ON OPERATOR IDEALS DEFINED BY A REFLEXIVE ORLICZ SEQUENCE SPACE

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Abstract

Classical theory of tensornorms and operator ideals studies mainly those defined by means of sequence spaces $\ell_p$. Considering Orlicz sequence spaces as natural generalization of $\ell_p$ spaces, in a previous paper [12] an Orlicz sequence space was used to define a tensornorm, and characterize minimal and maximal operator ideals associated, by using local techniques. Now, in this paper we give a new characterization of the maximal operator ideal to continue our analysis of some coincidences among such operator ideals. Finally we prove some new metric properties of tensornorm mentioned above.

Key words : Maximal operator ideals. Ultraproducts of spaces, Orlicz spaces.

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1. Introduction

Using ideas from \( \ell_p \) spaces, Saphar introduced a tensornorm \( g_p \) (see [18]) and given the great relationship between tensornorms and operator ideals, minimal and maximal operator ideals associated to the tensornorm in the sense of Defant and Floret were studied in the classical theory of tensornorm and operator ideals.

Since Orlicz spaces \( \ell_M \) are natural generalizations of \( \ell_p \) spaces, in [12] we introduced a tensornorm \( g_M^c \) by means of an Orlicz space \( \ell_M \) and using some aspects of local theory we characterized the minimal and maximal operator ideals associated to \( g_M^c \). Now in this paper our aim is to give another characterization of maximal operator ideal associated to this tensornorm to study the coincidence between components of the two operator ideals which in turn enables us to prove some metric properties of \( g_M^c \) and its dual.

Notation is standard. We will always consider Banach spaces over the real field, since we shall use results in the theory of Banach lattices. The canonical inclusion map from Banach space \( E \) into the bidual \( E'' \) will be denoted by \( J_E \). In general if \( E \) is a subspace of \( F \), the inclusion of \( E \) into \( F \) is denoted by \( I_{E,F} \). The set of finite dimensional subspaces of a normed space \( E \) will be denoted by \( FIN(E) \).

We recall the more relevant aspects on Banach lattices (we refer the reader to [1]). A Banach lattice \( E \) is order complete or Dedekind complete if every order bounded set in \( E \) has a least upper bound in \( E \), and it is order continuous if every order convergent filter is norm convergent. Every dual Banach sequence lattice \( E' \) is order complete and all reflexive spaces are even order continuous. A linear map \( T \) between Banach lattices \( E \) and \( F \) is said to be positive if \( T(x) \geq 0 \) in \( F \) for every \( x \in E, x \geq 0 \). \( T \) is called order bounded if \( T(A) \) is order bounded in \( F \) for every order bounded set \( A \) in \( E \).

Let \( \omega \) be the vector space of all scalar sequences and \( \varphi \) its subspace of the sequences with finitely many non zero coordinates. A sequence space \( \lambda \) is a linear subspace of \( \omega \) containing \( \varphi \) provided with a topology finer than the topology of coordinatewise convergence. A Banach sequence space will be a sequence space \( \lambda \) provided with a norm which makes it a Banach lattice and an ideal in \( \omega \), i.e. such that if \( |x| \leq |y| \) with \( x \in \omega \) and \( y \in \lambda \), then \( x \in \lambda \) and \( \|x\|_\lambda \leq \|y\|_\lambda \). A sectional subspace \( S_k(\lambda) \), \( k \in \mathbb{N} \), is the topological subspace of \( \lambda \) of those sequences \( (\alpha_i) \) such that \( \alpha_i = 0 \) for every \( i \geq k \). Clearly \( S_k(\lambda) \) is 1-complemented in \( \lambda \). A Banach sequence space
λ will be called regular whenever the sequence \( \{e_i\}_{i=1}^{\infty} \) where \( e_i := (\delta_{ij})_j \) (Kronecker’s delta) is a Schauder base in λ.

We now discuss Orlicz spaces. A non degenerated Orlicz function \( M \) is a continuous, non decreasing and convex function defined in \( \mathbb{R}^+ \) such that \( M(t) = 0 \) if and only if \( t = 0 \) and \( \lim_{t \to \infty} M(t) = \infty \). The Orlicz sequence space \( \ell_M \) is the space of all sequences \( a = (a_i)_{i=1}^{\infty} \) such that \( \sum_{i=1}^{\infty} M(|a_i|/c) < \infty \), for some \( c > 0 \). The functional

\[
\Pi_M(a) = \inf \{ c > 0 : \sum_{i=1}^{\infty} M(|a_i|/c) \leq 1 \}
\]

is a norm in \( \ell_M \) and \( (\ell_M, \Pi_M(.)) \) is a Banach space. We say that an Orlicz function \( M \) has the \( \Delta_2 \) property at zero if \( M(2t)/M(t) \) is bounded in a neighbourhood of zero. In general \( \ell_M \) is not regular, but this is the case if and only if \( M \) satisfies the \( \Delta_2 \) property at zero.

We say that the function \( M^* \) is the complementary of \( M \) if \( M^*(u) := \max\{ut - M(t) : 0 < t < \infty\} \) then \( H^* \) is also an Orlicz function. Associated to \( M^* \) we can introduce a new norm on \( \ell_M \), defined by

\[
\|a\|_M = \sup \{ \sum_{n=1}^{\infty} a_nb_n : \Pi_M((b_n)) \leq 1 \}
\]

if \( a = (a_i)_{i=1}^{\infty} \) which is equivalent to \( \Pi_M(.) \). Then if \( M \) has the \( \Delta_2 \)-property at zero, \( (\ell_M, \Pi_H(.)') = (\ell_{H^*}, \|\cdot\|_{M^*}) \) as isometric spaces. Moreover \( \ell_M \) is reflexive if and only if \( M \) and \( M^* \) have the \( \Delta_2 \)-property at zero. For more information on Orlicz functions and Orlicz spaces we refer to [13].

All Orlicz spaces in this paper are considered regular and reflexive. Moreover we will always suppose that \( M(1) = 1 \).

Let \( (\Omega, \Sigma, \mu) \) be a measure space, we denote by \( L_0(\mu) \) the space of equivalence classes, modulo equality \( \mu \)-almost everywhere, of \( \mu \)-measurable real-valued functions, endowed with the topology of local convergence in measure. And the space of all equivalence classes of \( \mu \)-measurable \( X \)-valued functions is denoted by \( L_0(\mu, X) \). By a Köthe function space \( K(\mu) \) on \( (\Omega, \Sigma, \mu) \), we shall mean an order dense ideal of \( L_0(\mu) \), which is equipped with a norm \( \|\cdot\|_{K(\mu)} \) that makes it a Banach lattice (if \( f \in L_0(\mu) \) and \( g \in K(\mu) \) \( |f| \leq |g| \), then \( f \in K(\mu) \) with \( \|f\|_{K(\mu)} \leq \|g\|_{K(\mu)} \)). Similarity, \( K(\mu, X) = \{ f \in L_0(\mu, X) : \|f(.)\|_X \in K(\mu) \} \), endowed with the norm \( \|f\|_{K(\mu, X)} = \|\|f(.)\|_X\|_{K(\mu)} \).

On theory of operator ideals and tensor norms we refer to the books [16] and [4] of Pietsch and Defant and Floret respectively.
If \( E \) and \( F \) are Banach spaces and \( \alpha \) is a tensor norm, then \( E \otimes_\alpha F \) represents the space \( E \otimes F \) endowed with the \( \alpha \)-norm topology. The completion of \( E \otimes_\alpha F \) is denoted by \( \hat{E} \otimes_\alpha \hat{F} \), and the norm of \( z \) in \( \hat{E} \otimes_\alpha \hat{F} \) by \( \alpha(z; E \otimes F) \). If there is no risk of confusion we write \( \alpha(z) \) instead of \( \alpha(z; E \otimes F) \).

2. The tensor norm \( g_M^\alpha \) and \( M \)-nuclear operators associated to an Orlicz function \( M \)

First we establish some notation. Given a Banach space \( E \) and an Orlicz function \( M \) with \( M(1)=1 \) such that \( \ell_M \) is reflexive, we say that a sequence \( (x_n)_{n=1}^\infty \in E^\mathbb{N} \) is strongly \( M \)-summing if \( \|x_n\| \in \ell_M \) and we write \( \pi_M((x_i)) := \Pi_H((\|x_n\|)) \) and it is said to be weakly \( M \)-summing if \( \varepsilon_M((x_i)) : = \sup_{\|x\| \leq 1} \|(x_n, x')\|_M \). We denote by \( \ell_M[E] \) (resp. \( \ell_M(E) \)) the space of all strongly (resp. weakly) \( M \)-summing in \( E \) with the norm \( \pi_M(.) \) (resp. \( \varepsilon_M(.) \)).

The more natural approach to define a tensor norm in analogy to Saphar’s tensor norm is as follows. Let \( E \) and \( F \) be Banach spaces and \( z \in E \otimes F \), we define

\[
g_M(z; E \otimes F) := \inf \pi_M((x_n)) \varepsilon_M^*(\|y_n\|)
\]

taking the infimum over all representations of \( z \) as \( \sum_{n=1}^\infty x_n \otimes y_n \). We will write \( g_M(z) \) instead of \( g_M(z; E \otimes F) \) if there is no possibility of confusion.

It is possible that for some \( M \) the functional \( g_M \) does not satisfy the triangle inequality, but it is always a reasonable quasi norm on \( E \otimes F \), see [3] and [6]. We denote \( E \hat{\otimes}_M F \) the corresponding quasi Banach space.

To have a tensor norm \( g_M^\alpha \) in [12] we took the Minkowski functional, denoted \( g_M^\alpha(z; E \otimes F) \), of the absolutely convex hull of the unit closed ball \( B_{g_M} := \{ z \in E \otimes F / g_M(z) \leq 1 \} \) of the quasi norm \( g_M \) in \( E \otimes F \), such that

\[
g_M^\alpha(z; E \otimes F) := \inf \sum_{i=1}^n \pi_M((x_{ij})) \varepsilon_M^*(\|y_{ij}\|)
\]

taking the infimum over all representations of \( z \) as \( \sum_{i=1}^n \sum_{j=1}^m x_{ij} \otimes y_{ij} \). Again, we will write \( g_M(z) \) instead of \( g_M(z; E \otimes F) \) if there is no possibility of confusion.

It is easy to see that \( g_M^\alpha \) is a tensor norm on the class of all Banach spaces (using criterion 12.2 in [4] and bearing in mind that \( \pi_M(e_i) = \|e_i\|_{\ell^\infty} = 1 \) for every \( i \in \mathbb{N} \), and that \( \forall z \in E \otimes F \quad g_M^\alpha(z; E \otimes F) \leq g_M(z; E \otimes F) \). We denote \( E \hat{\otimes} g_M^\alpha F \) the corresponding Banach space.
Proceeding as in [3] and [18], it is easy to see that if \( z \in E \hat{\otimes}_{g_M} F \), there are \( (x_i)_{i=1}^{\infty} \in \ell_M[E] \) and \( (y_i)_{i=1}^{\infty} \in \ell_M[F] \) such that \( \pi_M((x_i)) \in \ell_M[(y_i)] < \infty \) and \( z = \sum_{j=1}^{\infty} x_i \otimes y_i \). Moreover the quasi norm of \( z \) in \( E \hat{\otimes}_{g_M} F \) (again denoted by \( g_M(z) \)) is given by \( g_M(z) = \inf \pi_M((x_i)) \in \ell_M[(y_i)] \) taking the infimum over all such representations of \( z \) as \( \sum_{n=1}^{m} x_n \otimes y_n \). Similarly, if \( z \in E \hat{\otimes}_{g_M'} F \) then \( z \) can be represented as \( z = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} \otimes y_{ij} \) where \( (x_{ij})_{i=1}^{\infty} \in \ell_M[E] \) for each \( j \in \mathbb{N} \), \( (y_{ij})_{i=1}^{\infty} \in \ell_M[F] \) for each \( j \in \mathbb{N} \) and \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \pi_M((x_{ij})) \in \ell_M[(y_{ij})] < \infty \). Moreover, the norm of \( z \) in \( E \hat{\otimes}_{g_M'} F \) is \( g_M'(z) = \inf \sum_{i=1}^{\infty} \pi_M((x_{ij})) \in \ell_M[(y_{ij})] \) taking the infimum over all representations of \( z \) as \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} \otimes y_{ij} \).

The topology defined by the quasi norm \( g_M \) on \( E \otimes F \) is normable with norm equivalent to \( g_M' \). In fact, being \( \ell_M \) a reflexive Orlicz space and following the arguments of proposition 16 of [3], we consider the bilinear onto map \( R : \ell_M[E] \times \ell_M[F] \to E \hat{\otimes}_{g_M} F \) such that \( R((x_i), (y_i)) = \sum_{i=1}^{\infty} x_i \otimes y_i \). \( R \) is continuous with quasi norm less or equal one. Then there exists a unique linear and continuous map \( \ell_M[E] \otimes \ell_M[F] \to E \hat{\otimes}_{g_M} F \) (see [20]). This map can be extended to a continuous linear and onto map \( \ell_M[E] \otimes \ell_M[F] \to E \hat{\otimes}_{g_M} F \) which is open by the open mapping theorem. Then \( E \hat{\otimes}_{g_M} F \) is isomorphic to a quotient of a Banach space and so it is a Banach space itself. In this way there is a norm \( w_M(\cdot, E \otimes F) \) equivalent to the quasi norm \( g_M(\cdot, E \otimes F) \) furthermore it is easy to see that \( w_M(\cdot, E \otimes F) \), \( g_M(\cdot, E \otimes F) \) and \( g_M'(\cdot, E \otimes F) \) are equivalent with \( g_M'(\cdot, E \otimes F) \leq w_M(\cdot, E \otimes F) \). Given the last equivalence, \( g_M \) seems appropriate for our purposes, but we need \( g_M' \) for our main results.

To introduce \( M \)-nuclear operators, bearing in mind that every representation of \( z \in E' \hat{\otimes}_{g_M} F \) as \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} \otimes y_{ij} \) defines a map \( T_z \in \mathcal{L}(E, F) \) such that \( \forall x \in E, T_z(x) := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle x, y_{ij} \rangle x_{ij} \). Furthermore, \( T_z \) is well defined and independent of the chosen representation for \( z \). Let \( \Phi_{EF} : E' \hat{\otimes}_{g_M} F \to \mathcal{L}(E, F) \) be defined by \( \Phi_{EF}(z) := T_z \).

**Definition 1.** An operator between Banach spaces \( T : E \to F \) is said to be \( M \)-nuclear if \( T = \Phi_{EF}(z) \), for some \( z \in E' \hat{\otimes}_{g_M} F \).

Given any pair of Banach spaces \( E \) and \( F \), the space of the \( M \)-nuclear operators \( T : E \to F \) endowed with the topology of the norm \( N_M(T) := \inf \{ g_M(z) / \Phi_{EF}(z) = T \} \) or with the equivalent quasi-norm \( N_M(T) := \inf \{ g_M(z) / \Phi_{EF}(z) = T \} \). Also \( (N_M(E, F), N_M) \) denotes a component of the minimal Banach operator ideal \( (N_M, N_M) \) associated to the tensor norm \( g_M' \). Analogously as in [12] we obtain the following result.
Theorem 2. Let $E, F$ be any pair of Banach spaces and an operator $T \in \mathcal{L}(E, F)$. Then the following are equivalent:

1) $T$ is $M$-nuclear.

2) $T$ factors continuously in the following way:

\[
\begin{array}{c}
E \xrightarrow{T} F \\
A \xleftarrow{\ell_\infty} B \xrightarrow{\ell_M} 
\end{array}
\]

where $B$ is a diagonal multiplication operator defined by a positive sequence $(b_i) \in \ell_M$.

Furthermore $N_M(T) = \inf\{C \|B\|\|A\|\}$, infimum taken over all such factors.

3) $T$ factors continuously in the following way:

\[
\begin{array}{c}
E \xrightarrow{T} F \\
A \xleftarrow{\ell_\infty[\ell_\infty]} B \xrightarrow{\ell_1[\ell_M]} 
\end{array}
\]

where $B$ is a diagonal multiplication operator defined by a positive sequence $(b_i) \in \ell_1[\ell_M]$.

Furthermore $N_M^c(T) = \inf\{C \|B\|\|A\|\}$, infimum taken over all such factors.

Associated to $g_M$ and $g_M^c$, there are other important operator ideal.

Definition 3. Let $T \in \mathcal{L}(E, F)$, we say that $T$ is $M$-absolutely summing if exist a real number $C > 0$, such that for all sequences $(x_i)$ in $E$, with $\varepsilon_M((x_i)) < \infty$, it satisfies that

\[
\|T(x_i)\|_M \leq C\varepsilon_M((x_i))
\]
For $\mathcal{P}_M(E,F)$ we denote the Banach ideal of the $M$-absolutely summing operators $T : E \to F$ endowed with the topology of the norm $\mathcal{P}_M(T) := \inf C$, taking the infimum over all $C$ that satisfies (2.1)

**Theorem 4.** Let $E$ and $F$ be Banach spaces.\( \left(E \otimes_{\gamma_M} F\right)^{'} = \mathcal{P}_{\gamma_M}(F,E') \) isometrically.

### 3. $M$-integral operators

The characterization of maximal operator ideal obtained in [12] was given in terms of theory of finite representability of Banach spaces and/or Banach Lattices.

In present paper, we give another characterization of such ideals by considering the structure of finite dimensional subspaces of Orlicz spaces involved. The behavior of the Orlicz sequences spaces under ultraproducts is also crucial.

On ultraproducts of Banach spaces we refer to [8]. We only set the notation we will use. Let $D$ be a non empty index set and $\mathcal{U}$ a non-trivial ultrafilter in $D$. Given a family \( \{X_d, \ d \in D\} \) of Banach spaces, \( (X_d)_{\mathcal{U}} \) denotes the corresponding ultraproduct Banach space. If every $X_d$, $d \in D$, coincides with a fixed Banach space $X$ the corresponding ultraproduct is named an ultrapower of $X$ and is denoted by \((X)_{\mathcal{U}}\). Recall that if every $X_d$, $d \in D$ is a Banach lattice, \((X_d)_{\mathcal{U}}\) has a canonical order which makes it a Banach lattice. If we have another family of Banach spaces \( \{Y_d, \ d \in D\} \) and a family of operators \( \{T_d \in \mathcal{L}(X_d,Y_d), \ d \in D\} \) such that $\sup_{d \in D} \|T_d\| < \infty$, then \((T_d)_{\mathcal{U}} \in \mathcal{L}((X_d)_{\mathcal{U}}, (Y_d)_{\mathcal{U}})\) denotes the canonical ultraproduct operator.

We now give a local definition which has been inspired in Gordon and Lewis definition of local unconditional structure.

**Definition 5.** Given a sequence space $\lambda$, we say that a Banach space $X$ has an $S_k(\lambda)$-local unconditional structure if there exists a real constant $c > 0$ such that for every finite dimensional subspace $F$ of $X$, there is a section $S_n(\lambda)$ of $\lambda$ and linear operators $u : F \to S_n(\lambda)$ and $v : S_n(\lambda) \to X$ such that $\|u\|\|v\| \leq c$ and $vu = I_{F,X}$.

The constant $c$ which appears in above definition is called a $S_k(\lambda)$-local unconditional structure constant of $X$ and in this case we say that $X$ has $c$-$S_k(\lambda)$-local unconditional structure. If a Banach space $X$ has $c$-$S_k(\lambda)$-local unconditional structure for every $c > C$ we say that it has $C^+$-$S_k(\lambda)$-local unconditional structure.
The following definition was introduced by Pelczyński and Rosenthal [15] in 1975.

**Definition 6.** A Banach space $X$ has the uniform projection property if there is a $b > 0$ such that for each natural number $n$ there is a natural number $m(n)$ such that for every $n$-dimensional subspace $M \subset X$ there exists a $k$-dimensional and $b$-complemented subspace $Z$ of $X$ containing $M$ with $k \leq m(n)$.

The constant $b$ of the above definition is called a uniform projection property constant of $X$, and in this case we say that $X$ has the $b$-uniform projection property. If $X$ has the $b$-uniform projection property for every $b > B$ we say that $X$ has the $B^+$-uniform projection property.

We need to remark the following aspects involving Orlicz spaces.

The class of Banach spaces with the uniform projection property is quite large and includes, for example the reflexive Orlicz spaces, see [14]. In particular they have the $1 + \varepsilon$-uniform projection property for every $\varepsilon > 0$. Furthermore if $1 \leq p \leq \infty$ then, the Bochner space $L_p(\mu, E)$ and $\ell_p(E)$ has the $b$-uniform projection property if $E$ does, see [8]. We highlight that the uniform projection property is stable under ultrapowers, see [8]. Moreover from [17].

**Proposition 7.** If $\ell_M$ is reflexive, then every ultrapower of $\ell_1[\ell_M]$ ( of $\ell_M$) has $1^+-S_k(\ell_1)[S_k(\ell_M)]$-local unconditional structure (resp. $1^+-S_k(\ell_M)$-local unconditional structure).

According to the general theory of tensor norms and operator ideals, the normed ideal of $M$-integral operators $(\mathcal{I}_M, \mathcal{I}_M)$ is the maximal operator ideal associated to the tensor norm $g^c_M$ in the sense of Defant and Floret [4], or in an equivalent way, the maximal normed operator ideal associated to the normed ideal of $M$-nuclear operators in the sense of Pietsch [16]. From [4], for every pair of Banach spaces $E$ and $F$, an operator $T : E \to F$ is $M$-integral if and only if $J_F T \in (E \otimes g^c_M F)'$.

For every pair of Banach spaces $E, F$ we define $\mathcal{I}_M(T)$ to be the norm of $J_F T$ considered as an element of the topological dual of the Banach space $E \otimes g^c_M F'$. Remark that
I_M(T) = I_M(J_{F'} T) as a consequence of $F'$ be canonically complemented in $F''$.

First we give a non trivial example of $M$-integral operators.

**Theorem 8.** Let $(\Omega, \Sigma, \mu)$ a measure space and let $\ell_M$ be a reflexive Orlicz sequence space. Then every order bounded operator $S : L_\infty(\mu) \to \ell_M$ and every order bounded operator $S : L_\infty(\mu) \to \ell_1[\ell_M]$ are $M$-integral with $I_M(S) = \|S\|$.  

**Proof.** We will only give the proof if $S : L_\infty(\mu) \to \ell_M$ is an order bounded operator since the proof in the other case is similar.

The predual space of $\ell_M$ is $\ell_M^*$, which is regular space because $M^*$ has the $\Delta_2$ property at zero. Then, the linear span $T$ of the set $\{e_i, i \in \mathbb{N}\}$ is dense in $\ell_M^*$ and by the representation theorem of maximal operator ideals (see 17.5 in [4]) and the density lemma (theorem 13.4 in [4]) we only have to see that $S \in (L_\infty(\mu) \otimes g_{\ell_M^*}^*)'$.  

Given $z \in L_\infty(\mu) \otimes g_{\ell_M^*}^T$ and $\varepsilon > 0$, let $X$ and $Y$ be finite dimensional subspaces of $L_\infty(\mu)$ and $T$ respectively such that $z \in X \otimes Y$ and

$$g_{\ell_M}(z; X \otimes Y) \leq g_{\ell_M}(z; L_\infty(\mu) \otimes T) + \varepsilon.$$  

Let $\{g_s\}_{s=1}^m$ be a basis for $Y$ and let $k \in \mathbb{N}$ be such that $1 \leq s \leq m$ $g_s = \sum_{i=1}^k c_{si}e_i$. Then $\forall f \in X$, $\forall 1 \leq s \leq m$

$$\langle S, f \otimes g_s \rangle = \langle f, S'(g_s) \rangle = \langle f, \left( \sum_{i=1}^k c_{si} \right) S'(e_i) \rangle = \left( f \otimes \sum_{j=1}^k c_{sj} e_j, \sum_{i=1}^k S'(e_i) \otimes e_i \right).$$

Then if $U$ denotes the tensor $U := \sum_{i=1}^k S'(e_i) \otimes e_i \in L_\infty(\mu)' \otimes \lambda$, by bilinearity we get $\forall z \in X \otimes Y$ $\langle z, S \rangle = \langle U, z \rangle$.  

Given $\nu > 0$, for every $1 \leq i \leq k$ there is $f_i \in L_\infty(\mu)$ such that $\|f_i\| \leq 1$ and $\|S'(e_i)\| \leq ||S'(e_i), f_i|| + \nu$. Then $f := \sup_{1 \leq i \leq k} f_i$ lies in the closed unit ball of $L_\infty(\mu)$. On the other hand, $\ell_M$ is a dual lattice and hence it is order complete. By the Riesz-Kantorovich theorem (see theorem 1.13 in [1] for instance), the modulus $|S|$ of the operator $S$ exists in $\mathcal{L}(L_\infty(\mu), \ell_M)$. By the lattice properties of $\ell_M$ we have

$$\pi_M((S'(e_i))) = \pi_M \left( \sum_{i=1}^k \|S'(e_i)\| e_i \right) \leq \pi_M \left( \sum_{i=1}^k |S'(e_i), f_i| e_i \right)$$

for every $f \in X$.
\[ \pi_M \left( \sum_{i=1}^{k} e_i \right) \leq \pi_M \left( \sum_{i=1}^{k} |(S(f_i), e_i)| e_i \right) + \nu \pi_M \left( \sum_{i=1}^{k} e_i \right) \]

\[ \leq \pi_M \left( \sum_{i=1}^{k} |(S(f_i), e_i)| \right) + \nu \pi_M \left( \sum_{i=1}^{k} e_i \right) \leq \pi_M \left( \sum_{i=1}^{k} \langle |S(f_i)|, e_i \rangle \right) + \nu \pi_M \left( \sum_{i=1}^{k} e_i \right) = \]

\[ \pi_M (|S(|f|)|) + \nu \pi_M \left( \sum_{i=1}^{k} e_i \right) \leq ||S|| + \nu \pi_M \left( \sum_{i=1}^{k} e_i \right). \]

Moreover \( \varepsilon_{M^*}(\{(e_i)_{i=1}^{k}\}) \leq 1. \) Hence, denoting by \( I_X \) and \( I_Y \) the corresponding inclusion maps into \( L_{\infty}(\mu) \) and \( \ell_H \) respectively, we have

\[ |\langle S, z \rangle| = |\langle U, z \rangle| = |\langle U, ((I_X)' \otimes (I_Y)'')(z) \rangle| \leq \]

\[ \leq g_M(U; X \otimes Y') g_M'(((I_X)' \otimes (I_Y)'')(z); X' \otimes Y'') \leq \]

\[ \leq g_M(U; X \otimes Y') g_M'(((I_X)' \otimes (I_Y)'')(z); X' \otimes Y'') \leq \]

\[ \leq (g_M(U; \ell_\infty \otimes (\ell_M)) + \varepsilon) g_M'(z; \ell_\infty(\mu) \otimes \ell_{M^*}) \leq \]

\[ \leq g_M'(z; \ell_\infty(\mu) \otimes \ell_{M^*} \left( \pi_M((S'(e_i)) \varepsilon_{M^*}((e_i)) + \varepsilon \right) \]

\[ \leq g_M'(z; \ell_\infty(\mu) \otimes \ell_{M^*}) \left( ||S|| + \nu \pi_M \left( \sum_{i=1}^{k} e_i \right) + \varepsilon \right) \]

and \( \nu \) being arbitrary \( |\langle S, z \rangle| \leq g_M'(z; \ell_\infty(\mu) \otimes \ell_{M^*}) (||S|| + \varepsilon). \) Finally, by since \( \varepsilon \) is arbitrary we get \( |\langle S, z \rangle| \leq g_M'(z; \ell_\infty(\mu) \otimes \ell_{M^*} ||S||. \) But from [1] theorem 1.10, \( |S|(\chi_\Omega) = \sup\{|S(f)|; |f| \leq \chi_\Omega \} \) and as \( \ell_M \) is order continuous

\[ ||S|| = ||S|(\chi_\Omega)|| = \sup\{||S(f)|| ; ||f|| \leq 1 \} = ||S||. \]

Then \( S \) is \( M \)-integral with \( I_M(S) \leq ||S||. \) But as \( (I_M, I_M) \) is a Banach operators ideal, \( ||S|| \leq I_M(S) \), hence \( I_M(S) = ||S||. \)
Corollary 9. Let \((\Omega, \Sigma, \mu)\) be a measure space and \(n, k \in \mathbb{N}\). Then every operator \(T : L_\infty(\mu) \to S_k(\ell_M)\) and every operator \(T : L_\infty(\mu) \to S_n(\ell_1)[S_k(M)]\) satisfy that \(I_M(T) = \|T\|\).

Proof. The result follows easily from theorem 3, since every operator \(T : L_\infty(\mu) \to S_k(\ell_M)\) (resp. \(T : L_\infty(\mu) \to S_n(\ell_1)[S_k(M)]\)) in the other case is order bounded and \(S_k(\ell_M)\) is reflexive hence order continuous.

For our next theorem we need a very deep technical result of Lindenstrauss and Tzafriri [14] which gives us a kind of ”uniform approximation” of finite dimensional subspaces by finite dimensional sublattices in Banach lattices.

Lemma 10. Let \(\varepsilon > 0\) and \(n \in \mathbb{N}\) be fixed. There is a natural number \(h(n, \varepsilon)\) such that for every Banach lattice \(X\) and every subspace \(F \subset X\) of dimension \(\dim(F) = n\) there are \(h(n, \varepsilon)\) disjoint elements \(\{z_i, 1 \leq i \leq h(n, \varepsilon)\}\) and an operator \(A\) from \(F\) into the linear span \(G\) of \(\{z_i, 1 \leq i \leq h(n, \varepsilon)\}\) such that

\[
\forall x \in F \quad \|A(x) - x\| \leq \varepsilon \|x\|.
\]

Theorem 11. Let \(\ell_M\) be a regular Orlicz sequence space, \(G\) an abstract \(M\)-space, and \(X\) a Banach space with \(c\)-\(S_k(\ell_M)\) or \(c\)-\(S_n(\ell_1)[S_n(\ell_M)]\)-local uniform structure . Then every operator \(T : G \to X\) is \(M\)-integral and \(I_M(T) \leq c \|T\|\).

Proof. We will prove the case where \(X\) has \(c\)-\(S_k(\ell_M)\)-local unconditional structure since the other case is similar. By the representation theorem of maximal operator ideals (see 17.5 in [4]), we only need to show that \(J_X T \in (G \otimes g_M^' X')'\).

Given \(z \in G \otimes X'\) and \(\varepsilon > 0\), let \(P \subset G\) and \(Q \subset X'\) be finite dimensional subspaces and let \(z = \sum_{i=1}^n f_i \otimes x'_i\) be a fixed representation of \(z\) with \(f_i \in P\) and \(x'_i \in Q\), \(i = 1, 2, \ldots, n\) such that

\[
g_M^'(z; G \otimes X') \leq g_M^'(z; P \otimes Q) \leq g_M^'(z; G \otimes X') + \varepsilon.
\]

From lemma 10 we have a finite dimensional sublattice \(P_1\) of \(G\) and an operator \(A : P \to P_1\) so that \(\forall f \in P\), \(\|A(f) - f\| \leq \varepsilon\|f\|\). Then, if \(id_G\) denotes the identity map on \(G\) we have

\[
\|J_X T z\| = \left\| \sum_{i=1}^n (T(f_i), x'_i) \right\| \leq \sum_{i=1}^n \left( T(id_G - A)(f_i), x'_i \right) + \left\| \sum_{i=1}^n T A(f_i), x'_i \right\|
\]
Let $X_1 := T(P_1)$. As $X$ has $S_k(\ell_M)$-local unconditional structure, hence there are $k \in \mathbb{N}$, $u : X_1 \to S_k(\ell_M)$ and $v : S_k(\ell_M) \to X$ such that $I_{X_1,X} = v u$ and $\|u\| \|v\| \leq c$. Let $X_2 := v u(X_1)$ which is a finite dimensional subspace of $X$ containing $X_1$ and $I_{X_1,X_2} = v u$. Put $K_2 : X'' \to X_2'' = X''/X_2^\circ$ be the canonical quotient map. Then

$$
\sum_{i=1}^{n} \langle T(A(f_i)), x'_i \rangle = \sum_{i=1}^{n} \langle I_{X_1,X_2} T(A(f_i)), K_2(x'_i) \rangle = \sum_{i=1}^{n} \langle v u T(A(f_i)), K_2(x'_i) \rangle = \sum_{i=1}^{n} \langle u T(A(f_i)), v' K_2(x'_i) \rangle = \langle u T, \sum_{i=1}^{n} A(f_i) \otimes v' K_2(x'_i) \rangle
$$

with $\sum_{i=1}^{n} A(f_i) \otimes v' K_2(x'_i) \in P_1 \otimes (S_k(\ell_M))'$ and $u T : P_1 \to S_k(\ell_M)$. Since $P_1$ is a reflexive abstract $M$-space it is lattice isometric to some $L_\infty(\mu)$ space, hence by corollary 9 this map is $M$-integral with $I_M(u T) \leq \|u\| \|T\|$. Then

$$
\left| \sum_{i=1}^{n} \langle T(A(f_i)), x'_i \rangle \right| = \left| \langle u T, \sum_{i=1}^{n} A(f_i) \otimes v' K_2(x'_i) \rangle \right| \leq I_M(u T) g_M' \left( \sum_{i=1}^{n} A(f_i) \otimes v' K_2(x'_i) ; P_1 \otimes S_k(\ell_M) \right) \leq \|u\| \|T\| \|A\| \|v'\| \|K_2\| g_M(z ; P \otimes Q) \leq (1 + \varepsilon) c \|T\| g_M'(z ; P \otimes Q) \leq (1 + \varepsilon) c \|T\| (g_M'(z ; G \otimes X') + \varepsilon)
$$

and since $\varepsilon$ is arbitrary we obtain $|\langle J_X T, z \rangle| \leq c \|T\| g_M'(z ; G \otimes X')$.

Concerning to characterization theorem of $M$-integral operators we have:

**Theorem 12.** Let $\ell_M$ be a regular Orlicz sequence space and let $E$ and $F$ be Banach spaces. The following statements are equivalent:

1) $T \in \mathcal{I}_M(E,F)$.

2) $J_F T$ factors continuously in the following way:

On operator ideals defined by a reflexive Orlicz sequence space

\[ E \xrightarrow{J_FT} F'' \]
\[ \xrightarrow{A} \]
\[ L_\infty(\mu) \xrightarrow{B} X \]

where \( X \) is an ultrapower of \( \ell_1[\ell_M] \) and \( B \) is a lattice homomorphism. Furthermore \( \mathbf{I}_M(T) \) is equivalent to \( \inf\{\|D\|\|B\|\|A\|\} \), taking it over all such factors.

**Proof.** 1) \( \implies \) 2). Let \( D := \{ (P,Q) : P \in \text{FIN}(E), Q \in \text{FIN}(F') \} \) where \( \text{FIN}(Y) \) is the set of finite dimensional subspace of a Banach space \( Y \), endowed with the natural inclusion order

\[ (P_1, Q_1) \leq (P_2, Q_2) \iff P_1 \subset P_2, \ Q_1 \subset Q_2. \]

For every \( (P_0, Q_0) \in D, R(P_0, Q_0) := \{ (P,Q) \in D : (P_0, Q_0) \subset (P,Q) \} \) and \( R = \{ R(P,Q), (P,Q) \in D \} \). \( R \) is filter basis in \( D \), and according to Zorn’s lemma, let \( D \) be an ultrafilter on \( D \) containing \( R \). If \( d \in D, P_d \) and \( Q_d \) denote the finite dimensional subspaces of \( E \) and \( F' \) respectively so that \( d = (P_d, Q_d) \). For every \( d \in D \), if \( z \in P_d \otimes Q_d, J_FT|_{P_d \otimes Q_d} \in (P_d \otimes g_M Q_d)' = M_d' \otimes g_M Q_d' = N_M(P_d, Q_d') \). Then from theorem 2 of characterization of \( M \)-nuclear operators, \( J_FT|_{P_d \otimes Q_d} \) factors as

\[ P_d \]
\[ \xrightarrow{A_d} \]
\[ \ell_\infty[\ell_\infty] \]
\[ \xrightarrow{B_d} \]
\[ \ell_1[\ell_M] \]

where \( B_d \) is a positive diagonal operator and \( \|A_d\| \|B_d\| \|C_d\| \leq \mathbf{N}_M(T|_{P_d \otimes Q_d}) + \varepsilon = \mathbf{I}_M(T|_{P_d \otimes Q_d}) + \varepsilon. \) Then

\[ \|A_d\| \|B_d\| \|C_d\| \leq \mathbf{I}_M(T|_{P_d \otimes Q_d}) + \varepsilon \leq \mathbf{I}_M(T) + \varepsilon \]

Without loss of generality we can suppose that \( \|A_d\| = \|C_d\| = 1 \). We define \( W_E : E \rightarrow (M_d)_D \) such that \( W_E(x) = (x_d)_D \) so that \( x_d = x \) if \( x \in M_d \)
and $x_d = 0$ if $x \notin M_d$. In the same way we define $W_{F'} : F' \to (Q_d)_D$ such that $W_{F'}(a) = (a_d)_D$ so that $a_d = a$ if $a \in Q_d$ and $a_d = 0$ if $a \notin Q_d$. Then we have the following commutative diagram:

$$
\begin{array}{c}
E & \xrightarrow{J_F T} & F'' \\
| & | & | \\
W_E & \downarrow & W'_{F'} \\
| & | & | \\
(\ell_\infty[\ell_\infty])_D & \xrightarrow{(Q_d)_D} & ((Q_d)_D)' \\
| & | & | \\
(\ell_1[\ell_M])_D & \xrightarrow{C_D} & (C_d)_D \\
| & | & | \\
(A_d)_D & \xrightarrow{(J_F T)_{P_d \otimes Q_d}} & (Q_d)_D \\
| & | & | \\
(P_d)_D & \xrightarrow{(P_d)_D} & (Q_d)_D \\
\end{array}
$$

where $I$ is the canonical inclusion map. As in [14] $((\ell_1[\ell_M])_D)'''$ is a 1-complemented subspace of some ultrapower $((\ell_1[\ell_M])_D)_U$ which from [19] is another ultrapower $(\ell_1[\ell_M])_{\mu_1}$ with projection $Q$, the result follows with $A = (A_d)_D$, $B = ((B_d)_D)'''$ which is a lattice homomorphism, $C = P_{F''} I (C_d)_D$ and $X = (\ell_1[\lambda])_{\mu_1}$, having in mind that $(\ell_\infty[\ell_\infty])_D$ is an abstract $M$-space, there is a measure space such that $L\infty(\mu) = ((\ell_\infty[\ell_\infty])_D)'''$, where equality means that the spaces are lattice isometric.

2) $\implies$ 1) As $(I_M, I_M)$ is an operator ideal, it follows easily from theorem 3 and proposition 3.

The following new formulation of the preceding characterization theorem is needed in our context:

**Theorem 13.** Let $\ell_M$ be an Orlicz space. For every pair of Banach spaces $E$ and $F$, the following statements are equivalent:

1) $T \in I_M(E, F)$.

2) There exists a $\sigma$-finite measure space $(\mathcal{O}, \mathcal{S}, \nu)$ and a Köthe function space $\mathcal{K}(\nu)$ which is complemented in a space with $S_k(\ell_1)[S_n(\ell_M)]$-local unconditional structure, such that $J_F T$ factors continuously in the following way:
where $B$ is a multiplication operator for a positive function of $K(\nu)$. Furthermore $I_M(T) = \inf\{\|C\|\|B\|\|A\|\}$, taking the infimum over all such factors.

**Proof.** Starting from the theorem 12, as $\ell_1[\ell_M]$ has finite cotype, it is order continuous ([7], 4.6), and for [13], theorem 1.a.9 $\ell_1[\ell_M]$ can be decomposed into an unconditional direct sum of a family of mutually disjoint ideals $\{X_h, h \in H\}$ having a positive weak unit, and then from 1.b.14 in [13], as every $X_h$ is order isometric to a Köthe space of functions defined on a probability space $(\Omega_h, S_h, \nu_h)$, then $(\ell_1[\ell_M])_H$ is contained in the unconditional direct sum of $\{X_h, i \in N\}$ which is order isometric to a space of Köthe function space $K(\nu^1)$ over a measure space $(\Omega^1, S^1, \nu^1)$, hence we can substitute $(\ell_1[\ell_M])_H$ for $K(\nu^1)$ in 12. If we denote $z := B(\chi_\Omega)$ with $z = \sum_{i=1}^\infty y_{hi}$ with $y_{hi} \in X_{hi}$ for every $i \in N$, then $B(L_\infty(\mu))$ is contained in the unconditional direct sum of $\{X_{hi}, i \in N\}$ which is order isometric to a space of Köthe function space $K(\nu)$ over a $\sigma$-finite measure space $(\Omega, S, \nu)$ which is 1-complemented in $K(\nu^1)$.

Now, since $K(\nu)$ is order complete, there exists $g := \sup_{\|f\|_{L_\infty(\nu)}} B(f)$ in $K(\nu)$. Then the operators $B_1 : L_\infty(\mu) \to L_\infty(\nu)$ and $B_2 : L_\infty(\nu) \to K(\nu)$, such that $B_1(f)(\omega) := B(f)(\omega)/g(\omega)$, for all $f \in L_\infty(\mu)$, $\omega \in \Omega$ with $g(\omega) \neq 0$ and $B_1(f)(\omega) = 0$ otherwise, and $B_2(h)(\omega) := g(\omega)h(\omega)$ for all $h \in L_\infty(\nu)$, $\omega \in \Omega$, satisfy that $B = B_2B_1$ and $B_2$ is a multiplication operator for a positive element $g \in K(\nu)$.

**4. On equality between $M$-nuclear and $M$-integral operators**

Finally, using the preceding characterization theorems we give some properties of $M$-nuclear and $M$-integral operators. Let us establish now a necessary condition for equality between components of $M$-nuclear and $M$-integral operator ideals. First, we introduce a new operator ideal, which is contained in the ideal of the $M$-integral operators.
Definition 14. Given $E$ and $F$ Banach spaces, let $\ell_M$ be a Orlicz sequence. We say that $T \in \mathcal{L}(E,F)$ is **strictly $M$-integral** if exist a $\sigma$-finite measure space $(\mathcal{O}, \mathcal{S}, \nu)$ and a Köthe function space $\mathcal{K}(\nu)$ which is complemented in a space with $S_k(\ell_1)[S_n(\ell_M)]$-local unconditional structure, such that $T$ factors continuously in the following way:

![Diagram](https://example.com/diagram.png)

where $B$ is a multiplication operator for a positive function of $\mathcal{K}(\nu)$. endowed with the topology of the norm $\mathbf{SI}_M(T) = \mathbf{I}_M(T)$.

Obviously, if $F$ is a dual space, or it is complemented in its bidual space, then $\mathbf{SI}_M(E,F) = \mathbf{I}_M(E,F)$.

Theorem 15. Let $\ell_H$ be a Orlicz sequence space, and let $E$ and $F$ be Banach spaces, such that $E'$ satisfies the Radon-Nikodým property then, $\mathcal{N}_M(E,F) = \mathbf{SI}_M(E,F)$.

Proof. Let $T \in \mathbf{SI}_M(E,F)$ Were $E'$ has the Radon-Nikodým property and.

a) First, we suppose that $B$ is an multiplication operator for a function $g \in \mathcal{K}(\nu)$ with finite measure support $D$. We denote $\nu_D$ the restriction of $\nu$ to $D$.

As $(\chi_D A) : E \to L_\infty(\nu_D)$, then $(\chi_D A)' : (L_\infty(\nu_D))' \to E'$ and the restriction of $(\chi_D A)'_{L_1(\nu_D)} : L_1(\nu_D) \to E'$, thus, for every $x \in E$ and $f \in L_1(\nu_D)$

$$<x,(\chi_D A)'(f)> = <\chi_D A(x) , f> = \int_D \chi_D A(x) f d(\nu_D).$$

As $E'$ has the Radon-Nikodým property, by III(5) of [2], we have that $(\chi_D A)'$ has a Riesz representation, therefore exist a function $\phi \in L_\infty(\nu_D, E')$ such that for every $f \in L_1(\nu_D)$

$$(\chi_D A)'(f) = \int_D f \phi d(\nu_D).$$
Then, for every $x \in E$, we have that $\chi_{D}(A)(x) = \langle \phi(t), x \rangle$, $\nu_{D}$-almost everywhere in $D$, and then $B(\chi_{D}(A)) = \langle g\phi(.), x \rangle$, $\nu_{D}$-almost everywhere in $D$. Let $g\phi$ this last operator, and we can consider it as $K(\nu_{D}, E')$ element.

As the simple functions are dense in $K(\nu_{D}, E')$, $g\phi$ can be approximated by a sequence of simple functions $((S_{k})_{k=1}^{\infty})$.

We suppose $S_{k} = \sum_{j=1}^{m_{k}} x_{kj} \chi_{A_{kj}}$, where $\{ A_{kj} : i = 1, \ldots, m_{k} \}$ is a family of $\nu$-measure set of $\Omega$ pairwise disjoint. For each $k \in \mathbb{N}$, we can interpret $S_{k}$ as a map $S_{k} : E \to K(\nu)$ such that $S_{k}(x) = \sum_{j=1}^{m_{k}} x_{kj}, x > \chi_{A_{kj}}$ with norm less or equal than the norm of $S_{k}$ in $K(\nu, E')$.

Clearly for all $k \in \mathbb{N}$, $S_{k}$ is $M$-nuclear since it has finite rank, but we need to evaluate its $M$-nuclear norm coinciding with it $M$-integral norm. Let $S_{k}^{1} : E \to L_{\infty}(\nu)$ be such that

$$S_{k}^{1}(x) = \sum_{j=1}^{m_{k}} \frac{\langle x_{kj}, x \rangle}{\| x_{kj} \|} \chi_{A_{kj}}$$

and let $S_{k}^{2} : L_{\infty}(\nu) \to K(\nu)$ be such that $S_{k}^{2}(f) = \sum_{j=1}^{m_{k}} \| x_{kj} \| f(\chi_{A_{kj}})$.

Then $\| S_{k}^{1} \| \leq 1$ and $\| S_{k}^{2} \| \leq \| S_{k} \|_{K(\nu, E')}$ and $S_{k} = S_{k}^{2} S_{k}^{1}$. But as $K(\nu)$ is a complemented subspace of space with $S_{k}(\ell_{1})(S_{n}(\ell_{1}))$-local unconditional structure, from 11, there is $K > 0$ such that $\| S_{k}^{1} \|_{K(\nu, E')}$, hence $N_{M}(S_{k}^{2}) \leq K \| S_{k}^{2} \|_{K(\nu, E')}$, hence $N_{M}^{\infty}(S_{k}) \leq K \| S_{k} \|_{K(\nu, E')}$.

Then, as $(S_{k})_{k=1}^{\infty}$ converges in the $K(\nu_{D}, E')$ space, it is a Cauchy sequence in $N_{M}(E, K(\nu_{D}))$, and since this is complete, $(S_{k})_{k=1}^{\infty}$ converges to $g\phi$, that is to say, $g\phi \in N_{M}(E, K(\nu_{D}))$. Therefore, $g\phi = B\chi_{D}A$ is $M$-nuclear and so $T$ is also $M$-nuclear.

b) Now, if $g$ is any element of $K(\nu)$, $g$ it can be approximated in norm by means of a sequence $(t_{n})_{n=1}^{\infty}$ of simple functions with finite measure support, and therefore by a), the sequence $T_{n} = CB_{t_{n}}A$ is a Cauchy sequence in $N_{M}(E, F)$ converging to $T$ in $L(E, F)$, and then $T \in N_{M}(E, F)$.

As consequence of the former result and of the factorization theorems 13 and 2, we obtain the following metric properties of $g_{M}^{\infty}$ and $(g_{M}^{\infty})'$.

**Theorem 16.** $(g_{M}^{\infty})'$ is a totally accessible tensor norm.

**Proof.** Since $(g_{M}^{\infty})'$ is finitely generated, it is sufficient to prove that the map $F \otimes (g_{M}^{\infty})' E \to \mathcal{M}^{*}(E', F'')$, is isometric.

Let $z = \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij} \otimes x_{ij} \in F \otimes (g_{M}^{\infty})' E$, and let $H_{z} \in \mathcal{M}^{*}(E', F'')$ be the canonical map associated to $z$, that is to say,
\[ H_\alpha(x') = \sum_{i=1}^{n} \sum_{j=1}^{l_i} (x_{ij}, x') y_{ij} \text{ for all } x' \in E', \text{ con } H_\alpha \in \mathcal{L}(E', F) \subset \mathcal{L}(E', F''). \]

Applying the theorem 15.5 of [4] for \( \alpha = (g_M')' \), the theorem 4, and the equality \((g_M')'' = g_M'\) since \(g_M'\) finitely generated, we have that inclusion

\[ F \otimes_{(g_M')'} E \hookrightarrow \left( F' \otimes_{g_M'} E' \right)' \rightarrow P_{M'}(E', F'') \]

is an isometry, and therefore by proposition 12.4 in [4] we obtain

\[ P_{M'}(H_\alpha) = (g_M')'(z; F \otimes E) \leq (g_M')'(z; F \otimes E). \]

Now, given \( N \), a finite dimensional subspace of \( F \) such that \( z \in N \otimes (g_M')' E \), there exists \( V \in (N \otimes (g_M')') E' = \mathcal{T}(N, E') \) such that \( \mathbf{I}_M(V) \leq 1 \) and \((g_M')'(z; N \otimes E) = \langle z, V \rangle \). Clearly enough \( V \in \mathbf{S} \mathcal{T}_M(N, E') = \mathcal{T}_M(N, E') \) because \( E' \) is a dual space, and \( N' \), being finite dimensional, has the Radon-Nikodým property. Therefore by theorem 15, \( V \in \mathcal{N}_M(N, E') \) and by theorem 2, given \( \epsilon > 0 \), there is a factorization \( V \) in the way

\[
\begin{array}{ccc}
N & \xrightarrow{T} & E' \\
\downarrow & & \downarrow \\
A & \xrightarrow{W} & \ell_1[\ell_M] \\
\end{array}
\]

such that \( \|C\|\|B\|\|A\| \leq N_M(V) + \epsilon = \mathbf{I}_M(V) + \epsilon \leq 1 + \epsilon. \)

As \( \ell_\infty[\ell_\infty] \) has the extension metric property, (to see proposition 1, C.3.2. in [16]), \( A \) can be extended to a continuous map \( \overline{A} \in \mathcal{L}(F, \ell_\infty[\ell_\infty]) \) such that \( \|\overline{A}\| = \|A\| \). By theorem 2 again, \( W := CBA \) is in \( \mathcal{N}_M(F, E') \), so there is a representation \( w =: \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} y_i' \otimes x_{ij}' \in F' \otimes_{g_M'} E' \) of \( W \) verifying

\[ \sum_{i=1}^{\infty} \pi_M \left( (y_i') \right) e_M' \left( \left( x_{ij}' \right) \right) \leq N_M'(W) + \epsilon \leq \|C\|\|B\|\|A\| + \epsilon \leq 1 + 2\epsilon. \]

Then, \((g_M')'(z; F \otimes E) \leq (g_M')'(z; N \otimes E) = \langle z, V \rangle = \langle z, W \rangle\) it follows that

\[ (g_M')'(z; F \otimes E) \leq g_M'(w) P_{M'}(H_\alpha) \leq (1 + 2\epsilon) P_{M'}(H_\alpha) \]

whence \((g_M')'(z; F \otimes E) \leq P_{M'}(H_\alpha)\), and the equality is obvious.
Finally, as consequence of the former theorem and of proposition 15.6 of [4], we have:

**Corollary 17.** $g_M^c$ is an accessible tensor norm.

**References**


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