ON SOME FUNCTIONS CONCERNING FUZZY PG-CLOSED SETS

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Abstract

In this paper we consider new weak and strong forms of fuzzy preirresoluteness and fuzzy pre-closureness via the concept of fuzzy pg-closed sets which we call ap-Fp-irresolute functions, ap-Fp-closed functions and contra fuzzy pre-irresolute functions and we use it to obtain a characterization of fuzzy pre-$T_{1/2}$ spaces.

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1. Introduction and Preliminaries

In 1991 A. S. Binshahan [4] introduced and investigated the notions of fuzzy preopen and fuzzy preclosed sets. Since then the advent of these notions, several research papers with interesting results in different respects came to existence ([1],[3],[5],[6],[7],[10],[12]).

In this paper we shall introduce new generalizations of fuzzy pre-closedness called ap-fuzzy pre-closedness by using fuzzy pg-closed sets and study some of their basic properties. This definition enables us to obtain conditions under which functions and inverse functions preserve fuzzy pg-closed sets, also in this paper we present a new generalization of fuzzy pre-irresoluteness called contra fuzzy pre-irresolute. We define this last class of functions by the requirement that the inverse image of each fuzzy pre-open set in the co-domain is fuzzy preclosed in the domain. This notion is a stronger form of ap-fuzzy pre-irresolute and ap-fuzzy preclosed functions. Finally, we also characterize the class of fuzzy pre-$T_{1/2}$ spaces in terms of ap-fuzzy pre-irresolute and ap-fuzzy preclosed functions.

Throughout this paper $(X, \tau)$, $(Y, \sigma)$ and $(Z, \gamma)$ (or simply $X$, $Y$ and $Z$) represent non-empty fuzzy topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let $A$ be a fuzzy subset of $X$ then $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure of $A$ and the interior of $A$, respectively.

1.1 DEFINITION : The fuzzy set $A$ of $X$ is called fuzzy preopen [4] if $A \subseteq \text{Int}\text{Cl}(A)$.

The fuzzy subset $B$ of $X$ is called fuzzy preclosed if, its complement $B^c$ is fuzzy preopen in $X$.

By $\text{FPO}(X)$ (rep. $\text{FPC}(X)$), we denote the family of all fuzzy pre-open (resp. fuzzy preclosed) sets of $X$.

1.2 DEFINITION : The intersection of all fuzzy preclosed sets containing $A$ is called the preclosure of $A$ and is denoted by $\text{pCl}(A)$ [4].

1.3 DEFINITION : The fuzzy interior of $A$ [4] is denoted by $\text{pInt}(A)$, and is defined by the union of all fuzzy preopen sets of $X$ which are contained in $A$. 
1.4 DEFINITION: Recall that a fuzzy set $A$ is called fuzzy regular open (resp. regular closed) if $A = \text{IntCl}(A)$ (resp. $A = \text{ClInt}(A)$) [2].

1.5 DEFINITION: A function $f: (X, \tau) \to (Y, \sigma)$ is said to be:

(i) fuzzy preirresolute [8] if, $f^{-1}(V)$ is fuzzy preopen set in $(X, \tau)$ for each fuzzy open set $V$ of $(Y, \sigma)$,

(ii) fuzzy M-preclosed [11] (resp. fuzzy M-preopen [11]) if, for every fuzzy preclosed (resp. fuzzy preopen) set $B$ of $(X, \tau)$, $f(B)$ is fuzzy preclosed (resp. fuzzy preopen) set in $(Y, \sigma)$,

(iii) fuzzy contra preclosed if $f(A)$ is fuzzy preopen set in $(Y, \sigma)$ for every fuzzy closed set $A$ of $(X, \tau)$ [9].

2. AP-FP-IRRESOLUTE AND AP-FP-CLOSED FUNCTIONS

2.1 DEFINITION: A fuzzy set $A$ of $(X, \tau)$ is said to be fuzzy pre-generalized closed (briefly, Fpg-closed) if, $pCl(A) \subseteq H$ whenever $A \subseteq H$ and $H$ is fuzzy preopen in $X$. A fuzzy set $B$ is said to be fuzzy pre-generalized open (briefly, Fpg-open) in $X$ if its complement $B^c$ is Fpg-closed set in $X$.

2.2 REMARK: (i) Every fuzzy preclosed set is Fpg-closed but not conversely.

(ii) The concept of fuzzy g-closed sets and Fpg-closed sets are independent. For,

2.3 EXAMPLE: Let $X = \{a, b\}$, fuzzy sets $A, B, E, H$ and $K$ are defined as:

\[
\begin{align*}
A(a) &= 0.3, \ A(b) = 0.4; \\
B(a) &= 0.6, \ B(b) = 0.5; \\
E(a) &= 0.2, \ E(b) = 0.2; \\
H(a) &= 0.3, \ H(b) = 0.7; \\
K(a) &= 0.3, \ K(b) = 0.8. 
\end{align*}
\]

Let $\tau = f0, A, 1g$ and $\sigma = f0, H, 1g$. Then,

(i) $B$ is fuzzy pg-closed but not fuzzy preclosed set in $(X, \tau)$.

(ii) $E$ is fuzzy pg-closed but not fuzzy g-closed in $(X, \tau)$ and $K$ is fuzzy g-closed but not fuzzy pg-closed set in $(X, \sigma)$. 

2.4 DEFINITION : A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be approximately fuzzy preirresolute (briefly, \( \text{ap-Fp-irresolute} \)) if, \( p\text{Cl}(A) \cdot f^{-1}(H) \) whenever \( H \) is a fuzzy preopen subset of \((Y, \sigma)\), \( A \) is a fuzzy pre-\(\tau\)-closed subset of \((X, \tau)\) and \( A \cdot f^{-1}(H) \).

2.5 DEFINITION : A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be approximately fuzzy preclosed (briefly, \( \text{ap-Fp-closed} \)) if, \( f(B) \cdot p\text{Int}(A) \) whenever \( A \) is a fuzzy pre-\(\tau\)-closed subset of \((Y, \sigma)\), \( B \) is a fuzzy preclosed subset of \((X, \tau)\) and \( f(B) \cdot A \).

2.6 THEOREM : (i) A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \text{ap-Fp-irresolute} \) if, \( f^{-1}(A) \) is fuzzy preclosed in \((X, \tau)\) for every \( A \in FPO(Y) \).

(ii) \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \text{ap-Fp-closed} \) if, \( f(B) \in FPO(Y) \) for every fuzzy preclosed subset \( B \) of \((X, \tau)\).

PROOF : (i) Let \( H \cdot f^{-1}(A) \) where \( A \in FPO(Y) \) and \( H \) is a fuzzy pre-\(\tau\)-closed subset of \((X, \tau)\). Therefore \( p\text{Cl}(H) \cdot p\text{Cl}(f^{-1}(A)) = f^{-1}(A) \). Thus, \( f \) is \( \text{ap-Fp-irresolute} \).

(ii) Let \( f(B) \cdot A \), where \( B \) is a fuzzy preclosed subset of \((X, \tau)\) and \( A \) a fuzzy pre-\(\tau\)-open subset of \((Y, \sigma)\). Therefore, \( p\text{Int}(f(B)) \cdot p\text{Int}(A) \). Then, \( f(B) \cdot p\text{Int}(A) \). Thus, \( f \) is \( \text{ap-Fp-closed} \).

Clearly, fuzzy preirresolute functions are \( \text{ap-Fp-irresolute} \). Also, fuzzy M-preclosed functions are \( \text{ap-Fp-closed} \). The converse implication do not hold as it is shown in the following example.

2.7 EXAMPLE : Let \( X = \{a, b\} \) and \( Y = \{x, y\} \). Fuzzy sets \( E, A, B \) and \( H \) be defined as:

\[
E(a) = 0, E(b) = 0.8; \\
A(a) = 0.3, A(b) = 0.4; \\
H(x) = 0.3, H(y) = 0.4; \\
B(x) = 0, B(y) = 0.8.
\]

Let \( \tau = f0, E.1g \) and \( \sigma = f0, H.1g, \ i = f0, A.1g \) and \( \nu = f0, B.1g \). Then, the function \( f : (X, \tau) \rightarrow (Y, \sigma) \) defined by \( f(a) = x, f(b) = y \) is \( \text{ap-Fp-irresolute} \) but not fuzzy preirresolute since \( Z(x) = 0.1 \) and \( Z(y) = 0.1 \) are fuzzy pre-open in \( Y \) but \( f^{-1}(Z) \) is not fuzzy pre-open in \((X, \tau)\), and the function \( h : (X, \i) \rightarrow (Y, \nu) \) defined by \( h(a) = x, h(b) = y \), is \( \text{ap-Fp-closed} \) but not fuzzy M-preclosed.
2.8 REMARK : Converse of the Theorem 2.6 do not hold for .

2.9 EXAMPLE : The function f deﬁned in Example 2.3 is ap-Fp-irresolute. Fuzzy set \( K(x) = 0.9, K(y) = 0.9 \) are fuzzy preopen in \( Y \) but \( f^{-1}(K) \) is not fuzzy preclosed in \( (X, \tau) \). The function function h deﬁned in Example 2.7 is ap-Fp-closed and the fuzzy set \( W(a) = 0.1, W(b) = 0.1 \) is fuzzy preclosed set in \( (X, \tau) \), but \( h(W) \) is not fuzzy preopen set in \( (Y, \nu) \).

In the following theorem, we get under certain conditions the converse of Theorem 2.6 is true.

2.10 THEOREM : Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function. Then,

(i) Let all fuzzy subsets of \( (X, \tau) \) be fuzzy clopen, then \( f \) is ap-Fp-irresolute if and only if, \( f^{-1}(H) \) is fuzzy preclosed in \( (X, \tau) \) for every \( H \in \text{FPO}(Y) \).

(ii) Let all fuzzy subsets of \( (Y, \sigma) \) be fuzzy clopen, then \( f \) is ap-Fp-closed if and only if, \( f(B) \) is fuzzy preclosed for every fuzzy preclosed subset \( B \) of \( (X, \tau) \).

PROOF : (i) Assume \( f \) is ap-Fp-irresolute. Let \( A \) be an arbitrary fuzzy subset of \( (X, \tau) \) such that \( A \cdot Q \) where \( Q \in \text{FPO}(X) \). Then by hypothesis \( pCl(A) \cdot pCl(Q) = Q \). Therefore all fuzzy subsets of \( (X, \tau) \) are fuzzy pg-closed (and hence all are fuzzy pg-open). So for any \( H \in \text{FPO}(Y) \), \( f^{-1}(H) \) is fuzzy pg-closed in \( X \). Since \( f \) is ap-Fp-irresolute, \( pCl(f^{-1}(H)) \cdot f^{-1}(H) \). Therefore, \( pCl(f^{-1}(H)) = f^{-1}(H) \), i.e., \( f^{-1}(H) \) is fuzzy preclosed in \( (X, \tau) \).

The converse is clear by Theorem 2.6.

(ii) Assume \( f \) is ap-Fp-closed. Reasoning as in (i), we obtain that all fuzzy subsets of \( (Y, \sigma) \) are fuzzy pg-open. Therefore, for any fuzzy preclosed subset \( B \) of \( (X, \tau) \), \( f(B) \) is fuzzy pg-open in \( Y \). Since \( f \) is ap-Fp-closed, \( f(B) \cdot pInt(f(B)) \). Therefore, \( f(B) = pInt(f(B)) \), i.e., \( f(B) \) is fuzzy preopen.

The converse is clear by Theorem 2.6.

As an immediate consequence of Theorem 2.10, we have the following.

2.11 COROLLARY : Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function. Then,
(i) Let all the fuzzy subsets of \((X, \tau)\) be fuzzy clopen, then \(f\) is ap-F p-irresolute if and only if \(f\) is fuzzy preirresolute.

(ii) Let all the fuzzy subsets of \((Y, \sigma)\) be fuzzy clopen, the \(f\) is ap-F p-closed if and only if \(f\) is fuzzy M-preclosed.

3. CONTRA FUZZY PRE-IRRESOLUTE FUNCTIONS

3.1 DEFINITION : A function \(f : (X, \tau) \to (Y, \sigma)\) is called :

(i) contra fuzzy preirresolute if \(f^{-1}(A)\) is fuzzy preclosed in \((X, \tau)\) for each \(A \in \text{FPO}(Y)\),

(ii) contra fuzzy M-preclosed if \(f(B) \in \text{FPO}(Y)\) for each fuzzy pre-closed set \(B\) of \((X, \tau)\).

3.2 REMARK : The concepts of contra fuzzy preirresoluteness and fuzzy preirresoluteness are independent notions. For,

3.3 EXAMPLE : Let \(X = \{a, b\}\), \(Y = \{x, y\}\) and \(Z = \{p, q\}\). Define fuzzy sets \(A, E\) and \(B\) as:

\[ A(a) = 0.7, A(b) = 0.8; \]
\[ E(x) = 0.4, E(y) = 0.3; \]
\[ B(p) = 0.3, B(q) = 0.3. \]

Let \(\tau = \{0, A, 1g, 1g\}, \sigma = \{0, E, 1g\}\) and \(\tau = \{0, B, 1g\}\). Then the function \(f : (X, \tau) \to (Y, \sigma)\) defined by \(f(a) = x, f(b) = y\) is contra fuzzy preirresolute but not fuzzy preirresolute and the function \(h : (Y, \sigma) \to (Z, \sigma)\) defined by \(h(x) = p\) and \(h(y) = q\) is fuzzy preirresolute but not contra fuzzy preirresolute.

In the same manner one can prove that, contra fuzzy M-preclosed functions and fuzzy M-preclosed functions are independent notions.

3.4 DEFINITION : A function \(f : (X, \tau) \to (Y, \sigma)\) is called contra fuzzy precontinuous if \(f^{-1}(A)\) is fuzzy preclosed set in \((X, \tau)\) for each fuzzy open set \(A\) of \((Y, \sigma)\).

3.5 REMARK : Every contra fuzzy preirresolute function is contra fuzzy precontinuous, but not conversely. For,
3.6 EXAMPLE : Let $X = f a, bg$ and $Y = f x, yg$. Fuzzy sets $A$ and $B$ are defined as:

\[ A(a) = 0.7, A(b) = 0.8; \]
\[ B(x) = 0, B(y) = 0.6. \]

Let $\tau = f 0.A, 1g$ and $\sigma = f 0,B, 1g$. Then the function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = x$, $f(b) = y$ is contra fuzzy precontinuous but not contra fuzzy preirresolute.

The following result can be easily verified. Its proof is straightforward.

3.7 THEOREM : Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following conditions are equivalent:

(i) $f$ is contra fuzzy preirresolute,

(ii) The inverse image of each fuzzy preclosed set in $Y$ is fuzzy preopen in $X$.

3.8 REMARK : By Theorem 2.6, we have that every contra fuzzy preirresolute function is ap-Fp-irresolute and every contra fuzzy preclosed function is ap-Fp-closed, the converse implication do not hold (see Remark 2.8)

Now, we define the following.

3.9 DEFINITION : A function $f : (X, \tau) \to (Y, \sigma)$ is called perfectly contra fuzzy preirresolute if the inverse image of every fuzzy preopen set in $Y$ is fuzzy pre-clopen in $X$.

Every perfectly contra fuzzy preirresolute function is contra fuzzy preirresolute and fuzzy preirresolute. But the converse may not be true. For,

3.10 EXAMPLE : The functions $f$ and $h$ defined in Example 3.3 are contra fuzzy preirresolute and fuzzy preirresolute respectively but not perfectly contra fuzzy preirresolute.

Clearly, the following diagram holds and none of its implications is reversible:

\[
p.c.f.p.i. \quad \rightarrow \quad c.f.p.i.
\]
\[
+ \quad +
\]
\[
f.p.i. \quad \rightarrow \quad ap \quad Fp \quad i
\]
Where contra f.p.i = contra fuzzy preirresolute, ap-F p-i. = ap-F p-
irresolute, perfectly contra f.p.i. = perfectly contra fuzzy preirresolute.

The following theorem is a decomposition of perfectly contra fuzzy preir-
resoluteness.

3.11 THEOREM : For a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) the following
conditions are equivalent:
(i) \( f \) is perfectly contra fuzzy preirresolute.
(ii) \( f \) is both contra fuzzy preirresolute and fuzzy preirresolute.

3.12 THEOREM : If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is ap-F p-irresolute
and fuzzy M-preclosed, then for every F pg-closed set \( A \) of \( (X, \tau) \), \( f(A) \) is
F pg-closed in \( (Y, \sigma) \).

PROOF: Let \( A \) be a F pg-closed subset of \( (X, \tau) \). Let \( f(A) \cdot B \)
where \( B \) 2 FPO\( (Y) \). Then \( A \cdot f^{-1}(B) \) holds. Since \( f \) is ap-F p-irresolute
\( pCl(A) \cdot f^{-1}(B) \) and hence \( f(pCl(A)) \cdot B \). Therefore, we have,
\( pCl(f(A)) \cdot pCl(f(pCl(A))) = f(pCl(A)) \cdot B \). Hence \( f(A) \) is fuzzy pg-
closed in \( (Y, \sigma) \).

However the following theorem holds. The easy proof of it is omitted.

3.13 THEOREM : A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) and
\( g : (Y, \sigma) \rightarrow (Z, \gamma) \) be two functions such that \( gof : (X, \tau) \rightarrow (Z, \gamma) \). Then,
(i) \( gof \) is contra fuzzy preirresolute, if \( g \) is fuzzy preirresolute and \( f \) is
contra fuzzy preirresolute.
(ii) \( gof \) is contra fuzzy preirresolute, if \( g \) is contra fuzzy preirresolute
and \( f \) is fuzzy preirresolute.

In an analogous way, we have the following.

3.14 THEOREM : Let function \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (X, \tau) \rightarrow
(Z, \gamma) \) be two functions such that \( gof : (X, \tau) \rightarrow (Z, \gamma) \). Then,
(i) \( gof \) is ap-F p-closed, if \( f \) is fuzzy M-preclosed and \( g \) is ap-F p-closed.
(ii) \( gof \) is ap-F p-closed, if \( f \) is ap-F p-closed and \( g \) is fuzzy M-preopen
and \( g \) preserves F pg-closed sets.
(iii) \( gof \) is ap-F p-irresolute, if \( f \) is ap-F p-irresolute and \( g \) is fuzzy preir-
resolute.

PROOF: (i) Suppose \( B \) is an arbitrary fuzzy preclosed subset in \( (X, \tau) \)
and \( A \) is a F pg-open subset of \( (Z, \gamma) \) for which \( (gof)(B) \cdot A \). Then \( f(B) \)
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is fuzzy preclosed in \((Y, \sigma)\) because \(f\) is fuzzy M-preclosed. Since \(g\) is ap-Fp-closed, \(g(f(B)) \cdot \text{pInt}(A)\). This implies that \(gof\) is ap-Fp-closed.

(ii) Suppose \(B\) is an arbitrary fuzzy preclosed subset of \((X, \tau)\) and \(A\) is a fuzzy preopen subset of \((Z, \gamma)\) for which \((gof)(B) \cdot A\). Hence \(f(B) \cdot gi^1(A)\). Then \(f(B) \cdot \text{pInt}(gi^1(A))\) since \(gi^1(A)\) is Fpg-open and \(f\) is ap-Fp-closed. Thus, \((gof)(B) = g(f(B)) \cdot g(pInt(gi^1(A)))) \cdot pInt(ggi^1(A)) \cdot \text{pInt}(A)\). This shows that \(gof\) is ap-Fp-closed.

(iii) Suppose \(H\) is an arbitrary Fpg-closed subset of \((X, \tau)\) and \(E \in \text{FPO}(Z)\) for which \(H \cdot (gof)^1(E)\). Then \(gi^1(E) \in \text{FPO}(Y)\) since \(g\) is fuzzy preirresolute. Since \(f\) is ap-Fp-irresolute, \(pCl(A) \cdot f^1(gi^1(E)) = (gof)^1(E)\). This proved that \(gof\) is ap-Fp-irresolute.

In the following result we offer a new characterization of the class of fuzzy pre-\(T_{1/2}\) spaces by using the concepts of ap-Fp-irresolute functions and ap-Fp-closed functions.

3.15 DEFINITION : A fuzzy topological space \((X, \tau)\) is said to be fuzzy pre-\(T_{1/2}\) space if every Fpg-closed set is fuzzy preclosed in it.

3.16 THEOREM : Let \((X, \tau)\) be a fuzzy topological space. Then the following statements are equivalent:

(i) \((X, \tau)\) is fuzzy pre-\(T_{1/2}\) space,

(ii) For every space \((Y, \sigma)\) and every function \(f : (X, \tau) \to (Y, \sigma)\), \(f\) is ap-Fp-irresolute.

PROOF : (i) \(\Rightarrow\) (ii). Let \(E\) be a Fpg-closed subset of \((X, \tau)\) and suppose that \(E \cdot f^1(H)\) where \(H \in \text{FPO}(Y)\). Since \((X, \tau)\) is fuzzy pre-\(T_{1/2}\) space, \(E\) is fuzzy preclosed (i.e., \(E = pCl(E)\)). Therefore, \(pCl(E) \cdot f^1(H)\). Then \(f\) is ap-Fp-irresolute.

(ii) \(\Rightarrow\) (i). Let \(B\) be a Fpg-closed subset of \((X, \tau)\) and let \(Y\) be the fuzzy set of \(X\) with the topology \(\sigma = 0, B, 1\). Finally, let \(f : (X, \tau) \to (Y, \sigma)\) be the identity function. By assumption \(f\) is ap-Fp-irresolute. Since \(B\) is Fpg-closed in \((X, \tau)\) and fuzzy preopen in \((Y, \sigma)\) and \(B \cdot f^1(B)\), it follows that \(pCl(B) \cdot f^1(B) = B\). Hence \(B\) is fuzzy preclosed in \((X, \tau)\) and therefore, \((X, \tau)\) is a fuzzy pre-\(T_{1/2}\) space.

3.17 THEOREM : Let \((Y, \sigma)\) be a fuzzy topological space. Then the following statements are equivalent :
(i) $(Y, \sigma)$ is a fuzzy pre-$T_{3/2}$ space,
(ii) For every space $(X, \tau)$ and every function $f : (X, \tau) \rightarrow (Y, \sigma)$, $f$ is ap-Fp-closed.

Proof: Analogous to Theorem 3.16 making the obvious changes.

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