$O_{R}$-CONVERGENCE AND WEAK $O_{R}$-CONVERGENCE OF NETS AND THEIR APPLICATIONS$^{*}$

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Abstract
In this paper, the theory of $O_{R}$-convergence and weak $O_{R}$-convergence of nets is introduced in $L$-topological spaces by means of neighborhoods and strong neighborhoods of fuzzy points based on Shi’s $O$-convergence. It can be used to characterize preclosed sets, preopen sets, $\delta$-closed sets, $\delta$-open sets, near compactness and near $S^*$-compactness.

Keywords: $L$-space; neighborhood; strong neighborhood; $O_{R}$-convergence; weak $O_{R}$-convergence

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1. Introduction

As is known now, the Moore-smith convergence theory plays an important role in general topology, it not only is an significantly basic theory of fuzzy topology and fuzzy analysis but also has wide applications in fuzzy inference and some other aspects. In [18], Pu and Liu introduced the concept of Q-neighborhoods and established a systematic Moore-Smith convergence theory of fuzzy nets in $[0,1]$-topology. It paved a new way for the study of the fuzzy topology. Wang extended this theory to $L$-fuzzy set theory in [22]. Later on, all kinds of convergence theory were presented [2, 3, 4, 7, 8, 9, 12, 14]. In [19], Shi introduced the $O$-convergence theory of nets in terms of neighborhoods of fuzzy points in $L$-space. It overcomes the difficulty which the neighborhood method meets.

In this paper, our aim is to introduce the theory of $O_R$-convergence and weak $O_R$-convergence of nets in $L$-spaces based on Shi’s $O$-convergence. We shall discuss its properties and use them to characterize preclosed sets, preopen sets, $\delta$-closed sets, $\delta$-open sets, near compactness and near $S^*$-compactness.

2. Preliminaries

Throughout this paper $(L, \vee, \wedge')$ is a completely distributive de Morgan algebra. $X$ a nonempty set. $L^X$ is the set of all $L$-fuzzy sets (or $L$-sets for short) on $X$. The smallest element and the largest element in $L^X$ are denoted by $0$ and $1$.

An element $a$ in $L$ is called prime if $a \geq b \land c$ implies that $a \geq b$ or $a \geq c$. An element $a$ in $L$ is called co-prime if $a'$ is a prime element [13]. The set of nonunit prime elements in $L$ is denoted by $P(L)$. The set of nonzero co-prime elements in $L$ is denoted by $M(L)$. The set of nonzero co-prime elements in $L^X$ is denoted by $M(L^X)$. Members in $M(L^X)$ are also called points.

The binary relation $\prec$ in $L$ is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \leq L$, the relation $b \sup D$ always implies that the existence of $d \in D$ with $ad \geq b$ [10]. In a completely distributive de Morgan algebra $L$, each member $b$ is a sup of $\{a \in L \mid a \prec b\}$. In the sense of [15, 23], $\{a \in L \mid a \prec b\}$ is the greatest minimal family of $b$, in symbol $\beta(b)$. Moreover for $b \in L$, define $\alpha(b) = \{a \in L \mid a' \prec b\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For an $L$-set $G \in L^X$, $\beta(G)$ denotes the greatest minimal family of $G$
and $\beta^*(G) = \beta(G) \cap M(L^X)$.

An $L$-topological space (or $L$-space for short) is a pair $(X, T)$, where $T$ is a subfamily of $L^X$ which contains $\emptyset$, 1 and is closed for any suprema and finite infima. $T$ is called an $L$-topology on $X$. Each member of $T$ is called an open $L$-set and its quasi-complement is called a closed $L$-set.

**Definition 2.1.** Let $(X, T)$ be an $L$-space. $A \in L^X$ is called

1. regularly open [1] if $A^{-\circ} = A$, the complement of a regularly open set is called regularly closed;
2. $\beta$-open [16] if $AA^{-\circ}$, the complement of a $\beta$-open set is called $\beta$-closed;
3. preopen [16] if $AA^{-\circ}$, the complement of a preopen set is called preclosed. If $A$ is not only preopen, but also preclosed, then we call it preclopen.

**Definition 2.2 ([19]).** $x_\lambda \in M(L^X)$ is said to be quasi-coincident with $B \in L^X$ if $x_\lambda \not\subseteq B'$.

**Definition 2.3 ([19]).** An (a regularly open, preopen, $\delta$-open, etc.) open $L$-set $U$ is called an (a regularly open, preopen, $\delta$-open, etc.) open neighborhood of $x_\lambda \in M(L^X)$ if $X_\lambda U$. All (regularly open, preopen, $\delta$-open, etc.) open neighborhoods of $x_\lambda$ are denoted by $(N^\delta_R(x_\lambda), N^\delta_P(x_\lambda), N^\delta_S(x_\lambda))$.

**Definition 2.4 ([20]).** Let $(X, T)$ be an $L$-space. An (a regularly open, preopen, $\delta$-open, etc.) open $L$-set $U$ is called a strongly (regularly open, preopen, $\delta$-open, etc.) open neighborhood of a fuzzy point $x_\lambda$, if $\lambda \in \beta(U(x))$.

**Definition 2.5.** Let $(X, T_1)$ and $(Y, T_2)$ be two $L$-spaces. A map $f : (X, T_1) \rightarrow (Y, T_2)$ is called (1) almost continuous [1] if $f^L_\delta(G) \in T_1$ for all regularly open $L$-set $G$ in $(Y, T_2)$; (2) completely continuous [5, 17] if $f^L_\delta(G)$ is regularly open $L$-set in $(X, T_1)$ for each $G \in T_2$; (3) $R$-irresolute [21] if $f^L_\delta(G)$ is regularly closed in $(X, T_1)$ for each regularly closed $L$-set $G$ in $(Y, T_2)$; (4) $\delta$-continuous [11] if $f^L_\delta(G)$ is $\delta$-open in $(X, T_1)$ for each regularly open $L$-set $G$ in $(Y, T_2)$.

**Definition 2.6 ([19]).** A net $S$ with index set $D$ is also denoted by $\{S(n) \mid n \in D\}$ or $\{S(n)\}_{n \in D}$. For $G \in L^X$, a net $S$ is said to quasi-coincide with $G$ if $\forall n \in D, S(n) \not\subseteq G'$.

**Definition 2.7 ([19, 22]).** Let $\alpha \in M(L)$. A net $\{s(n) \mid n \in D\}$ in $L^X$ is called an $\alpha^-$-net if there exists $n_0 \in D$ such that $\forall n, n_0, V(S(n)) \alpha$, where
V(S(n)) denotes the height of S(n). A net \( \{ S(n) \} \_{n \in D} \) in \( L^X \) is said to be a constant \( \alpha \)-net if the height of each \( S(n) \) is a constant value \( \alpha \).

**Definition 2.8 ([19, 22])**. Let \( \{ S(n) \mid n \in D \} \) be a net in \((X, T)\), \( x_\lambda \in M(L^X) \). \( S \) eventually possesses the property \( P \), if there exists \( n_0 \in D \) such that \( \forall n \geq n_0 \), \( S(n) \) always possesses the property \( P \). \( S \) frequently possesses the property \( P \), if for every \( n \in D \), there always exists \( n_0 \in D \) such that \( n_0 \geq n \) and \( S(n_0) \) possesses the property \( P \).

**Definition 2.9 ([19])**. \( x_\lambda \) is an \( O \)-cluster point of \( S \), if \( \forall U \in N^0(x_\lambda) \), \( S \) is frequently in \( U \). \( x_\lambda \) is an \( O \)-limit point of \( S \), if \( \forall U \in N^0(x_\lambda) \), \( S \) is eventually in \( U \), in this case we also say that \( S \) \( O \)-converges to \( x_\lambda \), denoted by \( S \xrightarrow{O} x_\lambda \).

**Definition 2.10 ([20])**. Let \((X, T)\) be an \( L \)-space, \( a \in M(L) \) and \( G \in L^X \). A subfamily \( U \) of \( L^X \) is called a \( \beta_a \)-cover of \( G \) if for any \( x \in X \) with \( a \notin \beta(G'(x)) \), there exists an \( A \in U \) such that \( a \in \beta(A(x)) \). A \( \beta_a \)-cover \( U \) of \( G \) is called open (regularly open, etc.) \( \beta_a \)-cover of \( G \) if each member of \( U \) is open (regularly open, etc.).

It is obvious that \( U \) is a \( \beta_a \)-cover of \( G \) if and only if for any \( x \in X \) it follows that \( a \in \beta \left( G'(x) \lor \bigvee_{A \in U} A(x) \right) \).

**Definition 2.11 ([20])**. Let \((X, T)\) be an \( L \)-space, \( a \in M(L) \) and \( G \in L^X \). A subfamily \( U \) of \( L^X \) is called a \( Q_a \)-cover of \( G \) if for any \( x \in X \) with \( G(x) \leq a' \), it follows that \( \bigvee_{A \in U} A(x) \geq a \). A \( Q_a \)-cover \( U \) of \( G \) is called open (regularly open, etc.) \( Q_a \)-cover of \( G \) if each member of \( U \) is open (regularly open, etc.).

**Definition 2.12 ([21])**. Let \((X, T)\) be an \( L \)-space. \( G \in L^X \) is called nearly compact if for every family \( U \leq T \), it follows that

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in U} A(x) \right) \lor \bigvee_{V \in 2^U} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in V} A^{-\circ}(x) \right).
\]

**Lemma 2.13 ([21])**. Let \((X, T)\) be an \( L \)-space and \( G \in L^X \). Then \( G \) is nearly compact if and only if for any \( a \in M(L) \) and any \( b \in \beta^*(a) \), each open \( Q_a \)-cover of \( G \) has a finite subfamily \( V \) such that \( V^{-\circ} \) is a \( Q_b \)-cover of \( G \).
Definition 2.14. Let \((X, T)\) be an \(L\)-space and \(G \in L^X\). Then \(G\) is called nearly \(S^*\)-compact if for any \(a \in M(L)\), each \(\beta_a\)-cover of \(G\) has a finite subfamily \(V\) such that \(V^{-\circ} = \{A^{-\circ} \mid A \in V\}\) is a \(Q_a\)-cover of \(G\). \((X, T)\) is said to be nearly \(S^*\)-compact if \(1\) is nearly \(S^*\)-compact.

For the sake of convenience, we introduced the following concept.

Definition 2.15. Let \(A \in L^X\). \(cl_\delta(A) = \bigwedge\{V \mid AV^{-\circ}, V \in T'\}\) is called \(\delta\)-closure of \(A\). The \(\delta\)-interior of \(A\), written as \(int_\delta(A)\), is defined to be \(cl_\delta(A')\).

It can be proved that Definition 2.15 is equivalent to the notion of \(\delta\)-closure in [11] when \(L = [0, 1]\).

Obviously we have the following theorem.

Lemma 2.16. For each \(A \in L^X\), \(cl_\delta(A) \in T'\) and \(int_\delta(A) \in T\).

Lemma 2.17. Let \(A \in L^X\), then \(cl_\delta(A) = \bigwedge\{V \mid AV, V \text{ is regularly closed}\}\).

Lemma 2.18. Let \(A \in L^X\), then \(A^{-} = cl_\delta(A)\) and \(int_\delta(A)A^\circ\).

Lemma 2.19. If \(A\) is \(\beta\)-open, then \(A^{-} = cl_\delta(A)\); If \(A\) is \(\beta\)-closed, then \(A^\circ = int_\delta(A)\).

Definition 2.20. An \(L\)-set \(G\) is called \(\delta\)-closed if \(A = cl_\delta(A)\); The complement of a \(\delta\)-closed set is called \(\delta\)-open.

Lemma 2.21. Each regular open \(L\)-set is \(\delta\)-open and each regular closed \(L\)-set is \(\delta\)-closed.

3. \(O_R\)-convergence and weak \(O_R\)-convergence of nets

Definition 3.1. \(x_\lambda \in M(L^X)\) is said to be weak quasi-coincident with \(B \in L^X\) if \(\lambda \notin \beta(B'(x))\).

Definition 3.2. Let \((X, T)\) be an \(L\)-space, \(x_\lambda \in M(L^X)\) and \(S = \{S(n) \mid n \in D\}\) a net in \(L^X\). Then

(1) \(x_\lambda\) is an \(O_R\)-cluster point of \(S\), if \(\forall U \in N^\circ(x_\lambda), S\) is frequently in \(U^{-\circ}\).

(2) \(x_\lambda\) is an \(O_R\)-limit point of \(S\), if \(\forall U \in N^\circ(x_\lambda), S\) is eventually in \(U^{-\circ}\), in this case we also say that \(S\) \(O_R\)-converges to \(x_\lambda\), denoted by \(S \xrightarrow{O_R} x_\lambda\).
**Definition 3.3.** Let \( \{S(n) \mid n \in D\} \) be a net in \((X, T)\), \( x_\lambda \in M(L^X) \). \( x_\lambda \) is called a weak \( O_R \)-cluster point of \( S \), if for each strongly open neighborhood \( U \) of \( x_\lambda \), \( S \) is frequently in \( U^{-o} \). \( x_\lambda \) is called a weak \( O_R \)-limit point of \( S \), if for each strongly open neighborhood \( U \) of \( x_\lambda \), \( S \) is eventually in \( U^{-o} \), in this case, we also say that \( S \) weakly \( O_R \)-converges to \( x_\lambda \), denoted by \( S \overset{WO_R}{\rightarrow} x_\lambda \).

**Theorem 3.4.** Let \( S \) be a net in \((X, T)\) and \( x_\lambda \in M(L^X) \). Then the following conditions are equivalent.

1. \( x_\lambda \) is an \( O_R \)-cluster point of \( S \).
2. \( \forall U \in N^o_P(x_\lambda) \), \( S \) is frequently in \( U^{-o} \).
3. \( \forall U \in N^o_R(x_\lambda) \), \( S \) is frequently in \( U \).

**Proof.** (1) \( \Rightarrow \) (2) Suppose that \( x_\lambda \) is an \( O_R \)-cluster point of \( S \). If \( U \in N^o_P(x_\lambda) \), then \( U^{-o} \in N^o(x_\lambda) \). By the hypothesis of (1) \( S \) is frequently in \( U^{-o-o} \). \( S \) is frequently in \( U^{-o} \) since \( U^{-o-o} \subseteq U^{-o} \).

(2) \( \Rightarrow \) (3) is obvious.

(3) \( \Rightarrow \) (1) Suppose that the given condition hold for a net \( S \) and let \( U \in N^o(x_\lambda) \), then \( U^{-o} \in N^o_P(x_\lambda) \). By the hypothesis of (3) \( S \) is frequently in \( U^{-o} \). Therefore \( x_\lambda \) is an \( O_R \)-cluster point of \( S \). \( \Box \)

Analogous to the proof of Theorem 3.4 we can easily obtain the following result.

**Theorem 3.5.** Let \( S \) be a net in \((X, T)\) and \( x_\lambda \in M(L^X) \). Then the following conditions are equivalent.

1. \( x_\lambda \) is an \( O_R \)-limit point of \( S \).
2. \( \forall U \in N^o_P(x_\lambda) \), \( S \) is eventually in \( U^{-o} \).
3. \( \forall U \in N^o_R(x_\lambda) \), \( S \) is eventually in \( U \).

For weak \( O_R \)-convergence, we have same conclusions as Theorem 3.4 and Theorem 3.5. We omit them.

**Theorem 3.6.** Let \( S \) be a net in \((X, T)\) and \( x_\lambda \in M(L^X) \). Then

1. \( x_\lambda \) is a weak \( O_R \)-cluster point of \( S \) if and only if for each strongly \( \delta \)-open neighborhood \( U \) of \( x_\lambda \), \( S \) is frequently in \( U \).
2. \( x_\lambda \) is a weak \( O_R \)-limit point of \( S \) if and only if for each strongly \( \delta \)-open neighborhood \( U \) of \( x_\lambda \), \( S \) is eventually in \( U \).

**Proof.** (1) Sufficiency. Suppose that \( U \) is a strongly open neighborhood of \( x_\lambda \), then \( U^{-o} \) is a strongly \( \delta \)-open neighborhood of \( x_\lambda \). By the hypothesis, \( S \) is frequently in \( U^{-o} \). Therefore \( x_\lambda \) is a weak \( O_R \)-cluster point of \( S \).
Theorem 3.7. Let $S$ be a net in $(X, T)$, $T$ a subnet of $S$ and $x_\lambda, x_\mu \in M(L^X)$. Then

1. $S \overset{O}{\rightarrow} x_\lambda$ implies that $S \overset{OR}{\rightarrow} x_\lambda$;
2. $S \overset{OR}{\rightarrow} x_\lambda$ implies that $S \overset{WOR}{\rightarrow} x_\lambda$;
3. $S \overset{OR}{\rightarrow} x_\lambda$ implies that $x_\lambda$ is an $OR$-cluster point of $S$;
4. $S \overset{WOR}{\rightarrow} x_\lambda$ implies that $x_\lambda$ is a weak $OR$-cluster point of $S$;
5. $x_\lambda$ is an $O$-cluster point of $S$ implies that $x_\lambda$ is an $OR$-cluster point of $S$;
6. $x_\lambda$ is an $OR$-cluster point of $S$ implies that $x_\lambda$ is a weak $OR$-cluster point of $S$;
7. If $x_\lambda x_\mu$ and $x_\lambda$ is an $OR$-cluster point of $S$, then $x_\mu$ is also an $OR$-cluster point of $S$;
8. $S \overset{OR}{\rightarrow} x_\lambda x_\mu \Rightarrow S \overset{OR}{\rightarrow} x_\mu$;
9. If $x_\lambda x_\mu$ and $x_\lambda$ is a weak $OR$-cluster point of $S$, then $x_\mu$ is also a weak $OR$-cluster point of $S$;
10. $S \overset{WOR}{\rightarrow} x_\lambda x_\mu \Rightarrow S \overset{WOR}{\rightarrow} x_\mu$;
11. $S \overset{OR}{\rightarrow} x_\lambda \Rightarrow T \overset{OR}{\rightarrow} x_\lambda$;
12. $S \overset{WOR}{\rightarrow} x_\lambda \Rightarrow T \overset{WOR}{\rightarrow} x_\lambda$;
13. $x_\lambda$ is an $OR$-cluster point of $T$ implies that $x_\lambda$ is an $OR$-cluster point of $S$;
14. $x_\lambda$ is a weak $OR$-cluster point of $T$ implies that $x_\lambda$ is a weak $OR$-cluster point of $S$;
15. $x_\lambda$ is an $OR$-cluster point of $S$ if and only if $S$ has a subnet $R$ such that $R \overset{OR}{\rightarrow} x_\lambda$;
16. $x_\lambda$ is a weak $OR$-cluster point of $S$ if and only if $S$ has a subnet $R$ such that $R \overset{WOR}{\rightarrow} x_\lambda$. 
**Theorem 3.8.** Let \( x_\lambda \in M(L^X) \), \( B \) be \( \beta \)-open. Then the following conditions are equivalent.

1. \( x_\lambda \) quasi-coincides with \( B^- \).
2. There exists a net \( S \) quasi-coinciding with \( B \) such that \( S \xrightarrow{O_R} x_\lambda \).
3. There exists a net \( S \) quasi-coinciding with \( B \) such that \( x_\lambda \) is an \( O_R \)-cluster point of \( S \).

**Proof.** (1) \( \Rightarrow \) (2) Suppose that \( x_\lambda \) quasi-coincides with \( B^- \). Then \( \forall U \in N_B^0(x_\lambda), U \not\subseteq B' \), i.e., \( B^- \cup U' \). Hence \( B \not\subseteq U' \). This implies that \( U \not\subseteq B' \). Take \( S(U) \in M(L^X) \) such that \( S(U) \cup U \not\subseteq B' \). We obtain a net \( \{S(U) \mid U \in N_B^0(x_\lambda)\} \) \( O_R \)-converging to \( x_\lambda \) and it quasi-coincides with \( B \).

(2) \( \Rightarrow \) (3) is obvious by Theorem 3.7(3).

(3) \( \Rightarrow \) (1) Let \( \{S(n)\}_{n \in D} \) be a net quasi-coinciding with \( B \) and \( x_\lambda \) is an \( O_R \)-cluster point of \( S \). If \( x_\lambda (B^-)' \), then \( \forall n \in D \), there exists \( n_0 \in D \) such that \( n_0 \geq n \) and \( S(n_0)(B^-)' = B^- \cap B' \) since \( B \) is \( \beta \)-open, which contradicts that \( S \) quasi-coincides with \( B \). \( \square \)

**Corollary 3.9.** Let \((X,T)\) be an \( L \)-space and \( A \in L^X \). Then the following conditions are equivalent:

1. \( A \) is preclosed.
2. For any net \( S \) quasi-coinciding with \( A^\circ \), if \( S \xrightarrow{O_R} x_\lambda \), then \( x_\lambda \not\subseteq A' \).
3. For any net \( S \) quasi-coinciding with \( A^\circ \), if \( x_\lambda \) is an \( O_R \)-cluster point of \( S \), then \( x_\lambda \not\subseteq A' \).

**Proof.** (1) \( \Rightarrow \) (2) Suppose that \( x_\lambda A' \). Then \( A' \in N_B^0(x_\lambda) \). By Theorem 3.5 there exists \( n_0 \in D \) such that \( \forall n \geq n_0 \), \( S(n)A^\circ - A^\circ = A^\circ - A', \) which contradicts that \( S \) quasi-coincides with \( A^\circ \). Therefore \( x_\lambda \not\subseteq A' \).

(2) \( \Rightarrow \) (1) \( \forall x_\lambda \not\subseteq A^\circ - A', \) by Theorem 3.8 there exists a net quasi-coinciding with \( A^\circ \) such that \( S \xrightarrow{O_R} x_\lambda \). By the hypothesis of (2) \( x_\lambda \not\subseteq A' \). It implies that \( A' A^\circ - A, \) i.e., \( A^\circ - A \). Therefore \( A \) is preclosed.

(1) \( \Leftrightarrow \) (3) is analogous to the proof of (1) \( \Leftrightarrow \) (2). \( \square \)

**Corollary 3.10.** Let \((X,T)\) be an \( L \)-space and \( A \in L^X \). Then the following conditions are equivalent:

1. \( A \) is preopen.
2. \( \forall x_\lambda A, S \xrightarrow{O_R} x_\lambda \) implies that \( S \) is eventually in \( A^- \).
3. \( \forall x_\lambda A, \) if \( x_\lambda \) is \( O_R \)-cluster point of \( S \), then \( S \) is frequently in \( A^- \).

**Proof.** (1) \( \Rightarrow \) (2) is obvious by Theorem 3.5.
(2) \( \Rightarrow \) (1) \( \forall x_\lambda A^- = A'_-\), by Theorem 3.8 there exists a net quasi-coinciding with \( A'\) such that \( S \overset{O_R}{\rightarrow} x_\lambda\). If \( x_\lambda A\), by the hypothesis of (2) \( S\) is eventually in \( A^- A^-\), which contradicts that \( S\) quasi-coincides with \( A'\). Thus \( x_\lambda \not\subseteq A\). It implies that \( AA^-\). Therefore \( A\) is preopen.

(1) \( \Leftrightarrow \) (3) is analogous to the proof of (1) \( \Leftrightarrow \) (2). \( \square \)

**Corollary 3.11.** Let \( (X, T) \) be an \( L\)-space and \( A \in L^X\). Then \( A\) is preclopen if one of the following conditions is true.

1. For any net \( S\) quasi-coinciding with \( A'\), if \( S \overset{O_R}{\rightarrow} x_\lambda\), then \( x_\lambda \not\subseteq A'\)
2. \( \forall x_\lambda A, S \overset{O_R}{\rightarrow} x_\lambda\) implies that \( S\) is eventually in \( A^-\)
3. For any net \( S\) quasi-coinciding with \( A'\), if \( x_\lambda\) is an \( O_R\)-cluster point of \( S\), then \( x_\lambda \not\subseteq A'\)
4. \( \forall x_\lambda A,\) if \( x_\lambda\) is \( O_R\)-cluster point of \( S\), then \( S\) is frequently in \( A^-\).

**Proof.** Suppose that the condition (1) is satisfied. By Corollary 3.9 \( A\) is preclosed. Now we prove that \( A\) is preopen, i.e., \( AA^-\). \( \forall x_\lambda A^- = A'_-\), there exists a net \( S\) quasi-coinciding with \( A'\) such that \( S \overset{O_R}{\rightarrow} x_\lambda\). By the hypothesis of (1) and \( A'_- = A'\), it follows that \( x_\lambda \not\subseteq A\). This implies that \( AA^-\). Therefore \( A\) is preclopen.

Suppose that the condition (2) is satisfied. By Corollary 3.10, \( A\) is preopen. Now we prove that \( A\) is preclosed, i.e., \( A^- A^-\). \( \forall x_\lambda A^- = A'O\), there exists a net \( S\) quasi-coinciding with \( A'\) such that \( S \overset{O_R}{\rightarrow} x_\lambda\). If \( x_\lambda A'\), by the hypothesis of (2) \( S\) is eventually in \( A'O^- = A'O'\), which contradicts that \( S\) quasi-coincides with \( A^-\). Thus \( x_\lambda \not\subseteq A'\). It implies that \( A'A^-\), i.e., \( A^- A\). Therefore \( A\) is preclopen.

The other cases can achieved from the similar progress. \( \square \)

**Theorem 3.12.** Let \( x_\lambda \in M(L^X), B \in L^X\). Then the following conditions are equivalent.

1. \( x_\lambda\) weak quasi-coincides with \( \text{cl}_\delta(B)\).
2. There exists a net \( S\) quasi-coinciding with \( B\) such that \( S \overset{WOR}{\rightarrow} x_\lambda\).
3. There exists a net \( S\) quasi-coinciding with \( B\) such that \( x_\lambda\) is a weak \( O_R\)-cluster point of \( S\).

**Proof.** (1) \( \Rightarrow \) (2) Suppose that \( x_\lambda\) weak quasi-coincides with \( \text{cl}_\delta(B)\). Then for each strongly open neighborhood \( U\) of \( x_\lambda\), \( U \not\subseteq \text{cl}_\delta(B)'\), i.e., \( U \not\subseteq \bigcup\{C \mid CB', C\) is regularly open\}. Thus \( U^- \not\subseteq B'\). Take \( S(U) \in M(L^X)\) such that \( S(U)U^- = S(U) \not\subseteq B'\). We obtain a net

\[\{S(U) \mid U \text{ is a strongly open neighborhood of } x_\lambda\}.\]
It weak $O_R$-converges to $x_\lambda$ and quasi-coincides with $B$.

(2) $\Rightarrow$ (3) is obvious by Theorem 3.7(4).

(3) $\Rightarrow$ (1) Let $\{S(n)\}_{n \in D}$ be a net quasi-coinciding with $B$ and $x_\lambda$ is a weak $O_R$-cluster point of $S$. If $x_\lambda$ does not weak quasi-coincides with $cl_\delta(B)$, then $x_\lambda \notin \beta(cl_\delta(B'))$. Hence there exists a regularly open $L$-set $C$ such that $CB'$ and $x_\lambda \in \beta(C)$ since

$$x_\lambda \in \beta(cl_\delta(B')) = \beta((\bigwedge\{A \mid BA \text{ is regularly closed}\})')$$
$$= \beta(V\{A' \mid BA \text{ is regularly closed}\})$$
$$\cup \{\beta(C) \mid CB', C \text{ is regularly open}\}.$$

Then $\forall n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $S(n_0)CB'$, which contradicts that $S$ quasi-coincides with $B$. □

**Corollary 3.13.** Let $(X,T)$ be an $L$-space and $A \in L^X$. Then the following conditions are equivalent.

1. $A$ is $\delta$-closed.
2. For any net $S$ quasi-coinciding with $A$, if $S \xrightarrow{WO_R} x_\lambda$, then $x_\lambda \notin \beta(A')$.
3. For any net $S$ quasi-coinciding with $A$, if $x_\lambda$ is a weak $O_R$-cluster point of $S$, then $x_\lambda \in \beta(A')$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $x_\lambda \notin \beta(A')$. By Theorem 3.6, $S$ is eventually in $A'$ since $A$ is $\delta$-closed, which contradicts that $S$ quasi-coincides with $A$.

(2) $\Rightarrow$ (1) Suppose that $x_\lambda \notin \beta(cl_\delta(A'))$. Then there exists a net $S$ quasi-coinciding with $A$ such that $S \xrightarrow{WO_R} x_\lambda$. By the hypothesis of (2), it follows that $x_\lambda \notin \beta(A')$. Therefore $A'cl_\delta(A')$, i.e., $cl_\delta(A)A$. By Lemma 2.18 we know that $Acl_\delta(A)$. Therefore $A$ is $\delta$-closed.

(1) $\Leftrightarrow$ (3) This proof is analogous to the proof of (1) $\Leftrightarrow$ (2). □

**Corollary 3.14.** Let $(X,T)$ be an $L$-space and $A \in L^X$. Then the following conditions are equivalent.

1. $A$ is $\delta$-open.
2. $\forall x_\lambda \in \beta(A)$, $S \xrightarrow{WO_R} x_\lambda$ implies that $S$ is eventually in $A$.
3. $\forall x_\lambda \in \beta(A)$, if $x_\lambda$ is weak $O_R$-cluster point of $S$, then $S$ is frequently in $A$.

**Proof.** (1) $\Rightarrow$ (2) is obvious by Theorem 3.6.

(2) $\Rightarrow$ (1) Suppose that $x_\lambda \in \beta(A)$. If $x_\lambda \notin \beta(int_\delta(A)) = \beta(cl_\delta(A')')$, then there exists a net $S$ quasi-coinciding with $A'$ such that $S \xrightarrow{WO_R} x_\lambda$. 

By the hypothesis of (2), $S$ is eventually in $A$, which contradicts that $S$ quasi-coincides with $A'$. Thus $x_\lambda \in \beta(\text{int}_\delta(A))$. It implies that $\text{Aint}_\delta(A)$.

By Lemma 2.18 we know that $\text{int}_\delta(A)A$. Therefore $A$ is $\delta$-open.

(1) $\Leftrightarrow$ (3) This proof is analogous to the proof of (1) $\Leftrightarrow$ (2). □

**Theorem 3.15.** Let $f : (X, T_1) \to (Y, T_2)$ be a $R$-irresolute $L$-value Zadeh's type mapping. Then

1. For any net $S$ in $L^X$, if $S \xrightarrow{OR} x_\lambda$, then $f_{L^-}(S) \xrightarrow{OR} f_{L^-}(x_\lambda)$.
2. For any net $S$ in $L^X$, if $x_\lambda$ is an $O_R$-cluster point of $S$, then $f_{L^-}(x_\lambda)$ is an $O_R$-cluster point of $f_{L^-}(S)$.

**Proof.** (1) Suppose that $U \in N^0_R(f_{L^-}(x_\lambda))$. Then $f_{L^-}(U) \in N^0_R(x_\lambda)$.

Since $S \xrightarrow{OR} x_\lambda$, there exists $n_0 \in D$ such that $\forall n \geq n_0 \ S(n) f_{L^-}(U)$. This implies that $f_{L^-}(S) \xrightarrow{OR} f_{L^-}(x_\lambda)$ by

$$f_{L^-}(S(n)) f_{L^-}(f_{L^-}(U)) U.$$  

(2) This is analogous to the proof of (1). □

**Theorem 3.16.** Let $f : (X, T_1) \to (Y, T_2)$ be an almost continuous $L$-value Zadeh's type mapping. Then

1. For any net $S$ in $L^X$, if $S \xrightarrow{O} x_\lambda$, then $f_{L^-}(S) \xrightarrow{OR} f_{L^-}(x_\lambda)$.
2. For any net $S$ in $L^X$, if $x_\lambda$ is an $O$-cluster point of $S$, then $f_{L^-}(x_\lambda)$ is an $O_R$-cluster point of $f_{L^-}(S)$.

**Proof.** (1) Suppose that $U \in N^0_R(f_{L^-}(x_\lambda))$. Then $f_{L^-}(U) \in N^0_R(x_\lambda)$.

Since $S \xrightarrow{O} x_\lambda$, there exists $n_0 \in D$ such that $\forall n \geq n_0 \ S(n) f_{L^-}(U)$. This implies that $f_{L^-}(S) \xrightarrow{OR} f_{L^-}(x_\lambda)$ by

$$f_{L^-}(S(n)) f_{L^-}(f_{L^-}(U)) U.$$  

(2) This is analogous to the proof of (1). □

**Theorem 3.17.** Let $f : (X, T_1) \to (Y, T_2)$ be a completely continuous $L$-value Zadeh's type mapping. Then

1. For any net $S$ in $L^X$, if $S \xrightarrow{OR} x_\lambda$, then $f_{L^-}(S) \xrightarrow{O} f_{L^-}(x_\lambda)$.
2. For any net $S$ in $L^X$, if $x_\lambda$ is an $O_R$-cluster point of $S$, then $f_{L^-}(x_\lambda)$ is an $O$-cluster point of $f_{L^-}(S)$.  

\textbf{Proof.} (1) Suppose that $U \in N^\circ(f^-_L(x_\lambda))$. Then $f^-_L(U) \in N^\circ_R(x_\lambda)$. Since $S \xrightarrow{O_R} x_\lambda$, there exists $n_0 \in D$ such that $\forall n \geq n_0 \ S(n)f^-_L(U)$. This implies that $f^-_L(S) \xrightarrow{O} f^-_L(x_\lambda)$ by

$$f^-_L(S(n))f^-_L(f^-_L(U))U.$$ 

(2) This is analogous to the proof of (1). \hfill \Box

For weak $O_R$-convergence, we have the similar three conclusions as above since $f^-_L(x_\lambda) \in \beta(U)$ implies $x_\lambda \in \beta(f^-_L(U))$. They are also omitted here.

\textbf{Theorem 3.18.} Let $f : (X,T_1) \rightarrow (Y,T_2)$ be an $L$-value Zadeh’s type mapping. Then the following conditions are equivalent.

1. $f$ is $\delta$-continuous.
2. For any net $S$ in $L^X$, if $S \xrightarrow{WO_R} x_\lambda$, then $f^-_L(S) \xrightarrow{WO_R} f^-_L(x_\lambda)$.
3. For any net $S$ in $L^X$, if $x_\lambda$ is a weak $O_R$-cluster point of $S$, then $f^-_L(x_\lambda)$ is a weak $O_R$-cluster point of $f^-_L(S)$.

\textbf{Proof.} (1) $\Rightarrow$ (2) Suppose that $U$ is a strongly regularly open neighborhood of fuzzy point $f^-_L(x_\lambda)$ and net $S \xrightarrow{WO_R} x_\lambda$. Then $x_\lambda \in \beta(f^-_L(U))$. Thus $f^-_L(U)$ is a strongly $\delta$-open neighborhood of fuzzy point $x_\lambda$ since $f$ is $\delta$-continuous. Therefore exists $n_0 \in D$ such that $\forall n \geq n_0, S(n)f^-_L(U)$. Thus $f^-_L(S(n))f^-_L(f^-_L(U))U$ for any $n \geq n_0$. Therefore $f^-_L(S) \xrightarrow{WO_R} f^-_L(x_\lambda)$.

(2) $\Rightarrow$ (1) Suppose that $A$ is a regularly open $L$-set in $(Y,T_2)$. \forall x_\lambda \in \beta(f^-_L(A))$, let $S \xrightarrow{WO_R} x_\lambda$. By the hypothesis of (2) $f^-_L(S) \xrightarrow{WO_R} f^-_L(x_\lambda)$. There exists $n_0 \in D$ such that $\forall n \geq n_0, f^-_L(S(n))A$ since $f^-_L(x_\lambda) \in \beta(f^-_L(A)) \leq \beta(A)$. It implies that $S(n)f^-_L(A)$. Thus $f^-_L(A)$ is $\delta$-open by Corollary 3.14. Therefore $f$ is $\delta$-continuous.

(1) $\iff$ (3) This is analogous to the proof of (1) $\iff$ (2). \hfill \Box

4. Characterizations of near (compactness) $S^*$-compactness

\textbf{Theorem 4.1.} An $L$-set $G$ is nearly compact in $(X,T)$ if and only if $\forall a \in M(L), \forall b \in \beta^*(a)$, each constant $b$-net quasi-coinciding with $G$ has an $O_R$-cluster point $x_a$ quasi-coinciding with $G$.

\textbf{Proof.} Suppose that $G$ is nearly compact. For $a \in M(L)$ and $b \in \beta^*(a)$, let $\{S(n) \mid n \in D\}$ be a constant $b$-net quasi-coinciding with $G$. Suppose that $S$ has no $O_R$-cluster point $x_a$ quasi-coinciding with $G$. Then for each
such that \( \Psi_n \) is nearly compact, that there exists a subfamily \( \Psi = \{ U_x \mid x \notin G' \} \), then \( \Phi \) is an open \( Q_a \)-cover of \( G \). Since \( G \) is nearly compact, \( \Phi \) has a finite subfamily \( \Psi = \{ U_{x_i} \mid i = 1, 2, \cdots, k \} \) such that \( \Psi^{-\circ} \) is a \( Q_b \)-cover of \( G \). Since \( D \) is a directed set, there exists \( n_0 \in D \) such that \( n_0 \geq n_{x_i} \) for each \( i \). Thus we can obtain that \( \forall n \geq n_0, S(n) \cup \{ U_{x_i}^{-\circ} \mid i = 1, 2, \cdots, k \} \). This contradicts that \( \Psi^{-\circ} \) is a \( Q_b \)-cover of \( G \). Therefore \( S \) has an \( O_R \)-cluster point \( x_a \notin G' \).

Conversely suppose that \( \forall a \in M(L), \forall b \in \beta^*(a) \), each constant \( b \)-net quasi-coinciding with \( G \) has an \( O_R \)-cluster point \( x_a \notin G' \). We now prove that \( G \) is nearly compact. Let \( \Phi \) be an open \( Q_a \)-cover of \( G \). If for each finite subfamily \( \Psi \) of \( \Phi \), \( \Psi^{-\circ} \) is not a \( Q_b \)-cover of \( G \), then for each finite subfamily \( \Psi \) of \( \Phi \), there exists \( S(\Psi) \in \mathcal{M}(L^X) \) with height \( b \) such that \( S(\Psi) \notin G' \) and \( S(\Psi) \notin \bigvee \Psi^{-\circ} \). Take \( S = \{ S(\Psi) \mid \Psi \text{ is a finite subfamily of } \Phi \} \), then \( S \) is a constant \( b \)-net quasi-coinciding with \( G \). By \( b \in \beta^*(a) \) we can take \( s \in \beta^*(a) \) such that \( b \in \beta^*(s) \). Then \( S \) has an \( O_R \)-cluster point \( x_s \notin G' \). Hence for each finite subfamily \( \Psi \) of \( \Phi \) we have that \( x_s \notin \bigvee \Psi \) (because if \( x_s \notin \bigvee \Psi \), then there exists an \( A \in \Psi \) such that \( x_s \notin A \), i.e., \( A \) is an open neighborhood of \( x_s \), hence there exists a finite subfamily \( \Psi_0 \) of \( \Phi \) such that \( \Psi \leq \Psi_0 \) and \( S(\Psi_0)A^{-\circ} \in \bigvee \Psi^{-\circ} \cup \Psi_0^{-\circ} \), this contradicts the definition of \( S \) ), in particular \( x_s \notin B \) for each \( B \in \Phi \). But since \( \Phi \) is an open \( Q_a \)-cover of \( G \), we know that there exists \( B \in \Phi \) such that \( x_s \notin B \), this yields a contradiction with \( x_s \notin B \). So \( G \) is nearly compact. \( \Box \)

**Theorem 4.2.** An \( L \)-set \( G \) is nearly compact in \((X, T)\) if and only if \( \forall a \in M(L), \forall b \in \beta^*(a), \) each \( b^- \)-net quasi-coinciding with \( G \) has an \( O_R \)-cluster point \( x_a \) quasi-coinciding with \( G \).

**Proof.** The sufficiency is obvious, we need only to prove the necessity.

Let \( G \) be nearly compact, \( a \in M(L), b \in \beta^*(a) \) and \( \{ S(n) \mid n \in D \} \) be an \( b^- \)-net quasi-coinciding with \( G \). Then there exists \( n_0 \in D \) such that \( \forall n \geq n_0 \), \( S(n)b \). Put \( E = \{ n \in D \mid n \geq n_0 \} \) and

\[
T = \{ T(n) \mid n \in E, V(T(n)) = b \}, \text{ the support point of } T(n) \text{ is same as } S(n). \]

Then \( T \) is a constant \( b \)-net quasi-coinciding with \( G \). Let \( x_a \) be an \( O_R \)-cluster point of \( T \). It is easy to see that \( x_a \) is also an \( O_R \)-cluster point of \( S \). \( \Box \)

Analogous to the proof of Theorem 4.1 and Theorem 4.2 we can easily obtain the following two results.
Theorem 4.3. An $L$-set $G$ is near $S^*$-compact in $(X, T)$ if and only if $\forall a \in M(L)$, each constant $a$-net quasi-coinciding with $G$ has a weak $O_R$-cluster point $x_a / \notin \beta(G')$.

Theorem 4.4. An $L$-set $G$ is near $S^*$-compact in $(X, T)$ if and only if $\forall a \in M(L)$, each $a^-$-net quasi-coinciding with $G$ has a weak $O_R$-cluster point $x_a / \notin \beta(G')$.

References


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