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ON THE LOCAL HYPERCENTER OF A GROUP

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Abstract

We introduce a local hypercenter of an arbitrary group and study its basic properties. With this concept, it turns out that classical theorems of Baer, Mal'cev and McLain on locally nilpotent groups can be obtained as special cases of statements which are valid in any group. Furthermore, we investigate the connection between the local hypercenter of a group and the intersection of its maximal locally nilpotent subgroups.

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The local hypercenter of a group

An important canonical characteristic subgroup of an arbitrary group G is its *hypercenter* $\mathbf{H}(G)$, the union of the terms of the transinitely continued upper central series of G (see [6], [7]). Following BAER [1], $\mathbf{H}(G)$ has several equivalent descriptions: It can be seen as the smallest normal subgroup of G with center-free quotient and also as the largest normal subgroup M of G that is G -hypercentral in the sense that for every $N \trianglelefteq G$ such that $N < M$, we have that M/N contains a nontrivial central element of G/N . A group G is *hypercentral*, if $\mathbf{H}(G) = G$. This happens, if and only if every of its nontrivial homomorphic images has a nontrivial center. Clearly, nilpotent groups are hypercentral and hypercentral groups are locally nilpotent (see [6, pg. 364/365]. $\mathfrak{H}$ and $\mathfrak{N}$ indicate the classes of hypercentral and nilpotent groups, respectively, $L\mathfrak{N}(= L\mathfrak{H})$ is the class of locally nilpotent groups. Since there are locally nilpotent groups with trivial (hyper-) center - any restricted regular wreath product of a group of order a prime p with an infinite abelian p -group is an example - it is desirable to extend the concept of the hypercenter of a group in some way to a *local hypercenter*. A natural possibility to try this is given with the following definition, suggested to us by Prof. O. H. KEGEL :

Definition A. For any group G , the *local hypercenter* of G is defined to be

$$\mathbf{K}(G) = \left\{ x \in G \mid x \in \mathbf{H}(\langle x, g_1, \dots, g_r \rangle) \ \forall g_1, \dots, g_r \in G \right\} .$$

This means that $\mathbf{K}(G)$ consists of all elements x which lie in the hypercenter of any finitely generated subgroup of G containing x .

1. Basic properties

The (general) *hypercenter* $\mathbf{H}(G)$ of a group, the union of the terms of the transinitely continued upper central series of G , has the following equivalent descriptions (compare [7, pg. 28, 1.39.3]): Consider the two families of normal subgroups of G (by $\mathbf{Z}(G)$ we denote the center of G),

$$\mathfrak{F}_1(G) = \left\{ X \trianglelefteq G \mid \mathbf{Z}\left(\frac{G}{X}\right) = \frac{X}{X} \right\} \quad \text{and}$$

$$\mathfrak{F}_2(G) = \left\{ M \trianglelefteq G \mid \frac{M}{N} \cap \mathbf{Z}\left(\frac{G}{N}\right) > \frac{N}{N} \quad \forall \quad N \trianglelefteq G \text{ with } N < M \right\}$$

Then

$$\mathbf{H}(G) = \bigcap_{X \in \mathfrak{F}_1(G)} X = \prod_{M \in \mathfrak{F}_2(G)} M$$

Moreover, $\mathbf{H}(G) \in \mathfrak{F}_1(G) \cap \mathfrak{F}_2(G)$.

With these alternative descriptions of $\mathbf{H}(G)$, we can easily verify the following well known basic facts:

Lemma 1.1. *Let G be a group, T a subgroup and N a normal subgroup of G . Then*

- a) $T \cap \mathbf{H}(G) \leq \mathbf{H}(T)$.
- b) $\mathbf{H}(G)N/N \leq \mathbf{H}(G/N)$.
- c) The radical property: *If $N \leq \mathbf{H}(G)$, then $\mathbf{H}(G)/N = \mathbf{H}(G/N)$.*
In particular, $\mathbf{H}(G/\mathbf{H}(G))$ is trivial, that is, $G/\mathbf{H}(G)$ is $\mathbf{H}$ -free.
- d) $\mathbf{H}(G) = \mathbf{H}(A) \times \mathbf{H}(B)$, *if $G = A \times B$ is a direct product.*
- e) $S\mathbf{H}(G) \in H$ *whenever $S = \mathbf{H}(S)$ is a hypercentral subgroup of G .*
- f) *If $T\mathbf{H}(G) = G$, then $\mathbf{H}(T) = T \cap \mathbf{H}(G)$.*

Proof: a) For this assertion we have to show that $T \cap \mathbf{H}(G) \in \mathfrak{F}_2(T)$: We put $D = T \cap \mathbf{H}(G)$ and let $R \trianglelefteq T$ with $R < D$. Among the normal subgroups K of G with $K \leq \mathbf{H}(G)$ and $K \cap D \leq R$ choose a maximal one N (Zorn's Lemma). Then $N < \mathbf{H}(G)$ and $Z/N = \frac{\mathbf{H}(G)}{N} \cap \mathbf{Z}\left(\frac{G}{N}\right) > N/N$, since $\mathbf{H}(G) \in \mathfrak{F}_2(G)$. Therefore $Z \cap D \not\leq R$. For $z \in (Z \cap D) \setminus R$ we have $[z, T] \leq [z, G] \cap D \leq N \cap D \leq R$. So $\frac{D}{R} \cap \mathbf{Z}\left(\frac{T}{R}\right) > R/R$, i. e. $T \cap \mathbf{H}(G) = D \in \mathfrak{F}_2(T)$.

b) Since $\frac{G/N}{\mathbf{H}(G/N)}$ is center-free, we see that $\mathbf{H}(G)$ is contained in the pre-image of $\mathbf{H}(G/N)$. Therefore $\mathbf{H}(G)N/N \leq \mathbf{H}(G/N)$.

c) By b) we have $\mathbf{H}(G)/N \leq \mathbf{H}(G/N)$. Since $G/\mathbf{H}(G) \cong \frac{G/N}{\mathbf{H}(G)/N}$ is

center-free, it follows that also $\mathbf{H}(G/N) \leq \mathbf{H}(G)/N$.

d) Since $\mathbf{H}(A) \in \mathfrak{F}_2(G)$, one has $\mathbf{H}(A) \leq \mathbf{H}(G)$. Idem $\mathbf{H}(B) \leq \mathbf{H}(G)$ and therefore $\mathbf{H}(A) \times \mathbf{H}(B) \leq \mathbf{H}(G)$. Since $\mathbf{H}(A)B/B = \mathbf{H}(G/B)$, one sees that $G/\mathbf{H}(A)B$ is center-free. The same holds for $G/A\mathbf{H}(B)$. Therefore $\mathbf{H}(G) \leq \mathbf{H}(A)B \cap A\mathbf{H}(B) = \mathbf{H}(A)(B \cap A\mathbf{H}(B)) = \mathbf{H}(A)(B \cap A)\mathbf{H}(B) = \mathbf{H}(A) \times \mathbf{H}(B) \leq \mathbf{H}(G)$ and the assertion follows.

e) Let $L = \mathbf{H}(G)S$. Then $\mathbf{H}(G) \leq \mathbf{H}(L)$. Since $L/\mathbf{H}(L) = \mathbf{H}(L)S/\mathbf{H}(L) \cong S/S \cap \mathbf{H}(L)$ is hypercentral and $L/\mathbf{H}(L)$ is center-free, we see that $\mathbf{H}(L) = L$.

f) By a) we have $T \cap \mathbf{H}(G) \leq \mathbf{H}(T)$. With $G/\mathbf{H}(G)$ also $T/T \cap \mathbf{H}(G)$ is center-free. Therefore $\mathbf{H}(T) \leq T \cap \mathbf{H}(G)$.

The corresponding properties for the *local* hypercenter we have in

Proposition 1.2. *Let G be a group, T a subgroup and N a normal subgroup of G . Then*

- a) $\mathbf{K}(G)$ is a locally nilpotent characteristic subgroup of G with $\mathbf{H}(G) \leq \mathbf{K}(G)$.
- b) $T \cap \mathbf{K}(G) \leq \mathbf{K}(T)$.
- c) $\mathbf{K}(G)N/N \leq \mathbf{K}(G/N)$.
- d) The radical property : If $N \leq \mathbf{K}(G)$, then $\mathbf{K}(G/N) = \mathbf{K}(G)/N$.
In particular, $G/\mathbf{K}(G)$ is $\mathbf{K}$ -free.
- e) $\mathbf{K}(G) = \mathbf{K}(A) \times \mathbf{K}(B)$, if $G = A \times B$ is a direct product.
- f) $\mathbf{K}(G)S \in L\mathfrak{N}$ for every locally nilpotent subgroup S of G .
- g) If $T\mathbf{K}(G) = G$, then $\mathbf{K}(T) = T \cap \mathbf{K}(G)$.

Proof: For $x, y, g_1, g_2, \dots, g_r \in G$ we abbreviate

$$F_x = \langle x, g_1, \dots, g_r \rangle \text{ and } F_{x,y} = \langle x, y, g_1, \dots, g_r \rangle.$$

a) For all $x \in \mathbf{H}(G)$ and $g_1, \dots, g_r \in G$ we have $x \in \mathbf{H}(G) \cap F_x \leq \mathbf{H}(F_x)$ by Lemma 1.1 a). This means $x \in \mathbf{K}(G)$, i.e., $\mathbf{H}(G) \subseteq \mathbf{K}(G)$.

Let $x, y \in \mathbf{K}(G)$ and $g_1, \dots, g_r \in G$. We have $x, y \in \mathbf{H}(F_{x,y})$ and so $xy^{-1} \in \mathbf{H}(F_{x,y}) \cap F_{xy^{-1}} \leq \mathbf{H}(F_{xy^{-1}})$. So $xy^{-1} \in \mathbf{K}(G)$ and we see that $\mathbf{K}(G)$ is a subgroup of G . If $g_1, \dots, g_r \in \mathbf{K}(G)$, then $g_i \in \mathbf{H}(\langle g_1, \dots, g_r \rangle)$ for all $i = 1, \dots, r$. Therefore $\langle g_1, \dots, g_r \rangle = \mathbf{H}(\langle g_1, \dots, g_r \rangle)$ is a hypercentral, i.e. nilpotent subgroup and we see that $\mathbf{K}(G)$ is locally nilpotent. It is also clear that $\mathbf{K}(G)$ is invariant under all automorphisms of G .

b) For all $x \in T \cap \mathbf{K}(G)$ and $g_1, \dots, g_r \in T$ we have $x \in T \cap \mathbf{H}(F_x) \leq \mathbf{H}(T)$, i.e. $x \in \mathbf{K}(T)$. So $T \cap \mathbf{K}(G) \leq \mathbf{K}(T)$.

c) Let $N \trianglelefteq G$. For all $x \in \mathbf{K}(G)$ and $g_1, \dots, g_r \in G$ we have $x \in \mathbf{H}(F_x)$. It follows $x(N \cap F_x) \in \mathbf{H}(F_x)(N \cap F_x)/N \cap F_x \leq \mathbf{H}(F_x/N \cap F_x)$, using 1.1 b). Thus, by isomorphism, $xN \in \mathbf{H}(F_x N/N) = \mathbf{H}(\langle xN, g_1 N, \dots, g_r N \rangle)$. Therefore $xN \in \mathbf{K}(G/N)$, which means $\mathbf{K}(G)N/N \leq \mathbf{K}(G/N)$.

d) Let $N \leq \mathbf{K}(G)$. Clearly $\mathbf{K}(G)/N \leq \mathbf{K}(G/N)$ by c). For $xN \in \mathbf{K}(G/N)$, we have for all $g_1, \dots, g_r \in G$ that $xN \in \mathbf{H}(\langle xN, g_1 N, \dots, g_r N \rangle) = \mathbf{H}(F_x N/N)$. By isomorphism, $x(N \cap F_x) \in \mathbf{H}(F_x/N \cap F_x)$. Since $N \leq \mathbf{K}(G)$, we see $N \cap F_x \leq \mathbf{K}(G) \cap F_x \leq \mathbf{K}(F_x) = \mathbf{H}(F_x)$ as F_x is finitely generated. So $\mathbf{H}(F_x/N \cap F_x) = \mathbf{H}(F_x)/N \cap F_x$ by Lemma 1.1 c) and it follows that $x \in \mathbf{H}(F_x)$. This means $x \in \mathbf{K}(G)$ and therefore $xN \in \mathbf{K}(G)/N$. This shows $\mathbf{K}(G/N) = \mathbf{K}(G)/N$.

e) We have $\mathbf{K}(G) \leq \mathbf{K}(G)B \cap \mathbf{K}(G)A = (\mathbf{K}(G)B \cap A) \times (\mathbf{K}(G)A \cap B) \leq \mathbf{K}(A) \times \mathbf{K}(B)$, since $\mathbf{K}(G)B/B \leq \mathbf{K}(G/B)$ and $G/B \cong A$ such as $\mathbf{K}(G)A/A \leq \mathbf{K}(G/A)$ and $G/A \cong B$.

Conversely, let $x = ab \in \mathbf{K}(A) \times \mathbf{K}(B)$ with $a \in \mathbf{K}(A)$ and $b \in \mathbf{K}(B)$. If $g_1 = a_1 b_1, \dots, g_r = a_r b_r$ are any elements in G ($a_1, \dots, a_r \in A$, $b_1, \dots, b_r \in B$) and putting $A_1 = \langle a, a_1, \dots, a_r \rangle$, $B_1 = \langle b, b_1, \dots, b_r \rangle$ and $F_x = \langle x, g_1, \dots, g_r \rangle$, we have $a \in \mathbf{H}(A_1)$ and $b \in \mathbf{H}(B_1)$. Thus, $x = ab \in \mathbf{H}(A_1) \times \mathbf{H}(B_1) = \mathbf{H}(A_1 \times B_1)$, using 1.1 d). Moreover, $F_x \leq A_1 \times B_1$ and we see $x \in F_x \cap \mathbf{H}(A_1 \times B_1) \leq \mathbf{H}(F_x)$. Therefore, $x \in \mathbf{K}(G)$.

f) Let S be any locally nilpotent subgroup of G . Every finitely generated subgroup L_1 of $\mathbf{K}(G)S$ is contained in a subgroup of the form $L =$

$\langle K_1, S_1 \rangle$ with $K_1 = \langle k_1, \dots, k_r \rangle$ and $S_1 = \langle s_1, \dots, s_r \rangle$ where $k_1, \dots, k_r \in \mathbf{K}(G)$, $s_1, \dots, s_r \in S$. Now $k_i \in \mathbf{H}(L)$ and therefore $K_1 \leq \mathbf{H}(L)$. Since S_1 is nilpotent, we see by Lemma 1.1 e) that the subgroup $L = \mathbf{H}(L) S_1$ is hypercentral and therefore nilpotent. Thus L_1 is nilpotent and so $\mathbf{K}(G) S$ is locally nilpotent.

g) Since $G/\mathbf{K}(G) \cong T/\mathbf{K}(G) \cap T$, we see that $T/\mathbf{K}(G) \cap T$ is $\mathbf{K}$ -free by d). As $\mathbf{K}(G) \cap T \leq \mathbf{K}(T)$ and $\mathbf{K}(T/\mathbf{K}(G) \cap T) = \mathbf{K}(T)/\mathbf{K}(G) \cap T$, the assertion follows.

The surprising utility of the concept of the local hypercenter we shall see in the following section.

2. On theorems of BAER-McLAIN and MAL'CEV-McLAIN

In 1956, McLAIN [2], (see [6, 12.1.5/6, pg. 359]) published two necessary conditions for a group G to be *locally nilpotent*:

- 1) If maximal subgroups exist, then they are normal in G , and
- 2) Every chief factor of G is central in G .

Using our concept of the local hypercenter, we show that these properties of the locally nilpotent groups have a general meaning in every group. More exactly, we prove

Theorem 2.1. *Let U be a maximal subgroup and M/N a chief factor of a group G such that $M \leq \mathbf{K}(G)$. Then*

- a) U is normalized by $\mathbf{K}(G)$.
- b) M/N is central in G/N .

We remark that for $G = \mathbf{K}(G)$, the above results 1) and 2) are returned.

For the proof of our theorem we first have to show that the corresponding assertions are valid for the hypercenter $\mathbf{H}(G)$.

Lemma 2.2.

Let L be a subgroup and A/B a chief factor of a group W such that $A \leq \mathbf{H}(W)$. Then

a) If $\mathbf{H}(W) \not\leq L$, then $\mathbf{N}_W(L) > L$. In particular, $\mathbf{H}(W)$ normalizes every maximal subgroup L of W .

b) A/B is central in W/B .

We mention that the first part of a) must not be expected for $\mathbf{K}(W)$, as there are locally nilpotent groups with proper selfnormalizing subgroups.

Proof: a) Let L be a subgroup of W and suppose $\mathbf{H}(W) \not\leq L$. Let

$$N = (L \cap \mathbf{H}(W))_W = \bigcap_{g \in W} (L \cap \mathbf{H}(W))^g$$

denote the core of $L \cap \mathbf{H}(W)$ in W . As $N < \mathbf{H}(W)$, there is $x \in \mathbf{H}(W) \setminus N$ such that $[x, W] \leq N$. Therefore, $L^x \leq [x, L]L \leq NL = L$ and also $L^{x^{-1}} \leq L$. So $x \notin L$ but $x \in \mathbf{N}_W(L)$.

b) Since $B \trianglelefteq W$ and $B < A \leq \mathbf{H}(W)$, there is $x \in A \setminus B$ such that $[x, W] \leq B$. As $\langle x \rangle B \trianglelefteq W$ and A/B is minimal normal in W/B , we see that $\langle x \rangle B = A$ and therefore $[A, W] \leq B$. This means that A/B is central in W/B .

Proof of 2.1 : a) Let U be a maximal subgroup of G and suppose $\mathbf{K}(G) \not\leq U$. We have to show that U is normal in G .

Suppose that this is not the case. Since $G = U\mathbf{K}(G)$, we have that there is a commutator $c = [u, x] \notin U$ with $u \in U$ and $x \in \mathbf{K}(G)$. Since $G = \langle U, c \rangle$, we may choose a finitely generated subgroup V of U with $u \in V$ and $x \in \langle V, c \rangle$. Therefore we have

$$u, x \in W = \langle V, c \rangle = \langle V, x \rangle$$

and W is finitely generated.

Since $W = \langle V, x \rangle$, there is a maximal subgroup L of W such that $x \notin L$ and $V \leq L$. Since $x \in \mathbf{H}(W)$, we see with Lemma 2.2 a)) that $L \triangleleft W$. Therefore $c \in L$ and we get the contradiction $W = \langle V, c \rangle \leq L < W$.

b) Let M/N be a chief factor of G with $M \leq \mathbf{K}(G)$ and suppose that $[M, G] \not\leq N$. Using 1.2 c), we may assume $N = 1$, i.e. M is a minimal normal, non-central subgroup of G contained in $\mathbf{K}(G)$. There is $x \in M$ and $g \in G$ such that $c = [x, g] \neq 1$. So $M = \langle c^G \rangle$ and there will exist $g_1, g_2, \dots, g_r \in G$ such that $x \in \langle c^{g_1}, c^{g_2}, \dots, c^{g_r} \rangle$. For the finitely generated subgroup $W = \langle x, g, g_1, \dots, g_r \rangle$ we have that $x \in \mathbf{H}(W)$ as $x \in \mathbf{K}(G)$.

With $A = \langle x^W \rangle$ we get $[A, W] \leq A$. But since $c \in [A, W]$ we have also $c^{g_i} \in [A, W]$ for $i = 1, 2, \dots, r$ as $[A, W] \trianglelefteq W$. Therefore $x \in [A, W]$ and then $[A, W] = A$. Since $A = \langle x^W \rangle > 1$, there is a chief factor A/B of W (pick B maximal among the subgroups X of A with $X \trianglelefteq W$ and $x \notin X$). Since $A \leq \mathbf{H}(W)$ we see with Lemma 2.2 b) that A/B is central in W/B . Thus $[A, W] \leq B < A$, a contradiction.

3 $\mathbf{K}(G)$ and the local $\mathfrak{N}$ -preserver $\Lambda_{\mathfrak{N}}(G)$

For any group G , let $\mathcal{N}(G)$ denote the family of its *maximal locally nilpotent subgroups* (which exist by ZORN's Lemma). Closely connected with the local hypercenter is the following characteristic subgroup:

Definition B. Let G be a group.

$$\Lambda_{\mathfrak{N}}(G) = \bigcap_{H \in \mathcal{N}(G)} H,$$

the intersection of all maximal among the locally nilpotent subgroups of G , we call the *preserver of the local nilpotency of G* .

This subgroup is studied with details in [4] and [5]. It can for example also be characterized as (see [5, Th. 7.1])

$$\Lambda_{\mathfrak{N}}(G) = \{x \in G \mid \langle x^G \rangle \langle g \rangle \in L\mathfrak{N} \quad \forall \quad g \in G\}.$$

$\Lambda_{\mathfrak{N}}(G)$ is closely related with and contains $\mathbf{K}(G)$ - in general properly (see 4.1). Similarly, $\Lambda_{\mathfrak{N}}(G)$ has likewise all the principal properties which

describe the behaviour of the hypercenter and the local hypercenter, established in 1.1 and 1.2. More exactly, we have.

Proposition 3.1.

Let G be a group, T a subgroup and N a normal subgroup of G . Then

- a) $\mathbf{K}(G) \leq \Lambda_{\mathfrak{N}}(G)$
- b) $T \cap \Lambda_{\mathfrak{N}}(G) \leq \Lambda_{\mathfrak{N}}(T)$
- c) $\Lambda_{\mathfrak{N}}(G)N/N \leq \Lambda_{\mathfrak{N}}(G/N)$
- d) $\Lambda_{\mathfrak{N}}(G) = \Lambda_{\mathfrak{N}}(A) \times \Lambda_{\mathfrak{N}}(B)$, if $G = A \times B$ is a direct product.
- e) $S\Lambda_{\mathfrak{N}}(G) \in L\mathfrak{N}$ for every locally nilpotent subgroup $S \leq G$

Proof: a) By 1.2 f), $\mathbf{K}(G) \leq H$ whenever H is a maximal among the locally nilpotent subgroups of G .

b) - d) see [5, 6.1].

e) is true since the locally nilpotent subgroup S is contained in some, $\Lambda_{\mathfrak{N}}(G)$ in every $H \in \mathcal{N}(G)$.

■

The hypercenter $\mathbf{H}(G)$ and also the local hypercenter $\mathbf{K}(G)$ of a group G have the so-called *radical property*, that is, $G/\mathbf{H}(G)$ is $\mathbf{H}$ -free and $G/\mathbf{K}(G)$ is $\mathbf{K}$ -free. More general (see 1.1 c) and 1.2 d)) :

$$\begin{aligned} \mathbf{H}\left(\frac{G}{N}\right) &= \frac{\mathbf{H}(G)}{N} \quad \text{whenever } N \trianglelefteq G \text{ and } N \leq \mathbf{H}(G), \\ \mathbf{K}\left(\frac{G}{N}\right) &= \frac{\mathbf{K}(G)}{N} \quad \text{whenever } N \trianglelefteq G \text{ and } N \leq \mathbf{K}(G). \end{aligned}$$

Also $\Lambda_{\mathfrak{N}}(G)$ has the radical property :

Proposition 3.2.

In every group G we have

$$\Lambda_{\mathfrak{N}}(G/N) = \Lambda_{\mathfrak{N}}(G)/N$$

whenever $N \trianglelefteq G$ and $N \leq \Lambda_{\mathfrak{N}}(G)$. In particular,

$$\Lambda_{\mathfrak{N}}(G/\Lambda_{\mathfrak{N}}(G)) \text{ is trivial, i. e. } G/\Lambda_{\mathfrak{N}}(G) \text{ is } \Lambda_{\mathfrak{N}}\text{-free.}$$

Corollary 3.3.

Let $G = T\Lambda_{\mathfrak{N}}(G)$ for some subgroup $T \leq G$. Then

$$\Lambda_{\mathfrak{N}}(T) = T \cap \Lambda_{\mathfrak{N}}(G).$$

Proof : We have that $G/\Lambda_{\mathfrak{N}}(G) \cong T/\Lambda_{\mathfrak{N}}(G) \cap T$ is $\Lambda_{\mathfrak{N}}$ -free. As $\Lambda_{\mathfrak{N}}(T/\Lambda_{\mathfrak{N}}(G) \cap T) = \Lambda_{\mathfrak{N}}(T)/\Lambda_{\mathfrak{N}}(G) \cap T$, the assertion follows.

To show 3.2, we first we prove

Lemma 3.4.

Let G be a group. If $G/\Lambda_{\mathfrak{N}}(G)$ is locally nilpotent, then so is G .

Proof : Let H be a finitely generated subgroup of G . Then $H/H \cap \Lambda_{\mathfrak{N}}(G) \cong \Lambda_{\mathfrak{N}}(G)H/\Lambda_{\mathfrak{N}}(G) \leq G/\Lambda_{\mathfrak{N}}(G) \in L\mathfrak{N}$. So $H/H \cap \Lambda_{\mathfrak{N}}(G)$ is nilpotent. Since $H \cap \Lambda_{\mathfrak{N}}(G) \leq \Lambda_{\mathfrak{N}}(H)$, we see that also $H/\Lambda_{\mathfrak{N}}(H)$ is nilpotent. For every $g \in H$ we have therefore that $\Lambda_{\mathfrak{N}}(H)\langle g \rangle \trianglelefteq H$ and $\Lambda_{\mathfrak{N}}(H)\langle g \rangle$ is locally nilpotent. So g is in the Hirsch-Plotkin radical of H^1 [6, pg. 357]. Therefore H is nilpotent and G is locally nilpotent.

Proof of 3.2: Let $N \trianglelefteq G$ such that $N \trianglelefteq \Lambda_{\mathfrak{N}}(G)$. By 3.1 c), we have $\Lambda_{\mathfrak{N}}(G/N) \geq \Lambda_{\mathfrak{N}}(G)/N$. Let $L/N = \Lambda_{\mathfrak{N}}(G/N)$. For every $g \in G$, we have that $Y/N = (L\langle g \rangle)/N = (L/N) \cdot \langle gN \rangle = \Lambda_{\mathfrak{N}}(G/N)\langle gN \rangle \in L\mathfrak{N}$. Since $\Lambda_{\mathfrak{N}}(G) \leq \Lambda_{\mathfrak{N}}(Y)$, also $Y/\Lambda_{\mathfrak{N}}(Y) \in LN$. Therefore, $Y = L\langle g \rangle$ is locally nilpotent, by Lemma 3.4, which means that $L \leq \Lambda_{\mathfrak{N}}(G)$ i. e. $L = \Lambda_{\mathfrak{N}}(G)$.

¹Owing to an unlucky language abuse, the Hirsch-Plotkin radical does not have the radical property.

4 Necessary and sufficient conditions for $H(G) = K(G)$ and for $K(G) = \Lambda_{\mathfrak{N}}(G)$

We saw that $H(G) \leq K(G) \leq \Lambda_{\mathfrak{N}}(G)$. In every finite group G we have $H(G) = K(G) = \Lambda_{\mathfrak{N}}(G)$ (see [5, Motiv. Thm.]).

Obviously, $H(G) < K(G) = G$, if G is a locally nilpotent, non-hypercentral group. Remarkable is, that $K(G)$ also does not coincide in general with $\Lambda_{\mathfrak{N}}(G)$, as in seen by the following.

Example 4.1. (A. KRASSILNIKOV)

In general,

$$K(G) \subsetneq \Lambda_{\mathfrak{N}}(G)$$

Proof: Let T_p be a TARSKI-group (i.e. an infinite simple group of exponent p) and consider the restricted regular wreath-product

$$G = C_p \wr T_p.$$

G is the semidirect product $G = [B]K$, where B is the base-group of G and $K \cong T_p$. G is finitely generated so that $K(G) = H(G) = Z(G) = 1$. We claim that $\Lambda_{\mathfrak{N}}(G) = B$: G is not (locally) nilpotent and if $1 \neq y \in K$, then the subgroups $M_y = B\langle y \rangle$ account for all maximal locally nilpotent subgroups of G and it follows $\Lambda_{\mathfrak{N}}(G) = B$. Clearly the M_y are some of them. Let Y be any maximal locally nilpotent subgroup of G . Certainly, $Y \not\leq B$, so that $B < BY$. Also $BY \neq G$, as $BY/B \cong Y/B \cap Y$ is, whereas $G/B \cong T_p$ is not locally nilpotent. So $BY = M_y$ for some $1 \neq y \in K$. It follows $Y = M_y$ by the maximality of Y .

Lemma 4.2.

Let G be a group and $M \trianglelefteq G$ such that $M \leq \Lambda_{\mathfrak{N}}(G)$. If M is finitely generated, then $M \leq \zeta_k(G)$ for some natural number $k \geq 0$, where $\zeta_k(G)$ is the k -th term of the upper central series of G .

Proof: M is nilpotent and polycyclic. We prove that, if $N \trianglelefteq G$ with $N < M$, then M/N contains a nontrivial central element of G/N . Knowing this, we conclude with the maximal condition in M , that there is a k such that $1 = \zeta_0(G) \cap M < \zeta_1(G) \cap M < \dots < \zeta_k(G) \cap M = M$. Therefore $M \leq \zeta_k(G)$.

For the proof of $M/N \cap \mathbf{Z}(G/N) > 1$ we may assume $N = 1 < M$. We have that $1 < \mathbf{Z}(M) \trianglelefteq G$. Let $Z = \mathbf{Z}(M)$.

Case 1 : If the torsion subgroup T of Z is nontrivial, we have for some prime p a finite nontrivial p -subgroup $P \trianglelefteq G$ with $P \leq T$. For every $g \in G$, we have the $P \langle g \rangle \leq \Lambda_{\mathfrak{N}}(G)$ and it follows that $P \langle g \rangle$ is nilpotent. g must induce a p -automorphism on P and therefore $G/C_G(P)$ is a finite p -group and some nontrivial element of P is in the center of G .

Case 2 : Z is torsion-free and is therefore the direct product of (finitely many) infinite cyclic groups. Let $D \trianglelefteq G$, $1 < D \leq Z$ with D of minimal rank r , say. For every prime p , let D_p denote the smallest subgroup of D with elementary abelian p -quotient. Then $|D/D_p| = p^r$ and $D_p \trianglelefteq G$. Let $D_p \leq X_p < D$ with $X_p \trianglelefteq G$ such that D/X_p is a chief factor of G . Since $M/X_p \leq \Lambda_{\mathfrak{N}}(G/X_p)$, we see by case 1 that D/X_p is G -central, i. e. $[D, G] \leq X_p$. Therefore $[D, G] \leq \bigcap_p X_p$. But $\bigcap_p X_p$, having infinite index in D , is of rank $< r$. So $\bigcap_p X_p = 1$, as r is minimal and it follows $1 < D \leq \mathbf{Z}(G)$. ■

Proposition 4.3.

Let G be a group. Equivalent are:

- a) $\mathbf{H}(G) = \mathbf{K}(G)$
- b) $\mathbf{K}(G)/\mathbf{H}(G)$ is finite.
- c) $\mathbf{K}(G)/\mathbf{H}(G)$ is finitely generated.

- d) The normal subgroups X of G with $\mathbf{H}(G) \leq X \leq \mathbf{K}(G)$ satisfy the minimal condition.

Proof: "a) $\Rightarrow$ b) $\Rightarrow$ c)" and "a) $\Rightarrow$ d)" are trivial.

"c) $\Rightarrow$ a)" : Since $\mathbf{K}(G/\mathbf{H}(G)) = \mathbf{K}(G)/\mathbf{H}(G)$ we may assume $\mathbf{H}(G) = 1$. We consider $M = \mathbf{K}(G)$ in Lemma 4.2. and see that $\mathbf{K}(G) \leq \zeta_k(G) \leq \mathbf{H}(G) = 1$.

"d) $\Rightarrow$ a)": If $\mathbf{H}(G) < \mathbf{K}(G)$, there will be $N \trianglelefteq G$, $N \leq \mathbf{K}(G)$, such that $N/\mathbf{H}(G)$ is a chief factor of G . By 4.2, $[N, G] \leq \mathbf{H}(G)$, against the radical property of the hypercenter (1.1 c)).

■

Proposition 4.4.

Let G be a group. Equivalent are :

- a) $\Lambda_{\mathfrak{N}}(G) = \mathbf{K}(G)$
- b) $\Lambda_{\mathfrak{N}}(G)/\mathbf{K}(G)$ is finite
- c) $\Lambda_{\mathfrak{N}}(G)/\mathbf{K}(G)$ is finitely generated

Proof : Only "c) $\Rightarrow$ a)" has to be justified : Since $\Lambda_{\mathfrak{N}}(G/\mathbf{K}(G)) = \Lambda_{\mathfrak{N}}(G)/\mathbf{K}(G)$, we may assume $\mathbf{K}(G) = 1$. Considering $M = \Lambda_{\mathfrak{N}}(G)$ in Lemma 4.2, we see $\Lambda_{\mathfrak{N}}(G) \leq \zeta_k(G) \leq \mathbf{K}(G)$, i. e. $\Lambda_{\mathfrak{N}}(G) = 1$.

Corollary 4.5.

If $G/\Lambda_{\mathfrak{N}}(G)$ is locally finite, then $\Lambda_{\mathfrak{N}}(G) = \mathbf{K}(G)$.

Proof : If $x \in \Lambda_{\mathfrak{N}}(G)$ and F_x is a finitely generated subgroup of G containing x , then $F_x/\Lambda_{\mathfrak{N}}(F_x)$ is finite, so that, by Schreier's theorem (see [6, 1.6.11]), also $\Lambda_{\mathfrak{N}}(F_x)$ is finitely generated. By 4.4 we see that $\Lambda_{\mathfrak{N}}(F_x) = \mathbf{K}(F_x) = \mathbf{H}(F_x)$, i. e. $x \in \mathbf{H}(F_x)$. This means $x \in \mathbf{K}(G)$. This means $x \in \mathbf{K}(G)$. Thus $\Lambda_{\mathfrak{N}}(G) = \mathbf{K}(G)$.

(We observe that in A. KRASSILNIKOV'S Example (4.1), $G/\Lambda_{\mathfrak{N}}(G)$ -in fact the whole group G -is a torsion group !).

■

To finish, we mention in this context, that BAER and B.H. NEUMANN characterize certain properties of groups by means of special finite coverings : A group G is finite over its (hyper-) center, if and only if G is covered by a finite family of abelian (hypercentral) subgroups (see [7, pg. 105/107]).

We close with the following theorem, concerning *finite coverings of groups by locally nilpotent subgroups*, which is analogue to BAER/NEUMANN's:

Theorem 4.6.

For a group G , the following statements are equivalent :

- a) $\mathbf{K}(G)$ has finite index in G
- b) $\Lambda_{\mathfrak{N}}(G)$ has finite index in G .
- c) $G = \bigcup_{k=1}^r H_k$ with locally nilpotent subgroups $H_1, H_2, \dots, H_r$ of G .

Proof : The equivalence " $b) \iff c)$ " is shown in [5, 8.3/8.4].

" $a) \implies b)$ " is trivial, as $\mathbf{K}(G) \leq \Lambda_{\mathfrak{N}}(G)$

" $b) \implies a)$ " : Since $\Lambda_{\mathfrak{N}}(G) = \mathbf{K}(G)$ by 4.5.

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