REGULARITY AND AMENABILITY OF THE SECOND DUAL OF WEIGHTED GROUP ALGEBRAS

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Abstract

For a wide variety of Banach algebras $A$ (containing the group algebras $L^1(G), M(G)$ and $A(G)$) the Arens regularity of $A^{**}$ is equivalent to that of $A$, and the amenability of $A^{**}$ is equivalent to the amenability and regularity of $A$. In this paper, among other things, we show that this variety contains the weighted group algebras $L^1(G,w)$ and $M(G,w)$.

Keywords : Arens product, Weighted group algebra, Amenability

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1. Introduction

Over fifty years ago, Arens in his elaborate work [A], pointed out that, for every Banach algebra $A$, there exist two (Arens) products $\circ$ and $\cdot$ on the second dual $A^{**}$, extending the product of $A$. If these two products coincide on $A^{**}$, then $A$ is said to be (Arens) regular. For further details on the properties of Arens products see the survey article [D-H]. It is readily verified that the regularity of $A^{**}$ (equipped with either $\circ$ or $\cdot$) implies that $A^{**}$ is regular if and only if $(A^{**}, \circ)$ is regular. However, it has been shown in [Y3] that there exists a regular Banach algebra whose second dual is not regular; for a more simple example of such a Banach algebra see [P]. Every $C^*$-algebra is regular [S], and its second dual is a von Neumann algebra, and so is regular. As a consequence of Young's result [Y1], (which asserts $L^1(G)$ is regular if and only if $G$ is finite) the regularity of $L^1(G)^{**}$ is equivalent to the regularity of $L^1(G)$. For a commutative, semisimple, completely continuous and weakly sequentially complete Banach algebra $A$ whose dual $A^*$ is a von Neumann algebra (for instance, for the Fourier algebra $A(G)$), it has been shown in [U2] that the regularity of $A^{**}$ is equivalent to that of $A$.

A Banach algebra $A$ is said to be amenable (resp. weakly amenable) if every continuous derivation $D : A \to X^*$ (resp. $D : A \to A^*$) is inner for every Banach $A$-module $X$. It has been shown in [Go] (see also [G-L-W]) that, if either $(A^{**}, \circ)$ or $(A^{**}, \cdot)$ is amenable then so is $A$. However, for an infinite amenable group $G$, $L^1(G)$ is amenable, but $L^1(G)^{**}$ is not; indeed in [G-L-W] they showed that $L^1(G)^{**}$ is amenable if and only if $G$ is finite. For the Fourier algebra $A(G)$ it is known that, $A(G)^{**}$ is amenable if and only if $G$ is finite, see [Gra]. Although, one can use the earlier result of Forrest and Runde [F-R], to give a simple proof for the latter fact; (indeed, if $A(G)^{**}$ is amenable then so is $A(G)$, and the main result of [F-R] implies that, $G$ has an abelian subgroup $H$ of finite index. It induces an epimorphism from $A(G)^{**}$ on $A(H)^{**}$, in particular $A(H)^{**} = L^1(\hat{H})^{**}$ is amenable. It follows by [G-L-W], that $\hat{H}$ is finite, and so $G$ is finite.)

If $A$ is commutative or if it possesses a continuous involution then as it is shown in [G-L], the amenability (resp. weak amenability) of $(A^{**}, \circ)$ is equivalent to that of $(A^{**}, \cdot)$. It seems still not known if there exists a Banach algebra $A$ for which the amenability of $(A^{**}, \circ)$ is not equivalent to that of $(A^{**}, \cdot)$.

The main theme of this paper is to investigate the regularity and amenable-
ity of the second dual of the weighted group algebras $L^1(G, w)$ and $M(G, w)$.

2. Preliminaries

Throughout this paper, $G$ is a locally compact (topological) group, and $w$ is a weight on $G$; (which is a continuous function $w : G \rightarrow (0, \infty)$ with $w(xy) \leq w(x)w(y)$, for all $x, y \in G$, for convenience we shall assume that $w(e) = 1$, where $e$ is the identity of $G$. We define $\Omega : G \times G \rightarrow (0, 1]$ by $\Omega(x, y) = w(xy)/w(x)w(y)$.

A function $h : X \times Y \rightarrow \mathbb{C}$ is said to be $0$–cluster if $\lim_n \lim_m h(x_n, y_m) = 0 = \lim_m \lim_n h(x_n, y_m)$ for every two sequences $\{x_n\} \subseteq X$ and $\{y_m\} \subseteq Y$ of distinct points, provided the involved limits exist.

We define $w^*$ on $G$ by $w^*(x) = w(x)w(x^{-1}), (x \in G)$. It can be simply verified that $w^*$ is also a weight on $G$; moreover $w^*$ is bounded on $G$ if and only if $w$ is semi-multiplicative (that is, there exists $c > 0$ such that $cw(x)w(y) \leq w(xy)$, for all $x, y \in G$). Therefore, $\Omega$ can not be $0$–cluster when $w^*$ is bounded.

Define $L^1(G, w), L^\infty(G, w), C_0(G, w)$ and $LUC(G, w)$ as follows:

\[
\begin{align*}
L^1(G, w) &= \{ f : fw \in L^1(G) \}, \\
L^\infty(G, w) &= \{ f : f/w \in L^\infty(G) \}, \\
C_0(G, w) &= \{ f : f/w \in C_0(G) \}, \text{ and} \\
LUC(G, w) &= \{ f : f/w \in LUC(G) \}.
\end{align*}
\]

We norm these spaces in such a way the multiplication or division by $w$ becomes an isometry between the non-weighted and the corresponding weighted spaces (whose norm will denote by $\| \cdot \|_w$). Thus the non-weighted and the corresponding weighted spaces are isometrically isomorphic as Banach spaces, but quite different as Banach algebras. Recall the inclusion relations of non-weighted cases of these spaces and the fact that $L^1(G)^* = L^\infty(G)$, we have:

\[C_0(G, w) \subseteq LUC(G, w) \subseteq L^\infty(G, w) = L^1(G, w)^*.\]

We refer the reader to [R2], for more study of different subalgebras of $L^\infty(G, w)$, and their equalities.

We define $M(G, w)$ such that $M(G, w)$ becomes isometric isomorphic to the Banach space $C_0(G, w)^*$. For this sake, let $M^+(G, w)$ be the set of
all positive regular measures on $G$ for which $\mu w$ is again a positive regular measure on $G$; where $d(\mu w) = wd\mu$. Define an equivalence relation on $M^+(G, w) \times M^+(G, w)$ by $(\mu_1, \nu_1) \sim (\mu_2, \nu_2)$ if and only if $\mu_1 + \nu_2 = \mu_2 + \nu_1$. Now define $M(G, w)$ by

$$M(G, w) = \{[\mu, \nu] : \mu, \nu \in M^+(G, w)\},$$

where $[\mu, \nu]$ is the equivalence class of $(\mu, \nu)$. For a full discussion on $M(G, w)$ from this point of view and the fact that $C_0(G, w) = M(G, w)$ see [R1] and also [B].

It should be remarked that, if $w$ is multiplicative (i.e. $w(xy) = w(x)w(y)$, for all $x, y \in G$, or equivalently, $w$ is a positive character on $G$) then $L^1(G, w) \cong L^1(G)$ and $M(G, w) \cong M(G)$ as Banach algebras. Indeed it can be readily verified that $f \rightarrow fw$ and $[\mu, \nu] \rightarrow \mu w - \nu w$ are algebra isomorphism from $L^1(G, w)$ on $L^1(G)$ and $M(G, w)$ on $M(G)$, respectively.

As a ground reference for the second dual of weighted group algebras, one may refer to [D-L].

3. Main Results

We start with the next lemma.

**Lemma 1.** If $G$ is infinite (discrete ) and $\Omega$ is 0–cluster, then $F \circ G = 0 = F \circ G$, for every $F, G \in l^1(G, w)^{**} \setminus l^1(G, w)$.

**Proof.** Since $\Omega$ is 0-cluster, the mapping $(x, y) \rightarrow (\frac{\phi}{w})(xy)\Omega(x, y)$ is 0-cluster for every $\phi \in L^\infty(G, w)$. Using the Example 2 in page 312 of [Y2], the mapping $(f, g) \rightarrow \phi(f \ast g) = \sum \sum (\frac{\phi}{w})(xy)(fw)(x)(gw)(y)\Omega(x, y)$ is 0–cluster on $l^1(G, w) \times l^1(G, w)$ (the sums are taken on $x, y \in G$). Now for $F, G \in l^1(G, w)^{**} \setminus l^1(G, w)$ there exist two nets $\{f_\alpha\}$ and $\{g_\beta\}$, each consisting of distinct points in $l^1(G, w)$ such that $f_\alpha \rightarrow F$ and $g_\alpha \rightarrow G$, in the weak* topology, with

$$< F \circ G, \phi > = \lim_\alpha \lim_\beta \phi(f_\alpha \ast g_\beta),$$

for every $\phi \in L^\infty(G, w)$. One can construct two subsequences $\{f_{\alpha_m}\}$ and $\{g_{\beta_n}\}$ of $\{f_\alpha\}$ and $\{g_\beta\}$, respectively, such that, $< F \circ G, \phi > = \lim_m \lim_n \phi(f_{\alpha_m} \ast g_{\beta_n}) = \lim_n \lim_m \phi(f_{\alpha_m} \ast g_{\beta_n}) = < F \circ G, \phi >$, as required.

Now, we come to the one of the main results.
Theorem 2. The following statements are equivalent.

(i) $L^1(G, w)$ is regular,
(ii) $G$ is finite or $G$ is discrete and $\Omega$ is 0−cluster,
(iii) $L^1(G, w)^{**}$ is regular.

Proof. For (i)$\Rightarrow$(ii), suppose that $L^1(G, w)$ be regular. Since $L^1(G, w)$ is weakly sequentially complete and admits a bounded approximate identity, it is unital by theorem 3.3 of [U1]. Therefore $G$ is discrete. If $G$ is infinite, then by corollary 3.8 of [B-R] $\Omega$ must be 0−cluster. For (ii)$\Rightarrow$(iii), if $G$ is finite then $L^1(G, w)$ is reflexive; for the infinite case (iii) follows from Lemma1.

Suppose that $G$ admits a multiplicative weight bounded by $w$, (for instance, it is the case if either $1 \leq w$ or $G$ is amenable (as a group), for the latter see Lemma 1 of [W]). Then, there exists a unique multiplicative weight on $G$ which is equivalent to $w$, provided $w^*$ is bounded. Indeed, $\varphi(x) = \lim_{n \to \infty} w(x^n)^{1/n}$ defines a multiplicative weight on $G$ with $\varphi \leq w \leq c\varphi$, in which $c = \sup_{x \in G} w^*(x)$; see [W] for further details. In particular, $L^1(G, w) = L^1(G, \varphi) \cong L^1(G)$ and $M(G, w) = M(G, \varphi) \cong M(G)$.

An elegant result of [Gro] states $L^1(G, w)$ is amenable if and only $G$ is amenable and $w^*$ is bounded. Therefore, $L^1(G, w)$ is amenable if and only if $G$ is amenable and $L^1(G, w) \cong L^1(G)$. Recently, it has been proved in [D-G-H] that $M(G)$ is amenable if and only if $G$ is amenable and discrete. As a weighted version of this we have;

Proposition 3. $M(G, w)$ is amenable if and only if $G$ is amenable, discrete and $w^*$ is bounded.

Proof. If $M(G, w)$ is amenable, then $L^1(G, w)$ is amenable, therefore $G$ is amenable and $w^*$ is bounded; and so by the discussion just before the proposition, there exists a unique multiplicative weight on $G$ equivalent to $w$. It implies that, $M(G, w) \cong M(G)$. In particular, $M(G)$ is amenable. By [D-G-H] $G$ must be discrete. Since in the discrete setting $M(G, w) = L^1(G, w)$, the converse follows from [Gro] .

As the second main result we have the next which is an extension of Theorem 1.3 of [G-L-W].

Theorem 4. The following statements are equivalent.

(i) $L^1(G, w)^{**}$ is amenable,
(ii) $L^1(G, w)$ is amenable and regular,
(iii) $L^1(G, w)$ is regular and $w^*$ is bounded,
(iv) $L^1(G, w)$ is reflexive and $w^*$ is bounded,
(v) $L^1(G, w)$ is a $C^*$-algebra,
(vi) $G$ is finite.

**Proof.** Trivially (vi) implies the other parts. If $L^1(G, w)^{**}$ is amenable, then so is $L^1(G, w)$, and so $L^1(G, w) \cong L^1(G)$. Now the amenability of $L^1(G)^{**}$ necessitates $G$ must be finite by Theorem 1.3 of [G-L-W]. Thus (i) $\Rightarrow$ (vi) follows. The implications (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (iii) are obvious. Let $L^1(G, w)$ be regular and $w^*$ be bounded; therefore $\Omega$ cannot be 0-cluster and by Theorem 2, $G$ is finite. Assume that $L^1(G, w)$ is a $C^*$-algebra, then it is regular and so $G$ is discrete. Moreover, the equality $\|\delta_x \ast \delta_x^*\|_w = \|\delta_x\|^2_w$, for every $x \in G$ implies that $w(x) = \Delta(x)^{1/2}$, for each $x \in G$ ( $\Delta$ is the modular function of $G$), and this implies that $w$ is multiplicative and so $\Omega = 1$. Now Theorem 2 implies that $G$ is finite and this completes the proof.

**Remarks.** (i) The conclusions of Theorems 2 and 4 remains valid if we replace $L^1(G, w)$ by $M(G, w)$.

(ii) For a Banach algebra $A$ if $A^{***} \cdot F = A^* \circ F$, for every $F \in A^{**}$, (where $< m \ast F, G > = < m, F \circ G >$ for every $m \in A^{**}, F, G \in A^{**}$) then it is not hard to prove that the regularity of $A^{**}$ is equivalent to that of $A$; (indeed if $A$ is regular, then for every $f \in A^*$, the mapping $F \to f \circ F : A^{**} \to A^*$ is weakly compact, and the equality $A^{***} \cdot F = A^* \circ F$ implies that for every $\Phi \in A^{***}$ the mapping $F \to \Phi \cdot F : A^{**} \to A^{***}$ is weakly compact, which is equivalent to the regularity of $A^{**}$). Using this fact, one may give a different proof to the Theorem 2.

(iii) For a Banach algebra $A$ with a bounded approximate identity of norm one, $A^*A$ is a closed subspace of $A^*$, and $A^{**} = (A^*A)^* \oplus (A^*A)^\perp$ (as Banach spaces), where $(A^*A)^\perp = \{ F \in A^{**} : A^{**} \circ F = 0 \}$ is a closed ideal of $(A^{**}, \circ)$ and $(A^*A)^*$ is a closed subalgebra of $(A^{**}, \circ)$. These observations together with the Lemma 2.3 of [L-L] imply that; if $(A^{**}, \circ)$ is weakly amenable then so is $(A^*A)^*$. Now for $A = L^1(G, w)$ it has been shown in Proposition 1.3 of [Gro] that $A^*A = LUC(G, w)$. On the other hand, using the methods of Lemma 1.1 of [G-L-L], we have $LUC(G, w)^* = M(G, w) \oplus C_0(G, w)^\perp$, and that $M(G, w), C_0(G, w)^\perp$ are closed subalgebra and closed ideal of $LUC(G, w)^*$, respectively. Again use Lemma 2.3 of [L-L] the weak amenability of $L^1(G, w)^{**}$ implies that of $M(G, w)$, which is an extension of Proposition 4.14 in [L-L].
(iv) The existing examples support the conjecture that, for a Banach algebra \( A \) if \( A^{**} \) is amenable then \( A \) is regular.

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