Abstract

The object of this paper is to establish some generating relations by using operational formulae for a class of polynomials $T_{kn}^{(a+s-1)}(x)$ defined by Mittal. We have also derived finite summation formulae for (1.6) by employing operational techniques. In the end several special cases are discussed.

Key Words: Operational formulae; generating relations; finite sum formulae.

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1. Introduction

Chak [1] defined a class of polynomials as:

\[ G_{n,k}^{(\alpha)}(x) = x^{-\alpha-kn+n}e^x(x^kD)^n[x^\alpha e^{-x}] \]

where \( D = \frac{d}{dx} \), \( k \) is constant and \( n = 0, 1, 2, \ldots \).

Chatterjea [2] studied a class of polynomials for generalized Laguerre polynomial as:

\[ T_{\alpha}^{(r)}(x,p) = \frac{1}{n!}x^{-\alpha-n-1}\exp(px^r)(x^2D)^n[x^\alpha \exp(-px^r)]. \]

Gould and Hopper [3] introduced generalized Hermite polynomials as:

\[ H^{(r)}_n(x,a,p) = (-1)^nx^{-a}\exp(px^r)D^n[x^\alpha \exp(-px^r)]. \]

Singh [10] obtained generalized Truesdell polynomials by using Rodrigues formula, which is defined as:

\[ T_{\alpha}^{(s)}(x,r,p) = x^{-\alpha}\exp(px^r)(xD)^n[x^\alpha \exp(-px^r)]. \]

In 1971, Mittal [5] proved the Rodrigues formula for a class of polynomials \( T_{kn}^{(\alpha)}(x) \) as:

\[ T_{kn}^{(\alpha)}(x) = \frac{1}{n!}x^{-\alpha}\exp\{p_k(x)\}D^n[x^\alpha \exp\{-p_k(x)\}] \]

where \( p_k(x) \) is a polynomial in \( x \) of degree \( k \).

Mittal [6] also proved the following relation for (1.5)

\[ T_{kn}^{(\alpha+s-1)}(x) = \frac{1}{n!}x^{-\alpha-n}\exp\{p_k(x)\}\theta^n[x^\alpha \exp\{-p_k(x)\}] \]

and an operator \( \theta \equiv x(s + xD) \), where \( s \) is constant.

The following well-known facts are prepared for studying (1.6).

**Generalised Laguerre polynomials** (Srivastava and Manocha[12]) defined as:

\[ L^{(\alpha)}_n(x) = \frac{x^{-\alpha-n-1} e^x}{n!} (x^2D)^n[x^\alpha+1 e^{-x}]. \]
Hermite polynomials (Rainville [9]) defined as:

\[ H_n(x) = (-1)^n \exp(x^2) D^n[\exp(-x^2)]. \]  

(1.8)

Konhauser polynomials of first kind (Srivastava [11]) defined as:

\[ Y^\alpha_n(x; k) = \frac{x^{-\alpha-1} \exp(x)}{k^n n!} (x^{k+1}D)^n[x^\alpha+1 e^{-x}]. \]  

(1.9)

Konhauser polynomials of second kind (Srivastava [11]) defined as:

\[ Z^\alpha_n(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + \alpha + 1)}. \]  

(1.10)

where \( k \) is a positive integer.

Srivastava and Manocha [12] verified following result by using induction method,

\[ (x^2D)^n\{f(x)\} = x^{n+1}D^n\{x^{n-1}f(x)\}. \]  

(1.11)

2. Definitions and Notations

McBride [4] defined generating function as:

Let \( G(x, t) \) be a function that can be expanded in powers of \( t \) such that

\[ G(x, t) = \sum_{n=0}^{\infty} c_n f_n(x) t^n, \]  

where \( c_n \) is a function of \( n \) that may contain the parameters of the set \( \{f_n(x)\} \), but is independent of \( x \) and \( t \). Then \( G(x, t) \) is called a generating function of the set \( \{f_n(x)\} \).

Remark: A set of functions may have more than one generating function.

In our investigation we used the following properties of the differential operators;

\[ \theta \equiv x(s + xD) \]  
\[ \theta_1 \equiv (1 + xD), \] where \( D \equiv \frac{d}{dx} \) (Mittal [7], Patil and Thakare [8]) which are useful to establish linear generating relations and finite sum formulae.
(2.1) \[ \theta^n = x^n(s + xD)(s + 1 + xD)(s + 2 + xD) \ldots (s + (n - 1) + xD) \]

(2.2) \[ \theta^n(x^\alpha) = (\alpha + s)_{n} x^{\alpha+n} \]

(2.3) \[ \theta^n(xuv) = x \sum_{m=0}^{\infty} \binom{n}{m} \theta^{n-m}(v)\theta^{m}_{1}(u) \]

(2.4) \[ e^{t\theta}(x^\alpha) = x^{\alpha}(1 - xt)^{-(\alpha+s)} \]

(2.5) \[ e^{t\theta}(xuv) = xe^{t\theta}(v)e^{t\theta}_{1}(u) \]

(2.6) \[ e^{t\theta}(x^{\alpha}f(x)) = x^{\alpha}(1 - xt)^{-(\alpha+s)} f\left[ x(1 - xt)^{-1} \right] \]

(2.7) \[ e^{t\theta}(x^{\alpha-n}f(x)) = x^{\alpha}(1 + t)^{-1+(\alpha+s)} f\left[ x(1 + t) \right] \]

(2.8) \[ (1 - at)^{-\alpha/a} = (1 - at)^{-\beta/a} \sum_{m=0}^{\infty} \binom{\alpha - \beta}{a} \frac{(at)^m}{m!} \]

3. Generating Relations

We obtained some generating relations of (1.6) as

(3.1) \[ \sum_{n=0}^{\infty} T_{kn}^{(\alpha+s-1)}(x)t^n = (1 - t)^{-(\alpha+s)} \exp[p_k(x) - p_k\{x(1 - t)^{-1}\}] \]
\[
\sum_{n=0}^{\infty} T_{kn}^{(\alpha-n+s-1)}(x)t^n = (1 + t)^{-1+(\alpha+s)} \exp[p_k(x) - p_k\{x(1 + t)\}]
\]

(3.2)

\[
\sum_{m=0}^{\infty} \binom{m + n}{n} T_{k(n+m)}^{(\alpha+s-1)}(x) t^m
= (1 - t)^{-(\alpha+s+n)} \exp[p_k(x) - p_k\{x(1 - t)^{-1}\}] T_{kn}^{(\alpha+s-1)}\{x(1 - t)^{-1}\}
\]

(3.3)

\[
\sum_{m=0}^{\infty} \binom{m + n}{n} T_{k(n+m)}^{(\alpha-m+s-1)}(x) t^m
= (1 + t)^{\alpha+s-1} \exp[p_k(x) - p_k\{x(1 + t)\}] T_{kn}^{(\alpha-m+s-1)}\{x(1 + t)\}
\]

(3.4)

**Proof of (3.1).** From (1.6), we consider

\[
\sum_{n=0}^{\infty} x^n T_{kn}^{(\alpha+s-1)}(x)t^n = x^{-\alpha} \exp\{p_k(x)\} e^{\theta \alpha} \exp\{-p_k(x)\}
\]

and using (2.6), above equation reduces to,

\[
\sum_{n=0}^{\infty} x^n T_{kn}^{(\alpha-s+1)}(x)t^n = x^{-\alpha} \exp\{p_k(x)\} x^\alpha (1-xt)^{-(\alpha+s)} \exp[-p_k\{x(1-xt)^{-1}\}]
\]

\[
= (1 - xt)^{-(\alpha+s)} \exp[p_k(x) - p_k\{x(1 - xt)^{-1}\}]
\]

replacing \(t\) by \(t/x\), which gives (3.1).
Proof of (3.2). From (1.6) we consider,

\[ T_{kn}^{(\alpha-n+s-1)}(x) = \frac{1}{n!} x^{-(\alpha-n)-n} \exp\{p_k(x)\} \theta^n \left[ x^{\alpha-n} \exp\{-p_k(x)\} \right] \]

or

\[ \sum_{n=0}^{\infty} T_{kn}^{(\alpha-n+s-1)}(x)t^n = (x)^{-\alpha} \exp\{p_k(x)\} e^{t\theta} \left[ x^{\alpha-n} \exp\{-p_k(x)\} \right] \]

by using (2.7), we get

\[ \sum_{n=0}^{\infty} T_{kn}^{(\alpha-n+s-1)}(x)t^n = x^{-\alpha} \exp\{p_k(x)\} x^{\alpha}(1+t)^{-1+(\alpha+s)} \exp\{-p_k\{x(1+t)\}\} \]

\[ = (1 + t)^{-1+(\alpha+s)} \exp[p_k(x) - p_k\{x(1 + t)\}]. \]

Proof of (3.3). Again from (1.6) we consider,

\[ \theta^n [x^\alpha \exp\{-p_k(x)\}] = n! x^{\alpha+n} \exp\{-p_k(x)\} T_{kn}^{(\alpha+s-1)}(x) \]

or

\[ e^{t\theta} (\theta^n [x^\alpha \exp\{-p_k(x)\}]) = n! e^{t\theta} \left[ x^{\alpha+n} \exp\{-p_k(x)\} \right] T_{kn}^{(\alpha+s-1)}(x) \]

using (2.6) we get,

\[ \sum_{m=0}^{\infty} \frac{t^m \theta^{m+n}}{m!} [x^\alpha \exp\{-p_k(x)\}] \]

\[ = n! x^{\alpha+n}(1 - xt)^{-(\alpha+s+n)} \exp[-p_k\{x(1 - xt)^{-1}\}] T_{kn}^{(\alpha+s-1)}(x(1 - xt)^{-1}) \]

therefore, we get
\[
\sum_{m=0}^{\infty} \frac{1}{m! n!} (m+n)! x^{\alpha+m+n} \exp\{-p_k(x)\} T_{k(m+n)}^{(\alpha+s-1)}(x)t^m
\]
\[
= x^{\alpha+n} (1-xt)^{-(\alpha+s+n)} \exp[-p_k\{x(1-xt)^{-1}\}] T_{kn}^{(\alpha+s-1)}\{x(1-xt)^{-1}\}
\]
hence above equation reduces to,
\[
\sum_{m=0}^{\infty} x^m \binom{m+n}{n} T_{k(m+n)}^{(\alpha+s-1)}(x)t^m
\]
\[
= (1-xt)^{-(\alpha+s+n)} \exp[p_k(x) - p_k\{x(1-xt)^{-1}\}] T_{kn}^{(\alpha+s-1)}\{x(1-xt)^{-1}\}
\]
replacing \( t \) by \( t/x \), which gives (3.3).

**Proof of (3.4).** Again from (1.6) we consider,
\[
\theta^n[x^\alpha \exp\{-p_k(x)\}] = n! \ x^{\alpha+n} \exp\{-p_k(x)\} T_{kn}^{(\alpha+s-1)}(x)
\]
replacing \( \alpha \) by \( \alpha - m \), we get
\[
\theta^n[x^{\alpha-m} \exp\{-p_k(x)\}] = n! \ x^{\alpha-n} \exp\{-p_k(x)\} T_{kn}^{(\alpha+s-1)}(x)
\]
or
\[
e^{t\theta}(\theta^n[x^{\alpha-m} E_\alpha\{-p_k(x)\}]) = n! e^{t\theta}[x^{(\alpha+n)-m} \exp\{-p_k(x)\} T_{kn}^{(\alpha+m-s-1)}(x)]
\]
using (2.7) we get,
\[
\sum_{m=0}^{\infty} \frac{t^m}{m!} \theta^{m+n} [x^{\alpha-m} \exp\{-p_k(x)\}]
\]
\[
= n! \ x^{\alpha+n} (1+t)^{\alpha+s-1} \exp[-p_k\{x(1+t)\}] T_{kn}^{(\alpha-m+s-1)}\{x(1+t)\}
\]
therefore, we get
\[
\sum_{m=0}^{\infty} \frac{1}{m! n!} (m+n)! x^{\alpha-m+m+n} \exp\{-p_k(x)\} T^{(\alpha-m+s-1)}_{k(m+n)}(x) t^n \\
= x^{\alpha+n}(1+t)^{\alpha+s-1} \exp[-p_k(x(1+t))] T^{(\alpha-m+s-1)}_{k n}(x(1+t))
\]
which reduces to (3.4).

4. Finite Summation Formulae

We obtained finite summation formula for (1.6) as
\[
(4.1) \quad T^{(\alpha+s-1)}_{k n}(x) = \sum_{m=0}^{n} (m!)^{-1} (\alpha - \beta)_m T^{(\beta+s-1)}_{k(n-m)}(x)
\]
\[
(4.2) \quad T^{(\alpha+s-1)}_{k n}(x) = \sum_{m=0}^{n} \frac{1}{m!} (\alpha)_m T^{(s-1)}_{k(n-m)}(x)
\]

Proof of (4.1). We can write (1.6) as,
\[
\sum_{n=0}^{\infty} x^n T^{(\alpha+s-1)}_{k n}(x) t^n = x^{-\alpha} \exp\{p_k(x)\} e^{t\theta} [x^\alpha \exp\{-p_k(x)\}]
\]
by using (2.6), we write
\[
\sum_{n=0}^{\infty} x^n T^{(\alpha+s-1)}_{k n}(x) t^n \\
= x^{-\alpha} \exp\{p_k(x)\} x^\alpha (1-xt)^{-(\alpha+s)} \exp[-p_k(x(1-xt)^{-1})]
\]
\[
= (1-xt)^{-(\alpha+s)} \exp\{p_k(x) - p_k(x(1-xt)^{-1})\}
\]
applying (2.8), which yields
\[
\sum_{n=0}^{\infty} x^n T^{(\alpha+s-1)}_{k n}(x) t^n
\]
\[
(1 - xt)^{-(\beta + s)} \sum_{m=0}^{\infty} (\alpha - \beta)_m \frac{(xt)^m}{m!} \exp[p_k(x) - p_k\{x(1 - xt)^{-1}\}]
\]

\[
= \sum_{m=0}^{\infty} (\alpha - \beta)_m \frac{x^m t^m}{m!} \exp[p_k(x)](1 - xt)^{-(\beta + s)} \exp[-p_k\{x(1 - xt)^{-1}\}]
\]

using (3.1), above equation reduces to,

\[
\sum_{n=0}^{\infty} x^n T_n^{(\alpha + s - 1)}(x) t^n = \sum_{m=0}^{\infty} (\alpha - \beta)_m \frac{x^m t^{n+m}}{m! n!} \exp[p_k(x)] x^{-\beta} e^{t\theta} [x^\beta \exp(-p_k(x))]
\]

\[
= \sum_{m,n=0}^{\infty} (\alpha - \beta)_m \frac{x^m t^{n+m}}{m! n!} \exp[p_k(x)] x^{-\beta} \theta^n [x^\beta \exp(-p_k(x))]
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{m!} (\alpha - \beta)_m \frac{x^{-\beta + m}}{(n - m)!} \exp[p_k(x)] \theta^{n-m} [x^\beta \exp{-p_k(x)}] t^n
\]

equating the coefficients of \(t^n\), we get

\[
x^n T_n^{(\alpha + s - 1)}(x) = \sum_{m=0}^{n} \frac{1}{m!} (\alpha - \beta)_m \frac{x^{-\beta + m}}{(n - m)!} \exp[p_k(x)] \theta^{n-m} [x^\beta \exp{-p_k(x)}]
\]

Therefore, we obtain

\[
T_n^{(\alpha + s - 1)}(x) = \sum_{m=0}^{n} \frac{1}{m!} (\alpha - \beta)_m \frac{x^{-\beta - (n - m)}}{(n - m)!} \exp[p_k(x)] \theta^{n-m} [x^\beta \exp{-p_k(x)}]
\]

and applying (1.6) then above equation immediately leads to (4.1).

**Proof of (4.2).** We can write (1.6) as,

\[
T_n^{(\alpha + s - 1)}(x) = \frac{1}{n!} x^{-\alpha - n} \exp[p_k(x)] \theta^n [x x^{\alpha - 1} \exp{-p_k(x)}]
\]
using (2.3) we get,

and by using (2.1) which yields,

\[
T^{(\alpha+s-1)}_{kn}(x) = \frac{1}{n!} x^{-\alpha-n} \exp\{p_k(x)\} x \sum_{m=0}^{n} \frac{n!}{m! (n-m)!} \\
\times x^{n-m}[(s+xD)(s+1+xD)(s+2+xD) \ldots (s+(n-m-1)+xD)] \exp\{-p_k(x)\} \\
\times x^{m}[(1+xD)(2+xD)(3+xD) \ldots (m+xD)] x^{\alpha-1}
\]

\[
T^{(\alpha+s-1)}_{kn}(x) = \exp\{p_k(x)\} \sum_{m=0}^{n} \frac{1}{m! (n-m)!} \prod_{i=0}^{n-m-1} (s+i+xD) \exp\{-p_k(x)\} (\alpha)_m
\]

(4.3)

Putting \(\alpha = 0\) and replacing \(n\) by \(n-m\) in (1.6) which reduces to

\[
T^{(s-1)}_{k(n-m)}(x) = \frac{1}{(n-m)!} x^{-(n-m)} \exp\{p_k(x)\} \theta^{n-m}[\exp\{-p_k(x)\}]
\]

thus, we have

\[
\frac{1}{(n-m)!} \theta^{n-m}[\exp\{-p_k(x)\}] = \frac{x^{n-m}}{\exp\{p_k(x)\}} T^{(s-1)}_{k(n-m)}(x)
\]

using (2.1), we get

\[
\frac{1}{(n-m)!} \prod_{i=0}^{n-m-1} (s+i+xD)[\exp\{-p_k(x)\}] = \frac{1}{\exp\{p_k(x)\}} T^{(s-1)}_{k(n-m)}(x).
\]

(4.4)

use of (4.4) and (4.3), gives complete proof of (4.2).
5. Concluding Remarks

Some special cases of $T_{kn}^{(\alpha+s-1)}(x)$ polynomials are given below:

If we replace $\alpha$ by $\alpha + 1$, $p_k(x) = p_1(x) = x$ and $s = 0$ in (1.6), then this equation reduces to

$$T_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) = Z_n^{(\alpha)}(x;1) = Y_n^{(\alpha)}(x;1). \quad (5.1)$$

Again replacing $\alpha$ by $\alpha + 1$, $p_k(x) = px^r$ and $s = 0$ in (1.6), which gives

$$T_r^{(\alpha)}(x) = T_r^{(\alpha)}(x,p). \quad (5.2)$$

Substituting $\alpha = 1 - n$, $p_k(x) = x^2$, $s = 0$ in (1.6) and using (1.11), which yields

$$T_{2n}^{(1-n)}(x) = \frac{(-x)^n}{n!} H_n(x). \quad (5.3)$$

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References


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