U} \subset \mathcal{L}^X$ is said to be an a -shading of G if $G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \not\leq a$.

$\mathcal{U}$ is said to be a strong a -shading of G if $\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \not\leq a$. If $\mathcal{U}$ is an (a strong) a -shading of G and $\mathcal{U} \subset \mathbf{SPO}(\mathbf{X})$, then it is called an SP-open (strong) a -shading of G .

Definition 2.11. Let $\alpha \in M(L)$ and $G \in L^X$. A subfamily $\mathcal{A}$ of L^X is said to have α -nonempty intersection in G if there exists an $x \in X$ such that $G(x) \wedge \bigwedge_{A \in \mathcal{A}} A(x) \geq \alpha$. $\mathcal{A}$ is said to have weak α -nonempty intersection in G if $\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{A \in \mathcal{A}} A(x) \right) \geq \alpha$. $\mathcal{A}$ is said to have the finite (countable) weak α -intersection property in G if every finite (countable) subfamily $\mathcal{F}$ of $\mathcal{A}$ has weak α -nonempty intersection in G .

It is easy to prove the following two theorems.

Theorem 2.12. Let $(X, \mathcal{T})$ be an L -space and $G \in L^X$. Then the following conditions are equivalent.

- (1) G is (countably) SPN-compact;
- (2) For any $a \in P(L)$, each (countable) SP-open a -shading $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ which is a strong a -shading of G ;
- (3) For any $\alpha \in M(L)$, each (countable) family of strongly preclosed L -sets which has the finite weak α -intersection property in G has α -nonempty intersection in G .

Theorem 2.13. Let $(X, \mathcal{T})$ be an L -space and $G \in L^X$. Then the following conditions are equivalent.

- (1) G has the SPN-Lindelöf property;
- (2) For any $a \in P(L)$, each SP-open a -shading $\mathcal{U}$ of G has a countable subfamily $\mathcal{V}$ which is a strong a -shading of G ;
- (3) For any $\alpha \in M(L)$, each family of strongly preclosed L -sets which has the countable weak α -intersection property in G has α -nonempty intersection in G .

Theorem 2.14. Let $(X, \mathcal{T})$ be an L -space and $G \in L^X$. If $\text{supp}G = \{x \in X : G(x) > 0\}$ is finite, then G is SPN-compact.

Proof. $\forall \alpha \in M(L)$, $G_{[\alpha]} = \{x \in X : G(x) \geq \alpha\}$ is a finite set since $\text{supp}G$ is a finite set. Let $G_{[\alpha]} = \{x_1, x_2, \dots, x_n\}$ and Φ be an SP-closed α -remote family of G . For each $i \leq n$, choose now $P_i \in \Phi$ so that $\alpha \not\leq P_i(x_i)$. Let $\Psi = \{P_1, P_2, \dots, P_n\}$, then Ψ is finite subfamily of Φ and $\bigvee_{i=1}^n (G(x_i) \wedge \bigwedge P_i(x_i)) \not\geq \alpha$. Hence Ψ is an SP-closed strong α -remote family of G . Thus G is SPN-compact. $\square$

Theorem 2.15. Let $(X, \mathcal{T})$ be an L -space, and $G \in L^X$ be (countably) SPN-compact. Then for every strongly preclosed L -set H in $(X, \mathcal{T})$, $G \wedge H$ is (countably) SPN-compact.

Proof. We only prove the theorem for SPN-compactness. Suppose that $\mathcal{U}$ is an SP-open a -shading of $G \wedge H$. Then $\mathcal{U} \cup \{\mathcal{H}'\}$ is an SP-open a -shading of G . By SPN-compactness of G we know that $\mathcal{U} \cup \{\mathcal{H}'\}$ has a finite subfamily $\mathcal{V}$ which is a strong a -shading of G . Take $\mathcal{W} = \mathcal{V} \setminus \{\mathcal{H}'\}$. Then $\mathcal{W}$ is a finite subfamily of $\mathcal{U}$ and it is a strong a -shading of $G \wedge H$. This shows that $G \wedge H$ is SPN-compact. $\square$

Analogously we have the following result.

Theorem 2.16. Let $(X, \mathcal{T})$ be an L -space, and let G have the SPN-Lindelöf property. Then for every strongly preclosed L -set H in $(X, \mathcal{T})$, $G \wedge H$ has the SPN-Lindelöf property.

Theorem 2.17. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L -spaces, $f : X \rightarrow Y$ be an SP-irresolute mapping. If G is an SPN-compact (a countable SPN-compact, an SPN-Lindelöf) L -set in $(X, \mathcal{T}_1)$, then so is $f_L^{\rightarrow}(G)$ in $(Y, \mathcal{T}_2)$.

Proof. We only prove that the theorem is true for SPN-compactness. Let Φ be an SP-closed α -remote family of $f_L^{\rightarrow}(G)$ ($\alpha \in M(L)$). It is easy to prove that $f_L^{\leftarrow}(\Phi) = \{f_L^{\leftarrow}(P) : P \in \Phi\}$ is an SP-closed α -remote family of G . Since G is SPN-compact, there exists finite subfamily Ψ of Φ such that

$$\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \Psi} f_L^{\leftarrow}(B)(x) \right) \not\geq \alpha,$$

By Lemma 1.5 we have that

$$\bigvee_{y \in Y} \left(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B \in \Psi} B(y) \right) = \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \Psi} f_L^{\leftarrow}(B)(x) \right) \not\geq \alpha.$$

This implies that Ψ is a strong α -remote family of $f_L^{\rightarrow}(G)$. Thus $f_L^{\rightarrow}(G)$ is SPN-compact. $\square$

Theorem 2.18. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L -spaces, $f : X \rightarrow Y$ be an SP-continuous mapping. If G is an SPN-compact (a countably SPN-compact, an SPN-Lindelöf) L -set in $(X, \mathcal{T}_1)$, then $f_L^{\rightarrow}(G)$ is an N -compact (a countably N -compact, an N -Lindelöf) L -set in $(Y, \mathcal{T}_2)$.

3. SP-convergence of nets

Definition 3.1. Let $(X, \mathcal{T})$ be an L -space, $x_\lambda \in M(L^X)$ and $S = \{S(n) : n \in D\}$ a net in L^X . Then

- (1) x_λ is said to be an SP-limit point of S , in symbols, $S \xrightarrow{SP} x_\lambda$ if for each $P \in \eta_{sp}(x_\lambda)$, $S(n) \not\leq P$ is eventually true;
- (2) x_λ is said to be an SP-cluster point of S , in symbols, $S \overset{SP}{\propto} x_\lambda$ if for each $P \in \eta_{sp}(x_\lambda)$, $S(n) \not\leq P$ is frequently true.

The union of all SP-limit points and the union of all SP-cluster points of S are denoted by $\lim_{sp} S$ and $\text{ad}_{sp} S$ respectively. Obviously $\lim_{sp} S \leq \text{ad}_{sp} S$.

Theorem 3.2. Let $(X, \mathcal{T})$ be an L -space, $x_\lambda, x_\mu \in M(L^X)$ and S be a net in L^X . Then the following statements are true.

- (1) Let $T = \{T(n) : n \in D\}$ be a net with the same domain as S and for each $n \in D$, $T(n) \geq S(n)$. If $S \xrightarrow{SP} x_\lambda$, then $T \xrightarrow{SP} x_\lambda$;
- (2) Let $T = \{T(n) : n \in D\}$ be a net with the same domain as S and for each $n \in D$, $T(n) \geq S(n)$. If $S \overset{SP}{\propto} x_\lambda$, then $T \overset{SP}{\propto} x_\lambda$;
- (3) If $S \xrightarrow{SP} x_\lambda$ and $x_\mu \leq x_\lambda$. Then $S \xrightarrow{SP} x_\mu$;
- (4) If $S \overset{SP}{\propto} x_\lambda$ and $x_\mu \leq x_\lambda$. Then $S \overset{SP}{\propto} x_\mu$.

Proof. It is simple and omitted. $\square$

Theorem 3.3. Let $(X, \mathcal{T})$ be an L -space, $x_\lambda \in M(L^X)$ and S be a net in L^X . Then

- (1) $S \xrightarrow{SP} x_\lambda$ if and only if $x_\lambda \leq \lim_{sp} S$;
- (2) $S \overset{SP}{\propto} x_\lambda$ if and only if $x_\lambda \leq \text{ad}_{sp} S$.

Proof. We only prove the sufficiency of (1). Suppose that $x_\lambda \leq \lim_{sp} S$ and $P \in \eta_{sp}(x_\lambda)$. Then $\lim_{sp} S \not\leq P$. By the definition of $\lim_{sp} S$, there exists an SP -limit point e of S such that $e \not\leq P$, i.e., $P \in \eta_{sp}(e)$. Hence S is eventually not in P , i.e., $S \xrightarrow{SP} x_\lambda$. $\square$

Theorem 3.4. Let $(X, \mathcal{T})$ be an L -space and S be a net in L^X . Then $\lim_{sp} S$ and $\text{ad}_{sp} S$ are strongly preclosed.

Proof. Let $x_\lambda \leq \text{cl}_{sp}(\lim_{sp} S)$. Then $\lim_{sp} S \not\leq P$ for each $P \in \eta_{sp}(x_\lambda)$. Hence there exists a point e such that $e \leq \lim_{sp} S$ and $e \not\leq P$. Then $P \in \eta_{sp}(e)$. By Theorem 3.3 $S \xrightarrow{SP} e$. Hence $S(n) \not\leq P$ is eventually true. Thus $x_\lambda \leq \lim_{sp} S$. This implies that $\lim_{sp} S$ is strongly preclosed. Similarly $\text{ad}_{sp} S$ is strongly preclosed. $\square$

It is easy to prove the following theorem.

Theorem 3.5. Let $(X, \mathcal{T})$ be an L -space, $x_\lambda \in M(L^X)$ and $S = \{S(n) : n \in D\}$ be a net in L^X . If S has a subnet T such that $T \xrightarrow{SP} x_\lambda$, then $S \overset{SP}{\propto} x_\lambda$.

Definition 3.6. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L -spaces. A mapping $f : X \rightarrow Y$ is said to be SP -irresolute at $x_\lambda \in M(L^X)$ if $\text{cl}_{sp}(f_L^{\leftarrow}(B)) \in \eta_{sp}(x_\lambda)$ for each $B \in \eta_{sp}(f_L^{\rightarrow}(x_\lambda))$.

Theorem 3.7. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L -spaces. A mapping $f : X \rightarrow Y$ is SP -irresolute if and only if f is SP -irresolute at each $x_\lambda \in M(L^X)$.

Proof. Suppose that f is SP-irresolute and $x_\lambda \in M(L^X)$. Then $f_L^\leftarrow(P)$ is strongly preclosed for each $P \in \eta_{sp}(f_L^\rightarrow(x_\lambda))$. Clearly $x_\lambda \not\leq f_L^\leftarrow(P)$. Hence $f_L^\leftarrow(P) = cl_{sp}(f_L^\leftarrow(P)) \in \eta_{sp}(x_\lambda)$. So f is SP-irresolute at x_λ .

Conversely, suppose that f is SP-irresolute at each $x_\lambda \in M(L^X)$. Let P be a strongly preclosed in $(Y, \mathcal{T}_2)$. We may assume that $f_L^\leftarrow(P) \neq \underline{1}$ and suppose that $x_\lambda \not\leq f_L^\leftarrow(P)$. Then $f_L^\rightarrow(x_\lambda) \not\leq P$, i.e., $P \in \eta_{sp}(f_L^\rightarrow(x_\lambda))$. Hence $cl_{sp}(f_L^\leftarrow(P)) \in \eta_{sp}(x_\lambda)$. i.e., $x_\lambda \not\leq f_L^\leftarrow(P)$ implies that $x_\lambda \not\leq cl_{sp}(f_L^\leftarrow(P))$. So $cl_{sp}(f_L^\leftarrow(P)) \leq f_L^\leftarrow(P)$. Thus $f_L^\leftarrow(P)$ is strongly preclosed in $(X, \mathcal{T}_1)$. This shows that f is SP-irresolute. $\square$

Theorem 3.8. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L -spaces. Suppose that $f : X \rightarrow Y$ is SP-irresolute at $x_\lambda \in M(L^X)$. If $S \xrightarrow{SP} x_\lambda$ in L^X , then $f_L^\rightarrow(S) \xrightarrow{SP} f_L^\rightarrow(x_\lambda)$ in L^Y .

Proof. Suppose that f is SP-irresolute at x_λ and $S \xrightarrow{SP} x_\lambda$. Let $P \in \eta_{sp}(f_L^\rightarrow(x_\lambda))$. Then $f_L^\leftarrow(P) \leq cl_{sp}(f_L^\leftarrow(P)) \in \eta_{sp}(x_\lambda)$ since f is SP-irresolute at x_λ and so $S(n) \not\leq f_L^\leftarrow(P)$ is eventually true from $S \xrightarrow{SP} x_\lambda$. Therefore $f_L^\rightarrow(S) \not\leq P$ is eventually true. Thus $f_L^\rightarrow(S) \xrightarrow{SP} f_L^\rightarrow(x_\lambda)$. $\square$

Corollary 3.9. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L -spaces. If a mapping $f : X \rightarrow Y$ is SP-irresolute, then for each net S in L^X ,

$$f_L^\rightarrow(\lim_{sp} S) \leq \lim_{sp} f_L^\rightarrow(S).$$

Theorem 3.10. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L -spaces. If a mapping $f : X \rightarrow Y$ is SP-irresolute, then for each net T in L^Y ,

$$f_L^\leftarrow(\lim_{sp} T) \leq \lim_{sp} f_L^\leftarrow(T).$$

Proof. Let $T = \{T(n) : n \in D\}$ be a net in L^Y . Then

$$f_L^\leftarrow(T) = \{f_L^\leftarrow(T(n)) : n \in D\}$$

is a net in L^X . Since f is SP-irresolute, according to Corollary 3.9 we have

$$f_L^\rightarrow(\lim_{sp} f_L^\leftarrow(T)) \leq \lim_{sp} f_L^\rightarrow(f_L^\leftarrow(T)) \leq \lim_{sp} T.$$

Hence $\lim_{sp} f_L^\leftarrow(T) \leq f_L^\leftarrow(\lim_{sp} T)$. $\square$

4. Characterizations of SPN-compactness

Theorem 4.1. *Let (X, T) be an L -space and $G \in L^X$. Then G is SPN-compact if and only if $\forall \alpha \in M(L)$, every α -net in G has an SP-cluster point in G with height α .*

Proof. *Necessity.* Let G be SPN-compact and $S = \{S(n) : n \in D\}$ be an α -net in G . If S has no SP-cluster point in G with height α , then for each point $x_\alpha \leq G$, there is $P_x \in \eta_{sp}(x_\alpha)$ such that S is eventually in P_x , i.e., there is $n(x) \in D$ such that $S(n) \leq P_x$ whenever $n \geq n(x)$. Put $\Phi = \{P_x : x_\alpha \leq G\}$, then Φ is an SP-closed α -remote family of G . Since G is SPN-compact, there is a finite subfamily $\Psi = \{P_{x_i} \mid i = 1, 2, \dots, k\}$ of Φ which is a strong α -remote family of G , i.e., $\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{i=1}^k P_{x_i}(x) \right) \not\geq \alpha$, hence there is $r \in \beta^*(\alpha)$ such that for each $x \in X$ satisfying $G(x) \geq r$, it follows that $P_{x_i}(x) \not\geq r$ for each $i \leq k$. Put $P = \bigwedge_{i=1}^k P_{x_i}$. Then we obtain the following fact:

(C) For each point $x_r \leq G$, we have that $x_r \not\leq P$.

Since D is a directed set, there is an $n_0 \in D$ such that $n_0 \geq n(x_i)$ ($i = 1, 2, \dots, k$). Thus $\forall n \geq n_0$, we have that $S(n) \leq P$. Since S is an α -net in G , we can take $n_1 \geq n_0$ such that $V(S(n)) \geq r$ whenever $n \geq n_1$, where $V(S(n))$ is the height of $S(n)$. Hence by (C) we know that $S(n) \not\leq P$. This contradicts that $S(n) \leq P$. Therefore S has at least an SP-cluster point in G with height α .

Sufficiency. Suppose that $\forall \alpha \in M(L)$, each α -net in G has an SP-cluster point in G with height α . Let Φ be an SP-closed α -remote family of G . Suppose none of the finite subfamily Ψ of Φ is strong α -remote family of G . Then for each $\Psi \in 2^{(\Phi)}$, it is follows that

$$\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{P \in \Psi} P(x) \right) \geq \alpha.$$

This implies that $\forall r \in \beta^*(\alpha)$, there is $x(r, \Psi) \in X$ such that

$$G(x(r, \Psi)) \wedge \bigwedge_{P \in \Psi} P(x(r, \Psi)) \geq r. \quad (*)$$

Put $D = \beta^*(\alpha) \times 2^{(\Phi)}$ and for $(r_1, \Psi_1), (r_2, \Psi_2) \in D$, define

$$(r_1, \Psi_1) \leq (r_2, \Psi_2) \quad \text{if and only if} \quad r_1 \leq r_2, \Psi_1 \subset \Psi_2.$$

Obviously D is a directed set. Let $S(r, \Psi) = x(r, \Psi)_r$. Then $\{S(r, \Psi) : (r, \Psi) \in D\}$ is an α -net in G . However, S has no SP-cluster point in G with height α . In fact, for each point $x_\alpha \leq G$, there exists $R \in \Phi$ such that $R \in \eta_{sp}(x_\alpha)$ since Φ is an α -remote family of G . Obviously $\{R\} \in 2^{(\Phi)}$. For any $s \in \beta^*(\alpha)$, when $(r, \Psi) \geq (s, \{R\})$, by $(*)$ and $R \in \Psi$ we know that $S(r, \Psi) = x(r, \Psi)_r \leq R$. This shows that S is eventually in R . Hence x_α is not an SP-cluster point of S . Therefore S has no SP-cluster point in G with height α , which contradicts the assumption. Thus G is SPN-compact. $\square$

An SPN-compact L -set in an $[0, 1]$ -space has an interesting property as follows:

Theorem 4.2. Let $(X, \mathcal{T})$ be an $[0, 1]$ -space and $G \in [0, 1]^X$. If G is SPN-compact, then there exists $x \in X$ such that

$$G(x) = \sup\{G(t) : t \in X\}.$$

Proof. Let $\alpha = \sup\{G(t) : t \in X\}$. Clearly we may assume $\alpha \in (0, 1]$. Choosing now $x^n \in X$ so that $G(x^n) > \alpha - 1/n$ and denoting by $S(n)$ the fuzzy point with support x^n and value $G(x^n)$, we get an α -net $S = \{S(n) : n \in \mathbb{N}\}$ contained in G . Since G is SPN-compact, S has an SP-cluster point x_α in G with height α . It is obvious that $G(x) = \alpha$. $\square$

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